Pontryagin maximum principle for optimal control problems with various boundary conditions
(Lecture 12)

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Reminder: Plan of previous lecture

1. Geometric statement of PMP for free time
2. PMP for optimal control problems
3. Statement of PMP with transversality conditions

Plan of this lecture

1. Proof of PMP with transversality conditions
2. PMP with mixed boundary conditions

Consider the optimal control problem:

$$
\begin{aligned}
&\dot{q} = f_u(q), \qquad q \in M, \quad u \in U \subset \mathbb{R}^m, && (1)\\
&q(0) \in N_0, \qquad q(t_1) \in N_1, && (2)\\
&t_1 > 0 \text{ fixed}, && (3)\\
&J(u) = \int_0^{t_1} \varphi(q(t), u(t))\, dt \to \min, && (4)
\end{aligned}
$$

where N_0 and N_1 are given immersed submanifolds of the state space M.

Theorem 1

Let $\widetilde{u}(t)$, $t \in [0, t_1]$, be an optimal control in problem (1)–(4). Define a family of Hamiltonians:

$$h_u^\nu(\lambda) = \langle \lambda, f_u(q) \rangle + \nu \varphi(q, u), \qquad \lambda \in T_q^* M, \ q \in M, \quad \nu \in \mathbb{R}, \ u \in U.$$

Then there exists a Lipschitzian curve $\lambda_t \in T^*_{\widetilde{q}(t)} M$, $t \in [0, t_1]$, and a number $\nu \in \mathbb{R}$ such that:

$$\dot{\lambda}_t = \overrightarrow{h^\nu_{\widetilde{u}(t)}}(\lambda_t), \tag{5}$$

$$h^\nu_{\widetilde{u}(t)}(\lambda_t) = \max_{u \in U} h_u^\nu(\lambda_t), \tag{6}$$

$$(\lambda_t, \nu) \neq (0, 0), \qquad t \in [0, t_1], \tag{7}$$

$$\nu \leq 0, \tag{8}$$

$$\lambda_0 \perp T_{\widetilde{q}(0)} N_0, \qquad \lambda_{t_1} \perp T_{\widetilde{q}(t_1)} N_1. \tag{9}$$

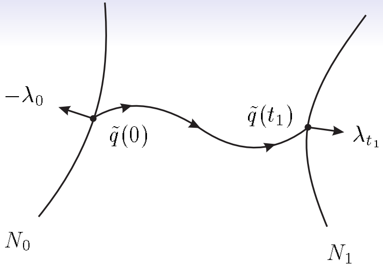

Figure: Transversality conditions (9)

Proof of Theorem 1.

- The scheme of proof developed in previous versions of PMP can be applied to much more general problems after appropriate modifications. Now we only indicate how the proofs of these theorems should be changed in order to cover the new boundary conditions $q(0) \in N_0$, $q(t_1) \in N_1$.
- (1) First consider the special case where the initial point is fixed: let $N_0 = \{q_0\}$ for some point $q_0 \in M$.

- Further, in the case of fixed terminal point $q(t_1)$, the necessary condition for optimality of the trajectory $q_{\widetilde{u}}(t)$ was the following:

$$\widehat{q}_1 \in \partial \widehat{\mathcal{A}}_{\widehat{q}_0}(t_1). \tag{11}$$

  Here $\widehat{\mathcal{A}}$ is the attainable set of the extended system (10) and $\widehat{q}_1 = \widehat{q}_{\widetilde{u}}(t_1)$.
- Now, when the target manifold N_1 is not a point, we should modify the argument. In a sense, we reduce the target manifold to a point defining it locally by an equation $\Phi = 0$.
- Choose a submersion

$$\Phi \, : \, O_{q_{\widetilde{u}(t_1)}} \to \mathbb{R}^p, \qquad p = \dim M - \dim N_1,$$

  of a small neighborhood $O_{q_{\widetilde{u}(t_1)}} \subset M$, so that

$$\Phi^{-1}(0) = N_1 \cap O_{q_{\widetilde{u}(t_1)}}.$$

- Further, extend the submersion: define the mapping

$$\widehat{\Phi} : \mathbb{R} \times O_{q_{\widetilde{u}(t_1)}} \to \mathbb{R}^{1+p}, \qquad \widehat{\Phi}\begin{pmatrix} y \\ q \end{pmatrix} = \begin{pmatrix} y \\ \Phi(q) \end{pmatrix}.$$

- Since the control $\widetilde{u}(t)$ is optimal in our problem (1)–(4), then

$$\widehat{\Phi}(\widehat{q}_1) \in \partial \widehat{\Phi}(\widehat{\mathcal{A}}_{\widehat{q}_0}(t_1)). \tag{12}$$

  So we replace the necessary optimality condition (11) by (12) and return to the scheme of proof of PMP for problems with two-point boundary condition.

- Take any $k \in \mathbb{N}$ and any needle-like variation of the optimal control:

$$u_s(t), \quad s \in \mathbb{R}^k_+, \qquad u_0(t) = \widetilde{u}(t), \quad t \in [0, t_1].$$

- Define the mappings

$$G : \mathbb{R}^k \to \mathbb{R} \times M, \qquad G(s) = \widehat{q}_{u_s}(t_1) = \widehat{q}_0 \circ \overrightarrow{\exp} \int_0^{t_1} \widehat{f}_{u_s(t)}\, dt, \qquad (13)$$

$$F : \mathbb{R}^k \to \mathbb{R}^{1+p}, \qquad F(s) = \widehat{\Phi}(G(s)) = \widehat{q}_0 \circ \overrightarrow{\exp} \int_0^{t_1} \widehat{f}_{u_s(t)}\, dt \circ \widehat{\Phi}. \qquad (14)$$

- Then it follows from inclusion (12) that

$$\widehat{\Phi}(\widehat{q}_1) = F(0) \in \partial F(\mathbb{R}^k_+). \qquad (15)$$

- By the first auxiliary lemma for the geometric statement of PMP,

$$F_0'(\mathbb{R}^k_+) = \operatorname{cone}\left\{ \left.\frac{\partial F}{\partial s_i}\right|_0 \mid i = 1, \ldots, k \right\} \neq \mathbb{R}^{1+p},$$

thus there exists a plane of support, i.e.,

$$\exists\, \widehat{\xi} \in \left(\mathbb{R}^{1+p}\right)^*, \quad \widehat{\xi} \neq 0,$$

such that

$$\left\langle \widehat{\xi}, \left.\frac{\partial F}{\partial s_i}\right|_0 \right\rangle \leq 0, \qquad i = 1, \ldots, k. \qquad (16)$$

- We compute the derivative by the chain rule:

$$\left.\frac{\partial F}{\partial s_i}\right|_0 = \widehat{\Phi}_* \left.\frac{\partial G}{\partial s_i}\right|_0, \tag{17}$$

and rewrite inequalities (16) as follows:

$$\left\langle \widehat{\Phi}^* \widehat{\xi}, \left.\frac{\partial G}{\partial s_i}\right|_0 \right\rangle = \left\langle \widehat{\xi}, \widehat{\Phi}_* \left.\frac{\partial G}{\partial s_i}\right|_0 \right\rangle \leq 0, \qquad i = 1, \dots, k. \tag{18}$$

- Then we denote the covector

$$\widehat{\lambda}_{t_1} = \widehat{\Phi}^* \widehat{\xi} = \begin{pmatrix} \nu \\ \lambda_{t_1} \end{pmatrix} \in T_{\widehat{q}_1}(\mathbb{R} \times M) \tag{19}$$

and obtain conclusions (5)–(8) in the same way as in PMP for optimal control problems with two-point boundary condition.

- The only distinction now is that the covector $\widehat{\lambda}_{t_1}$ is not arbitrary: equality (19) implies the second of the transversality conditions (9).
- Indeed, we have

$$\lambda_{t_1} = \Phi^* \xi, \qquad \xi \in \left(\mathbb{R}^p\right)^*,$$

thus

$$\langle \lambda_{t_1}, T_{q_{\widetilde{u}}(t_1)} N_1 \rangle = \langle \Phi^* \xi, T_{q_{\widetilde{u}}(t_1)} N_1 \rangle = \langle \xi, \underbrace{\Phi_* T_{q_{\widetilde{u}}(t_1)} N_1}_{=0} \rangle = 0.$$

- The first transversality condition (9) is now trivially satisfied, so the proof of this theorem in the case $N_0 = \{q_0\}$ is complete.

- (2) Let now the initial manifold N_0 be an arbitrary immersed submanifold of M. We can modify the scheme presented above to cover this case as well. Since now the initial point $q(0)$ is not fixed, we add variations of $q(0)$.
- Replace mappings (13), (14) by the following ones:

$$G : N_0 \times \mathbb{R}^k \to \mathbb{R} \times M, \qquad G(q,s) = \widehat{q} \circ \overrightarrow{\exp} \int_0^{t_1} \widehat{f}_{u_s(t)}\, dt,$$

$$F : N_0 \times \mathbb{R}^k \to \mathbb{R}^{1+p}, \qquad F(q,s) = \widehat{\Phi}(G(q,s)) = \widehat{q} \circ \overrightarrow{\exp} \int_0^{t_1} \widehat{f}_{u_s(t)}\, dt \circ \widehat{\Phi},$$

  where $\widehat{q} = (0,q) \in \mathbb{R} \times M$.
- Then the necessary optimality condition (15) is replaced by the inclusion

$$F(\widetilde{q}(0), 0) \in \partial F(N_0 \times \mathbb{R}^k_+). \tag{20}$$

- Apply the first auxiliary lemma before geometric statement of PMP to restriction of the mapping F to the space

$$\mathbb{R}^m \cong O_{\widetilde{q}(0)} \times \mathbb{R}^k, \qquad m = l + k, \quad l = \dim N_0,$$

where $O_{\widetilde{q}(0)} \subset N_0$ is a small neighborhood of $\widetilde{q}(0)$.
- By the remark after that lemma, inclusion (20) implies that

$$F'_{(\widetilde{q}(0),0)}(\mathbb{R}^l \oplus \mathbb{R}^k_+) \neq \mathbb{R}^{1+p},$$

i.e., there exists a covector

$$\widehat{\xi} \in \left(\mathbb{R}^{1+p}\right)^*, \qquad \widehat{\xi} \neq 0, \quad \widehat{\xi} = \begin{pmatrix} \nu \\ \xi \end{pmatrix},$$

such that

$$\begin{aligned} \left\langle \widehat{\xi}, \frac{\partial F}{\partial q} v \right\rangle &\leq 0, \qquad v \in T_{\widetilde{q}(0)} N_0, \\ \left\langle \widehat{\xi}, \frac{\partial F}{\partial s_i} \right\rangle &\leq 0, \qquad i = 1, \dots, k. \end{aligned} \tag{21}$$

- In the first inequality v belongs to a linear space, thus it turns into equality:

$$\left\langle \widehat{\xi}, \frac{\partial F}{\partial q} v \right\rangle = 0, \qquad v \in T_{\widetilde{q}(0)} N_0. \tag{22}$$

- Compute by Leibniz rule the partial derivative:

$$\begin{aligned}
\left.\frac{\partial F}{\partial q}\right|_{(\widetilde{q}(0),0)} &: \; T_{\widetilde{q}(0)} N_0 \to \mathbb{R}^{1+p}, \\
\left.\frac{\partial F}{\partial q}\right|_{(\widetilde{q}(0),0)} v &= \begin{pmatrix} 0 \\ v \end{pmatrix} \circ \overrightarrow{\exp} \int_0^{t_1} \widehat{f}_{\widetilde{u}(t)} \, dt \circ \widehat{\Phi} = \begin{pmatrix} 0 \\ v \circ P^{t_1} \circ \Phi \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ \Phi_* P_*^{t_1} v \end{pmatrix}, \qquad v \in T_{\widetilde{q}(0)} N_0,
\end{aligned}$$

where $P^{t_1} = \overrightarrow{\exp} \int_0^{t_1} f_{\widetilde{u}(t)} \, dt.$

- Then conditions (22), (21) read as follows:

$$\left\langle \xi, \Phi_* P_*^{t_1} v \right\rangle = 0, \qquad v \in T_{\widetilde{q}(0)} N_0, \tag{23}$$

$$\left\langle \widehat{\Phi}^* \widehat{\xi}, \left. \frac{\partial G}{\partial s_i} \right|_{(\widetilde{q}(0),0)} \right\rangle \leq 0, \qquad i = 1, \ldots, k.$$

- As before, introduce the covector $\widehat{\lambda}_{t_1} = (\nu, \lambda_{t_1})$ by equality (19), then conclusions (5)–(8) of this theorem and the second transversality condition (9) follows.

- The first transversality condition is also satisfied: equality (23) can be rewritten as

$$\left\langle \lambda_{t_1}, P_*^{t_1} v \right\rangle = 0, \qquad v \in T_{\widetilde{q}(0)} N_0.$$

- But $\lambda_0 = P_{t_1}^* \lambda_{t_1}$, thus

$$\langle \lambda_0, v \rangle = \left\langle P_{t_1}^* \lambda_{t_1}, v \right\rangle = 0, \qquad v \in T_{\widetilde{q}(0)} N_0.$$

- The theorem is completely proved.

- Now consider even more general problem with mixed boundary conditions, see inclusion (25) below. Pontryagin Maximum Principle easily generalizes to this case, both in formulation and in proof.
- We study optimal control problem of the form:

$$\dot{q} = f_u(q), \qquad q \in M, \quad u \in U \subset \mathbb{R}^m, \tag{24}$$

$$(q(0), q(t_1)) \in N \subset M \times M, \tag{25}$$

$$t_1 > 0 \text{ fixed}, \tag{26}$$

$$J(u) = \int_0^{t_1} \varphi(q(t), u(t))\, dt \to \min, \tag{27}$$

where N is a smooth immersed submanifold of $M \times M$.

Theorem 2

Let $\widetilde{u}$ be an optimal control in problem (24)–(27). *Then there hold all statements of Theorem* 1 *except its transversality condition* (9), *which is replaced now by the relation*

$$(-\lambda_0, \lambda_{t_1}) \perp T_{(\widetilde{q}(0), \widetilde{q}(t_1))} N. \tag{28}$$

Remarks

(1) We identify

$$T^*_{(q_0,q_1)}(M \times M) \cong T^*_{q_0}M \oplus T^*_{q_1}M,$$

so the transversality condition (28) makes sense.

(2) An important particular case of mixed boundary conditions (25) is the case of periodic trajectories:

$$q(t_1) = q(0). \tag{29}$$

Indeed, then

$$N = \Delta \stackrel{\text{def}}{=} \{(q,q) \mid q \in M\} \subset M \times M, \tag{30}$$

the diagonal of the product $M \times M$. In this case the transversality condition (28) reads

$$\langle(-\lambda_0, \lambda_{t_1}), (v, v)\rangle = -\langle\lambda_0, v\rangle + \langle\lambda_{t_1}, v\rangle = 0, \qquad v \in T_{q(0)}M = T_{q(t_1)}M,$$

i.e., $\lambda_0 = \lambda_{t_1}$. That is, an optimal trajectory in the problem with periodic boundary conditions (29) possesses a periodic Hamiltonian lift (extremal).

Proof of Theorem 2.

- We reduce our problem to the case of separated boundary conditions by introducing an auxiliary problem on $M \times M$:

$$\begin{cases} \dot{x} = 0, \\ \dot{q} = f_u(q), \end{cases} \qquad (x, q) \in M \times M, \quad u \in U,$$

$$(x(0), q(0)) \in \Delta, \qquad (x(t_1), q(t_1)) \in N,$$

(the diagonal Δ is defined in (30) above)

$$J(u) = \int_0^{t_1} \varphi(q(t), u(t))\, dt \to \min .$$

- It is obvious that this problem is equivalent to our problem (24)–(27).
- We apply a version of PMP (Theorem 1) to the auxiliary problem.
- The Hamiltonian is the same as for the initial problem:

$$h_u^\nu(\eta, \lambda) = h_u^\nu(\lambda) = \langle \lambda, f_u(q) \rangle + \nu\varphi(q, u), \qquad (\eta, \lambda) \in T^*M \oplus T^*M.$$

- The corresponding Hamiltonian system is

$$\begin{cases} \dot{\eta}_t = 0, \\ \dot{\lambda}_t = \overrightarrow{h^{\nu}_{\widetilde{u}(t)}}(\lambda_t). \end{cases} \tag{31}$$

- All required statements of PMP obviously follow, we should only check transversality conditions.
- At the initial instant $t = 0$ the first of conditions (9) reads:

$$\langle(\eta_0, \lambda_0), (v, v)\rangle = \langle \eta_0, v\rangle + \langle \lambda_0, v\rangle = 0, \qquad v \in T_{\widetilde{q}(0)}M,$$

i.e., $\eta_0 + \lambda_0 = 0$, or, taking into account the first of equations (31), $\eta_{t_1} = -\lambda_0$.
- And at the terminal instant $t = t_1$:

$$(\eta_{t_1}, \lambda_{t_1}) \perp T_{(\tilde{x}(t_1), \widetilde{q}(t_1))}N,$$

that is,

$$(-\lambda_0, \lambda_{t_1}) \perp T_{(\widetilde{q}(0), \widetilde{q}(t_1))}N,$$

which is the required transversality condition (28). □

Remarks

(1) Needless to say, if the terminal time t_1 is free, then one should add to statements of Theorem 2 the additional equality $h^{\nu}_{\widetilde{u}(t)}(\lambda_t) \equiv 0$.

(2) Pontryagin Maximum Principle withstands further generalizations to wider classes of cost functionals and boundary conditions. After certain modifications of argument, the general scheme provides necessary optimality conditions for more general problems.