Elements of Chronological Calculus-2

(Lecture 4)

Yuri Sachkov

yusachkov@gmail.com

«Elements of geometric control theory»

Lecture course in Dept. of Mathematics and Mechanics

Lomonosov Moscow State University

12 March 2025

Plan of previous lecture

1. Points, Diffeomorphisms, and Vector Fields
2. Seminorms and $C^\infty(M)$-Topology
3. Families of Functionals and Operators
4. ODEs with discontinuous right-hand side
5. Definition of the right chronological exponential
6. Formal series expansion
7. Estimates and convergence of the series
8. Left chronological exponential
9. Uniqueness for functional and operator ODEs

Plan of this lecture

1. Autonomous vector fields
2. Action of diffeomorphisms on vector fields
3. Commutation of flows
4. Variations formula
5. Derivative of flow with respect to parameter
6. Differential 1-forms

Autonomous vector fields

- For an *autonomous vector field*

$$V_t \equiv V \in \operatorname{Vec} M,$$

  the flow generated by a complete field is called the *exponential* and is denoted as e^{tV}.
- The asymptotic series for the exponential takes the form

$$e^{tV} \approx \sum_{n=0}^{\infty} \frac{t^n}{n!} V^n = \mathrm{Id} + tV + \frac{t^2}{2} V \circ V + \cdots \; ,$$

  i.e, it is the standard exponential series.
- The exponential of an autonomous vector field satisfies the ODEs

$$\frac{d}{d\,t} e^{tV} = e^{tV} \circ V = V \circ e^{tV}, \qquad e^{tV}\Big|_{t=0} = \mathrm{Id}\,.$$

- We apply the asymptotic series for exponential to find the Lie bracket of autonomous vector fields $V, W \in \operatorname{Vec} M$.
- We compute the first nonconstant term in the asymptotic expansion at $t = 0$ of the curve:

$$
\begin{aligned}
q(t) &= q \circ e^{tV} \circ e^{tW} \circ e^{-tV} \circ e^{-tW} \\
&= q \circ \left(\mathrm{Id} + tV + \frac{t^2}{2} V^2 + \cdots \right) \circ \left(\mathrm{Id} + tW + \frac{t^2}{2} W^2 + \cdots \right) \\
&\quad \circ \left(\mathrm{Id} - tV + \frac{t^2}{2} V^2 + \cdots \right) \circ \left(\mathrm{Id} - tW + \frac{t^2}{2} W^2 + \cdots \right) \\
&= q \circ \left(\mathrm{Id} + t(V + W) + \frac{t^2}{2} (V^2 + 2V \circ W + W^2) + \cdots \right) \\
&\quad \circ \left(\mathrm{Id} - t(V + W) + \frac{t^2}{2} (V^2 + 2V \circ W + W^2) + \cdots \right) \\
&= q \circ (\mathrm{Id} + t^2 (V \circ W - W \circ V) + \cdots) .
\end{aligned}
$$

- So the Lie bracket of the vector fields as operators (directional derivatives) in $C^\infty(M)$ is
$$[V,W] = V \circ W - W \circ V.$$
- This proves the formula in local coordinates: if
$$V = \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i}, \qquad W = \sum_{i=1}^{n} b_i \frac{\partial}{\partial x_i}, \qquad a_i,\ b_i \in C^\infty(M),$$
then
$$[V,W] = \sum_{i,j=1}^{n} \left(a_j \frac{\partial b_i}{\partial x_j} - b_j \frac{\partial a_i}{\partial x_j} \right) \frac{\partial}{\partial x_i} = \frac{dW}{dx} V - \frac{dV}{dx} W.$$
- Similarly,
$$\begin{aligned} q \circ e^{tV} \circ e^{sW} \circ e^{-tV} &= q \circ (\mathrm{Id} + tV + \cdots) \circ (\mathrm{Id} + sW + \cdots) \circ (\mathrm{Id} - tV + \cdots) \\ &= q \circ (\mathrm{Id} + sW + ts[V,W] + \cdots), \end{aligned}$$
and
$$q \circ [V,W] = \left. \frac{\partial^2}{\partial s \partial t} \right|_{s=t=0} q \circ e^{tV} \circ e^{sW} \circ e^{-tV}.$$

Action of diffeomorphisms on tangent vectors

- We have already found counterparts to points, diffeomorphisms, and vector fields among functionals and operators on $C^\infty(M)$. Now we consider action of diffeomorphisms on tangent vectors and vector fields.
- Take a tangent vector $v \in T_q M$ and a diffeomorphism $P \in \operatorname{Diff} M$. The tangent vector $P_* v \in T_{P(q)} M$ is the velocity vector of the image of a curve starting from q with the velocity vector v. We claim that

$$P_* v = v \circ P, \qquad v \in T_q M, \quad P \in \operatorname{Diff} M, \tag{1}$$

as functionals on $C^\infty(M)$.
- Take a curve

$$q(t) \in M, \qquad q(0) = q, \quad \left.\frac{d}{d\,t}\right|_{t=0} q(t) = v,$$

then

$$\begin{aligned} P_* v\, a &= \left.\frac{d}{d\,t}\right|_{t=0} a(P(q(t))) = \left(\left.\frac{d}{d\,t}\right|_{t=0} q(t) \right) \circ P a \\ &= v \circ P a, \qquad a \in C^\infty(M). \end{aligned}$$

Action of diffeomorphisms on vector fields

- Now we find expression for P_*V, $V \in \operatorname{Vec} M$, as a derivation of $C^\infty(M)$.
- We have

$$\begin{aligned} q \circ P \circ P_*V &= P(q) \circ P_*V = (P_*V)(P(q)) = P_*(V(q)) = V(q) \circ P \\ &= q \circ V \circ P, \quad q \in M, \end{aligned}$$

thus

$$P \circ P_*V = V \circ P,$$

i.e.,

$$P_*V = P^{-1} \circ V \circ P, \qquad P \in \operatorname{Diff} M, \; V \in \operatorname{Vec} M.$$

- So diffeomorphisms act on vector fields as similarities.
- In particular, diffeomorphisms preserve compositions:

$$P_*(V \circ W) = P^{-1} \circ (V \circ W) \circ P = (P^{-1} \circ V \circ P) \circ (P^{-1} \circ W \circ P) = P_*V \circ P_*W,$$

thus Lie brackets of vector fields:

$$P_*[V, W] = P_*(V \circ W - W \circ V) = P_*V \circ P_*W - P_*W \circ P_*V = [P_*V, P_*W].$$

Action of diffeomorphisms on vector fields

- If $B : C^\infty(M) \to C^\infty(M)$ is an automorphism, then the standard algebraic notation for the corresponding similarity is $\operatorname{Ad} B$:

$$(\operatorname{Ad} B)V \stackrel{\text{def}}{=} B \circ V \circ B^{-1}.$$

- That is,

$$P_* = \operatorname{Ad} P^{-1}, \qquad P \in \operatorname{Diff} M.$$

- Now we find an infinitesimal version of the operator Ad.
- Let P^t be a flow on M,

$$P^0 = \mathrm{Id}, \qquad \left.\frac{d}{d\,t}\right|_{t=0} P^t = V \in \operatorname{Vec} M.$$

- Then

$$\left.\frac{d}{d\,t}\right|_{t=0} \left(P^t\right)^{-1} = -V,$$

so

$$\begin{aligned} \left.\frac{d}{d\,t}\right|_{t=0} (\operatorname{Ad} P^t)W &= \left.\frac{d}{d\,t}\right|_{t=0} (P^t \circ W \circ (P^t)^{-1}) = V \circ W - W \circ V \\ &= [V, W], \qquad W \in \operatorname{Vec} M. \end{aligned}$$

- Denote

$$\operatorname{ad} V = \operatorname{ad}\left(\left.\frac{d}{d\,t}\right|_{t=0} P^t\right) \stackrel{\text{def}}{=} \left.\frac{d}{d\,t}\right|_{t=0} \operatorname{Ad} P^t,$$

then

$$(\operatorname{ad} V)W = [V, W], \qquad W \in \operatorname{Vec} M.$$

- Differentiation of the equality

$$\operatorname{Ad} P^t [X, Y] = [\operatorname{Ad} P^t X, \operatorname{Ad} P^t Y] \qquad X, Y \in \operatorname{Vec} M,$$

at $t = 0$ gives *Jacobi identity* for Lie bracket of vector fields:

$$(\operatorname{ad} V)[X, Y] = [(\operatorname{ad} V)X, Y] + [X, (\operatorname{ad} V)Y],$$

which may also be written as

$$[V, [X, Y]] = [[V, X], Y] + [X, [V, Y]], \qquad V, X, Y \in \operatorname{Vec} M,$$

or, in a symmetric way

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0, \qquad X, Y, Z \in \operatorname{Vec} M. \tag{2}$$

- The set $\operatorname{Vec} M$ is a vector space with an additional operation — Lie bracket, which has the properties:
  (1) bilinearity:

$$[\alpha X + \beta Y, Z] = \alpha[X, Z] + \beta[Y, Z],$$
$$[X, \alpha Y + \beta Z] = \alpha[X, Y] + \beta[X, Z], \qquad X, Y, Z \in \operatorname{Vec} M, \quad \alpha, \beta \in \mathbb{R},$$

  (2) skew-symmetry:

$$[X, Y] = -[Y, X], \qquad X, Y \in \operatorname{Vec} M,$$

  (3) Jacobi identity (2).
- In other words, the set $\operatorname{Vec} M$ of all smooth vector fields on a smooth manifold M forms a *Lie algebra*.

- Consider the flow $P^t = \overrightarrow{\exp} \int_0^t V_\tau \, d\tau$ of a nonautonomous vector field V_t. We find an ODE for the family of operators $\mathrm{Ad}\, P^t = (P^t)_*^{-1}$ on the Lie algebra $\mathrm{Vec}\, M$.

$$
\begin{aligned}
\frac{d}{d\,t}(\mathrm{Ad}\, P^t)X &= \frac{d}{d\,t}\left(P^t \circ X \circ (P^t)^{-1}\right) \\
&= P^t \circ V_t \circ X \circ (P^t)^{-1} - P^t \circ X \circ V_t \circ (P^t)^{-1} \\
&= (\mathrm{Ad}\, P^t)[V_t, X] = (\mathrm{Ad}\, P^t)\,\mathrm{ad}\, V_t\, X, \qquad X \in \mathrm{Vec}\, M.
\end{aligned}
$$

- Thus the family of operators $\mathrm{Ad}\, P^t$ satisfies the ODE

$$
\frac{d}{d\,t}\,\mathrm{Ad}\, P^t = (\mathrm{Ad}\, P^t) \circ \mathrm{ad}\, V_t \tag{3}
$$

with the initial condition

$$
\mathrm{Ad}\, P^0 = \mathrm{Id}\,. \tag{4}
$$

- So the family $\mathrm{Ad}\, P^t$ is an invertible solution for the Cauchy problem

$$
\dot{A}_t = A_t \circ \mathrm{ad}\, V_t, \quad A_0 = \mathrm{Id}
$$

for operators $A_t : \mathrm{Vec}\, M \to \mathrm{Vec}\, M$.

- We can apply the same argument as for the analogous Cauchy problem for flows to derive the asymptotic expansion

$$\operatorname{Ad} P^t \approx \mathrm{Id} + \int_0^t \operatorname{ad} V_\tau \, d\tau + \cdots + \underset{\Delta_n(t)}{\int \cdots \int} \operatorname{ad} V_{\tau_n} \circ \cdots \circ \operatorname{ad} V_{\tau_1} \, d\tau_n \, \ldots \, d\tau_1 + \cdots \qquad (5)$$

then prove uniqueness of the solution, and justify the following notation:

$$\overrightarrow{\exp} \int_0^t \operatorname{ad} V_\tau \, d\tau \overset{\text{def}}{=} \operatorname{Ad} P^t = \operatorname{Ad} \left(\overrightarrow{\exp} \int_0^t V_\tau \, d\tau \right).$$

- Similar identities for the left chronological exponential are

$$\overleftarrow{\exp} \int_0^t \operatorname{ad}(-V_\tau) \, d\tau \overset{\text{def}}{=} \operatorname{Ad} \left(\overleftarrow{\exp} \int_0^t (-V_\tau) \, d\tau \right)$$
$$\approx \mathrm{Id} + \sum_{n=1}^{\infty} \underset{\Delta_n(t)}{\int \cdots \int} (-\operatorname{ad} V_{\tau_1}) \circ \cdots \circ (-\operatorname{ad} V_{\tau_n}) \, d\tau_n \, \ldots \, d\tau_1 .$$

- For the asymptotic series (5), there holds an estimate of the remainder term similar to the estimate for the flow P^t.
- Denote the partial sum

$$T_m = \mathrm{Id} + \sum_{n=1}^{m-1} \underset{\Delta_n(t)}{\int \cdots \int} \operatorname{ad} V_{\tau_n} \circ \cdots \circ \operatorname{ad} V_{\tau_1} \, d\tau_n \dots d\tau_1,$$

then for any $X \in \operatorname{Vec} M$, $s \geq 0$, $K \Subset M$

$$\begin{aligned} &\left\| \left(\operatorname{Ad} \overrightarrow{\exp} \int_0^t V_\tau \, d\tau - T_m \right) X \right\|_{s,K} \\ &\qquad \leq C_1 e^{C_1 \int_0^t \|V_\tau\|_{s+1,K'} \, d\tau} \frac{1}{m!} \left(\int_0^t \|V_\tau\|_{s+m,K'} \, d\tau \right)^m \|X\|_{s+m,K'} \qquad (6) \\ &\qquad = O(t^m), \qquad t \to 0, \end{aligned}$$

where $K' \Subset M$ is some compactum containing K.

- For autonomous vector fields, we denote

$$e^{t\,\mathrm{ad}\,V} \stackrel{\mathrm{def}}{=} \mathrm{Ad}\, e^{tV},$$

thus the family of operators $e^{t\,\mathrm{ad}\,V} : \mathrm{Vec}\, M \to \mathrm{Vec}\, M$ is the unique solution to the problem

$$\dot{A}_t = A_t \circ \mathrm{ad}\, V, \qquad A_0 = \mathrm{Id},$$

which admits the asymptotic expansion

$$e^{t\,\mathrm{ad}\,V} \approx \mathrm{Id} + t\,\mathrm{ad}\, V + \frac{t^2}{2}\mathrm{ad}^2\, V + \cdots \ .$$

- Let $P \in \mathrm{Diff}\, M$, and let V_t be a nonautonomous vector field on M. Then

$$P \circ \overrightarrow{\exp} \int_0^t V_\tau \, d\tau \circ P^{-1} = \overrightarrow{\exp} \int_0^t \mathrm{Ad}\, P\, V_\tau \, d\tau \qquad (7)$$

since the both parts satisfy the same operator Cauchy problem.

Commutation of flows

Let $V_t \in \operatorname{Vec} M$ be a nonautonomous vector field and $P^t = \overrightarrow{\exp} \int_0^t V_\tau \, d\tau$ the corresponding flow. We are interested in the question: under what conditions the flow P^t preserves a vector field $W \in \operatorname{Vec} M$?

Proposition 1

$P^t_* W = W \quad \forall t \quad \Leftrightarrow \quad [V_t, W] = 0 \quad \forall t.$

Proof.

$$
\begin{aligned}
\frac{d}{d\,t}(P_t)^{-1}_* W &= \frac{d}{d\,t} \operatorname{Ad} P^t W = \left(\frac{d}{d\,t} \overrightarrow{\exp} \int_0^t \operatorname{ad} V_\tau \, d\tau \right) W \\
&= \left(\overrightarrow{\exp} \int_0^t \operatorname{ad} V_\tau \, d\tau \circ \operatorname{ad} V_\tau \right) W = \left(\overrightarrow{\exp} \int_0^t \operatorname{ad} V_\tau \, d\tau \right) [V_t, W] \\
&= (P^t)^{-1}_* [V_t, W],
\end{aligned}
$$

thus $(P^t)^{-1}_* W \equiv W$ if and only if $[V_t, W] \equiv 0$. □

- In general, flows do not commute, neither for nonautonomous vector fields V_t, W_t:

$$\overrightarrow{\exp} \int_0^{t_1} V_\tau \, d\tau \circ \overrightarrow{\exp} \int_0^{t_2} W_\tau \, d\tau \neq \overrightarrow{\exp} \int_0^{t_2} W_\tau \, d\tau \circ \overrightarrow{\exp} \int_0^{t_1} V_\tau \, d\tau,$$

nor for autonomous vector fields V, W:

$$e^{t_1 V} \circ e^{t_2 W} \neq e^{t_2 W} \circ e^{t_1 V}.$$

Proposition 2

In the autonomous case, commutativity of flows is equivalent to commutativity of vector fields: if $V, W \in \operatorname{Vec} M$, then

$$e^{t_1 V} \circ e^{t_2 W} = e^{t_2 W} \circ e^{t_1 V}, \quad t_1, t_2 \in \mathbb{R}, \qquad \Leftrightarrow \qquad [V, W] = 0.$$

Proof.

Necessity:

$$\frac{d^2}{dt^2} q \circ e^{tV} \circ e^{tW} \circ e^{-tV} \circ e^{-tW} = q \circ 2[V, W].$$

Sufficiency. We have $\left(\operatorname{Ad} e^{t_1 V}\right) W = e^{t_1 \operatorname{ad} V} W = W$. Taking into account equality (7), we obtain

$$e^{t_1 V} \circ e^{t_2 W} \circ e^{-t_1 V} = e^{t_2 \left(\operatorname{Ad} e^{t_1 V}\right) W} = e^{t_2 W}.$$

□

Variations formula

- Consider an ODE of the form

$$\dot{q} = V_t(q) + W_t(q). \tag{8}$$

  We think of V_t as an initial vector field and W_t as its perturbation.
- Our aim is to find a formula for the flow Q^t of the new field $V_t + W_t$ as a perturbation of the flow $P^t = \overrightarrow{\exp} \int_0^t V_\tau \, d\tau$ of the initial field V_t.
- In other words, we wish to have a decomposition of the form

$$Q^t = \overrightarrow{\exp} \int_0^t (V_\tau + W_\tau) \, d\tau = C_t \circ P^t.$$

- We proceed as in the method of variation of parameters; we substitute the previous expression to ODE (8):

$$\begin{aligned} \frac{d}{d\,t} Q^t &= Q^t \circ (V_t + W_t) \\ &= \dot{C}_t \circ P^t + C_t \circ P^t \circ V_t \\ &= \dot{C}_t \circ P^t + Q^t \circ V_t, \end{aligned}$$

cancel the common term $Q^t \circ V_t$:

$$Q^t \circ W_t = \dot{C}_t \circ P^t,$$

and write down the ODE for the unknown flow C_t:

$$\begin{aligned} \dot{C}_t &= Q^t \circ W_t \circ \left(P^t\right)^{-1} \\ &= C_t \circ P^t \circ W_t \circ \left(P^t\right)^{-1} \\ &= C_t \circ \left(\operatorname{Ad} P^t\right) W_t \\ &= C_t \circ \left(\overrightarrow{\exp} \int_0^t \operatorname{ad} V_\tau \, d\tau \right) W_t, \qquad C_0 = \operatorname{Id}. \end{aligned}$$

- This operator Cauchy problem is of the form $\dot{C}^t = C^t \circ V_t,\ C^0 = \mathrm{Id}$, thus it has a unique solution:

$$C_t = \overrightarrow{\exp} \int_0^t \left(\overrightarrow{\exp} \int_0^\tau \operatorname{ad} V_\theta \, d\theta \right) W_\tau \, d\tau.$$

- Hence we obtain the required decomposition of the perturbed flow:

$$\overrightarrow{\exp} \int_0^t (V_\tau + W_\tau) \, d\tau = \overrightarrow{\exp} \int_0^t \left(\overrightarrow{\exp} \int_0^\tau \operatorname{ad} V_\theta \, d\theta \right) W_\tau \, d\tau \circ \overrightarrow{\exp} \int_0^t V_\tau \, d\tau. \quad (9)$$

- This equality is called the *variations formula*.
- It can be written as follows:

$$\overrightarrow{\exp} \int_0^t (V_\tau + W_\tau) \, d\tau = \overrightarrow{\exp} \int_0^t (\operatorname{Ad} P^\tau) \, W_\tau \, d\tau \circ P^t.$$

- So the perturbed flow is a composition of the initial flow P^t with the flow of the perturbation W_t twisted by P^t.

- Now we obtain another form of the variations formula, with the flow P^t to the left of the twisted flow.
- We have

$$\overrightarrow{\exp} \int_0^t (V_\tau + W_\tau)\, d\tau = \overrightarrow{\exp} \int_0^t (\mathrm{Ad}\, P^\tau)\, W_\tau\, d\tau \circ P^t$$
$$= P^t \circ \left(P^t\right)^{-1} \circ \overrightarrow{\exp} \int_0^t (\mathrm{Ad}\, P^\tau)\, W_\tau\, d\tau \circ P^t$$
$$= P^t \circ \overrightarrow{\exp} \int_0^t \left(\mathrm{Ad}\, \left(P^t\right)^{-1} \circ \mathrm{Ad}\, P^\tau\right) W_\tau\, d\tau$$
$$= P^t \circ \overrightarrow{\exp} \int_0^t \left(\mathrm{Ad}\left(\left(P^t\right)^{-1} \circ P^\tau\right)\right) W_\tau\, d\tau.$$

- Notice that

$$\left(P^t\right)^{-1} \circ P^\tau = \overrightarrow{\exp} \int_t^\tau V_\theta\, d\theta.$$

- Thus

$$\overrightarrow{\exp} \int_0^t (V_\tau + W_\tau)\, d\tau = P^t \circ \overrightarrow{\exp} \int_0^t \left(\overrightarrow{\exp} \int_t^\tau \operatorname{ad} V_\theta\, d\theta \right) W_\tau\, d\tau$$
$$= \overrightarrow{\exp} \int_0^t V_\tau\, d\tau \circ \overrightarrow{\exp} \int_0^t \left(\overrightarrow{\exp} \int_t^\tau \operatorname{ad} V_\theta\, d\theta \right) W_\tau\, d\tau. \tag{10}$$

- For autonomous vector fields $V, W \in \operatorname{Vec} M$, the variations formulas (9), (10) take the form:

$$e^{t(V+W)} = \overrightarrow{\exp} \int_0^t e^{\tau \operatorname{ad} V} W\, d\tau \circ e^{tV} = e^{tV} \circ \overrightarrow{\exp} \int_0^t e^{(\tau - t) \operatorname{ad} V} W\, d\tau. \tag{11}$$

- In particular, for $t = 1$ we have

$$e^{V+W} = \overrightarrow{\exp} \int_0^1 e^{\tau \operatorname{ad} V} W\, d\tau \circ e^V.$$

Derivative of flow with respect to parameter

- Let $V_t(s)$ be a nonautonomous vector field depending smoothly on a real parameter s. We study dependence of the flow of $V_t(s)$ on the parameter s.
- We write

$$\overrightarrow{\exp} \int_0^t V_\tau(s+\varepsilon)\, d\tau = \overrightarrow{\exp} \int_0^t \left(V_\tau(s) + \delta_{V_\tau}(s,\varepsilon)\right)\, d\tau \qquad (12)$$

with the perturbation $\delta_{V_\tau}(s,\varepsilon) = V_\tau(s+\varepsilon) - V_\tau(s)$.

- By the variations formula (9), the previous flow is equal to

$$\overrightarrow{\exp} \int_0^t \left(\overrightarrow{\exp} \int_0^\tau \mathrm{ad}\, V_\theta(s)\, d\theta \right) \delta_{V_\tau}(s,\varepsilon)\, d\tau \circ \overrightarrow{\exp} \int_0^t V_\tau(s)\, d\tau.$$

- Now we expand in ε:

$$\begin{aligned}
\delta_{V_\tau}(s,\varepsilon) &= \varepsilon \frac{\partial}{\partial s} V_\tau(s) + O(\varepsilon^2), \qquad \varepsilon \to 0, \\
W_\tau(s,\varepsilon) &\stackrel{\text{def}}{=} \left(\overrightarrow{\exp} \int_0^\tau \operatorname{ad} V_\theta(s)\, d\theta \right) \delta_{V_\tau}(s,\varepsilon) \\
&= \varepsilon \left(\overrightarrow{\exp} \int_0^\tau \operatorname{ad} V_\theta(s)\, d\theta \right) \frac{\partial}{\partial s} V_\tau(s) + O(\varepsilon^2), \qquad \varepsilon \to 0,
\end{aligned}$$

thus

$$\begin{aligned}
\overrightarrow{\exp} \int_0^t W_\tau(s,\varepsilon)\, d\tau &= \mathrm{Id} + \int_0^t W_\tau(s,\varepsilon)\, d\tau + O(\varepsilon^2) \\
&= \mathrm{Id} + \varepsilon \int_0^t \left(\overrightarrow{\exp} \int_0^\tau \operatorname{ad} V_\theta(s)\, d\theta \right) \frac{\partial}{\partial s} V_\tau(s)\, d\tau + O(\varepsilon^2).
\end{aligned}$$

- Finally,

$$\overrightarrow{\exp} \int_0^t V_\tau(s+\varepsilon)\, d\tau = \overrightarrow{\exp} \int_0^t W_{s,\tau}(\varepsilon)\, d\tau \circ \overrightarrow{\exp} \int_0^t V_\tau(s)\, d\tau$$

$$= \overrightarrow{\exp} \int_0^t V_\tau(s)\, d\tau$$

$$+ \varepsilon \int_0^t \left(\overrightarrow{\exp} \int_0^\tau \operatorname{ad} V_\theta(s)\, d\theta \right) \frac{\partial}{\partial s} V_\tau(s)\, d\tau \circ \overrightarrow{\exp} \int_0^t V_\tau(s)\, d\tau + O(\varepsilon^2),$$

that is,

$$\frac{\partial}{\partial s} \overrightarrow{\exp} \int_0^t V_\tau(s)\, d\tau = \int_0^t \left(\overrightarrow{\exp} \int_0^\tau \operatorname{ad} V_\theta(s)\, d\theta \right) \frac{\partial}{\partial s} V_\tau(s)\, d\tau \circ \overrightarrow{\exp} \int_0^t V_\tau(s)\, d\tau. \quad (13)$$

- Similarly, we obtain from the variations formula (10) the equality

$$\frac{\partial}{\partial s} \overrightarrow{\exp} \int_0^t V_\tau(s)\, d\tau = \overrightarrow{\exp} \int_0^t V_\tau(s)\, d\tau \circ \int_0^t \left(\overrightarrow{\exp} \int_t^\tau \operatorname{ad} V_\theta(s)\, d\theta \right) \frac{\partial}{\partial s} V_\tau(s)\, d\tau. \quad (14)$$

- For an autonomous vector field depending on a parameter $V(s)$, formula (13) takes the form

$$\frac{\partial}{\partial s} e^{tV(s)} = \int_0^t e^{\tau \operatorname{ad} V(s)} \frac{\partial V}{\partial s}\, d\tau \circ e^{tV(s)},$$

and at $t = 1$:

$$\frac{\partial}{\partial s} e^{V(s)} = \int_0^1 e^{\tau \operatorname{ad} V(s)} \frac{\partial V}{\partial s}\, d\tau \circ e^{V(s)}. \quad (15)$$

Proposition 3

Assume that

$$\left[\int_0^t V_\tau \, d\tau, V_t\right] = 0 \qquad \forall t. \tag{16}$$

Then

$$\overrightarrow{\exp} \int_0^t V_\tau \, d\tau = e^{\int_0^t V_\tau \, d\tau} \qquad \forall t.$$

That is, we state that under the commutativity assumption (16), the chronological exponential $\overrightarrow{\exp} \int_0^t V_\tau \, d\tau$ coincides with the flow $Q^t = e^{\int_0^t V_\tau \, d\tau}$ defined as follows:

$$Q^t = Q_1^t,$$
$$\frac{\partial Q_s^t}{\partial s} = \int_0^t V_\tau \, d\tau \circ Q_s^t, \qquad Q_0^t = \mathrm{Id}\,.$$

Proof.

- We show that the exponential in the right-hand side satisfies the same ODE as the chronological exponential in the left-hand side.
- By (15), we have

$$\frac{d}{d\,t} e^{\int_0^t V_\tau \, d\tau} = \int_0^1 e^{\tau \, \mathrm{ad} \int_0^t V_\theta \, d\theta} \, V_t \, d\tau \circ e^{\int_0^t V_\tau \, d\tau}.$$

- In view of equality (16),

$$e^{\tau \, \mathrm{ad} \int_0^t V_\theta \, d\theta} \, V_t = V_t,$$

thus

$$\frac{d}{d\,t} e^{\int_0^t V_\tau \, d\tau} = V_t \circ e^{\int_0^t V_\tau \, d\tau}.$$

- By equality (16), we can permute operators in the right-hand side:

$$\frac{d}{d\,t} e^{\int_0^t V_\tau \, d\tau} = e^{\int_0^t V_\tau \, d\tau} \circ V_t.$$

- Notice the initial condition

$$e^{\int_0^t V_\tau \, d\tau}\Big|_{t=0} = \mathrm{Id}\,.$$

- Now the statement follows since the Cauchy problem for flows

$$\dot{A}_t = A_t \circ V_t, \qquad A_0 = \mathrm{Id}$$

has a unique solution:

$$A_t = e^{\int_0^t V_\tau \, d\tau} = \overrightarrow{\exp} \int_0^t V_\tau \, d\tau.$$

□

- Here we finish our excursion to Chronological Calculus.

Differential 1-forms

Linear forms

- E a real vector space of finite dimension n.
- A *linear form* on E is a linear function $\xi : E \to \mathbb{R}$.
- The set of linear forms on E has a natural structure of a vector space called the *dual space* to E and denoted by E^*.
- If vectors $e_1, \ldots, e_n$ form a basis of E, then the corresponding *dual basis* of E^* is formed by the covectors $e_1^*, \ldots, e_n^*$ such that

$$\langle e_i^*, e_j \rangle = \delta_{ij}, \qquad i,\ j = 1, \ldots n.$$

- So the dual space has the same dimension as the initial one:

$$\dim E^* = n = \dim E.$$

Cotangent bundle

- M a smooth manifold and T_qM its tangent space at a point $q \in M$.
- The space of linear forms on T_qM, i.e., the dual space $(T_qM)^*$ to T_qM, is called the *cotangent space* to M at q and is denoted as T_q^*M.
- The disjoint union of all cotangent spaces is called the *cotangent bundle* of M:

$$T^*M \stackrel{\text{def}}{=} \bigsqcup_{q \in M} T_q^*M.$$

- The set T^*M has a natural structure of a smooth manifold of dimension $2n$, where $n = \dim M$.
- Local coordinates on T^*M are constructed from local coordinates on M.
- Let $O \subset M$ be a coordinate neighborhood and let

$$\Phi : O \to \mathbb{R}^n, \qquad \Phi(q) = (x_1(q), \ldots, x_n(q)),$$

be a local coordinate system.

- Differentials of the coordinate functions

$$dx_i|_q \in T_q^* M, \qquad i = 1, \ldots, n, \quad q \in O,$$

form a basis in the cotangent space $T_q^* M$.

- The dual basis in the tangent space $T_q M$ is formed by the vectors

$$\left.\frac{\partial}{\partial x_i}\right|_q \in T_q M, \qquad i = 1, \ldots, n, \quad q \in O,$$

$$\left\langle dx_i, \frac{\partial}{\partial x_j} \right\rangle \equiv \delta_{ij}, \qquad i, \ j = 1, \ldots, n.$$

- Any linear form $\xi \in T_q^* M$ can be decomposed via the basis forms:

$$\xi = \sum_{i=1}^{n} \xi_i \, dx_i.$$

- So any covector $\xi \in T^* M$ is characterized by n coordinates $(x_1, \ldots, x_n)$ of the point $q \in M$ where ξ is attached, and by n coordinates $(\xi_1, \ldots, \xi_n)$ of the linear form ξ in the basis $dx_1, \ldots, dx_n$.

- Mappings of the form

$$\xi \mapsto (\xi_1, \ldots, \xi_n;\ x_1, \ldots, x_n)$$

  define local coordinates on the cotangent bundle. Consequently, T^*M is a $2n$-dimensional manifold.
- Coordinates of the form (ξ, x) are called *canonical coordinates* on T^*M.

- If $F : M \to N$ is a smooth mapping between smooth manifolds, then the differential
$$F_* : T_qM \to T_{F(q)}N$$
has the adjoint (dual) mapping
$$F^* \stackrel{\text{def}}{=} (F_*)^* : T^*_{F(q)}N \to T^*_qM$$
defined as follows:
$$\begin{aligned} F^*\xi &= \xi \circ F_*, \qquad \xi \in T^*_{F(q)}N, \\ \langle F^*\xi, v\rangle &= \langle \xi, F_*v\rangle, \qquad v \in T_qM. \end{aligned}$$
- A vector $v \in T_qM$ is pushed forward by the differential F_* to the vector $F_*v \in T_{F(q)}N$, while a covector $\xi \in T^*_{F(q)}N$ is pulled back to the covector $F^*\xi \in T^*_qM$.
- So a smooth mapping $F : M \to N$ between manifolds induces a smooth mapping $F^* : T^*N \to T^*M$ between their cotangent bundles.

Differential 1-forms

- A *differential 1-form* on M is a smooth mapping $q \mapsto \omega_q \in T_q^* M, \quad q \in M$, i.e, a family $\omega = \{\omega_q\}$ of linear forms on the tangent spaces $T_q M$ smoothly depending on the point $q \in M$.
- The set of all differential 1-forms on M has a natural structure of an infinite-dimensional vector space denoted as $\Lambda^1 M$.
- Like linear forms on a vector space are dual objects to vectors of the space, differential forms on a manifold are dual objects to smooth curves in the manifold.
- The pairing operation is the *integral* of a differential 1-form $\omega \in \Lambda^1 M$ along a smooth oriented curve $\gamma : [t_0, t_1] \to M$, defined as follows:

$$\int_\gamma \omega \stackrel{\text{def}}{=} \int_{t_0}^{t_1} \langle \omega_{\gamma(t)}, \dot{\gamma}(t) \rangle \, dt.$$

- The integral of a 1-form along a curve does not change under orientation-preserving smooth reparametrizations of the curve and changes its sign under change of orientation.

Plan of this lecture

1. Autonomous vector fields
2. Action of diffeomorphisms on vector fields
3. Commutation of flows
4. Variations formula
5. Derivative of flow with respect to parameter
6. Differential 1-forms