Krener's theorem and Optimal control problem
(Lecture 4)

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3. Seeing the Ox:

On a yonder branch perches a nightingale cheerfully singing;
The sun is warm, and a soothing breeze blows, on the bank the willows are green;
The ox is there all by himself, nowhere is he to hide himself;
The splendid head decorated with stately horns what painter can reproduce him?

Pu-ming, "The Ten Oxherding Pictures"

Reminder: Plan of the previous lecture

1. Proof of the Orbit theorem.
2. Corollaries of the Orbit theorem:
    - Rashevskii–Chow theorem,
    - Lie algebra rank condition,
    - Frobenius theorem.

Plan of this lecture

1. Examples of orbits of control systems
2. Krener's theorem
3. Statement of optimal control problem
4. Existence of optimal controls

Example: Orbits of different dimensions

- Let

$$M = \mathbb{R}_x, \qquad \mathcal{F} = \left\{ x\frac{\partial}{\partial x} \right\} \subset \mathrm{Vec}(M).$$

- We have:

$$\begin{aligned} x_0 > 0 \quad &\Rightarrow \quad \mathcal{O}_{x_0} = \{x > 0\}, \\ x_0 = 0 \quad &\Rightarrow \quad \mathcal{O}_{x_0} = \{x = 0\}, \\ x_0 < 0 \quad &\Rightarrow \quad \mathcal{O}_{x_0} = \{x < 0\}, \end{aligned}$$

- Thus the system has two one-dimensional orbits and one zero-dimensional orbit.

Example:
More orbits of different dimensions

- Let

$$M = \mathbb{R}^3_{x,y,z}, \qquad \mathcal{F} = \left\{ x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x},\ y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y},\ z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z} \right\} \subset \mathrm{Vec}(M).$$

- Then for any point $q \in \mathbb{R}^3$

$$\mathcal{O}_q = \{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = |q|^2\},$$

- This is a sphere for $q \neq 0$ and a point for $q = 0$.
- An orbit of a control system is a generalisation of a trajectory of a vector field to the case of more than one vector field.

Attainable sets of full-rank systems

- Let $\mathcal{F} \subset \mathrm{Vec}(M)$ be a full-rank system. The assumption of full rank is not very strong in the analytic case: if it is violated, we can consider the restriction of $\mathcal{F}$ to its orbit, and this restriction is full-rank.
- What is the possible structure of *attainable sets* of $\mathcal{F}$?
- It is easy to construct systems in the two-dimensional plane that have the following attainable sets:
  - a smooth full-dimensional manifold without boundary;
  - a full-dimensional manifold with smooth boundary;
  - a full-dimensional manifold with non-smooth boundary, with corner or cusp singularity.

Possible attainable sets of full-rank systems

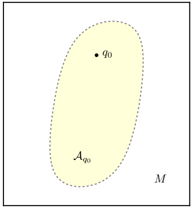


Figure: Attainable set — smooth manifold without boundary

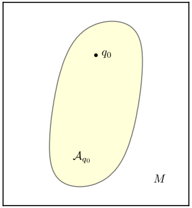


Figure: Attainable set — manifold with smooth boundary

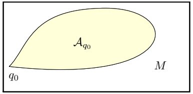


Figure: Attainable set — manifold with nonsmooth boundary

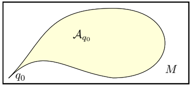


Figure: Attainable set — manifold with nonsmooth boundary

Impossible attainable sets of full-rank systems

- But it is impossible to construct an attainable set that is:
  - a lower-dimensional submanifold;
  - a set whose boundary points are isolated from its interior points.

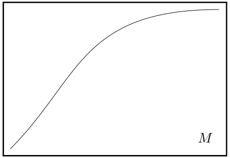


Figure: Forbidden attainable set: subset of lower dimension

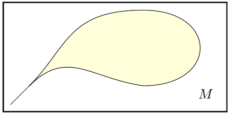


Figure: Forbidden attainable set: subset with isolated boundary points

- These possibilities are forbidden respectively by the following theorem.

Krener's theorem

Theorem (Krener)

Let $\mathcal{F} \subset \mathrm{Vec}(M)$*, and let* $\mathrm{Lie}_q \mathcal{F} = T_q M$ *for any* $q \in M$*. Then:*

(1) $\mathrm{int}\, \mathcal{A}_q \neq \varnothing$ *for any* $q \in M$

(2) $\mathrm{cl}(\mathrm{int}\, \mathcal{A}_q) \supset \mathcal{A}_q$ *for any* $q \in M$*.*

Proof of Krener's theorem: 1/2

- Since item (2) implies item (1), we prove item (2): $\mathrm{cl}(\mathrm{int}\,\mathcal{A}_q) \supset \mathcal{A}_q$.
- We argue by induction on dimension of M. If $\dim M = 0$, the statement is obvious. Let $\dim M > 0$.
- Take any $q_1 \in \mathcal{A}_q$, and fix any neighbourhood $q_1 \in W(q_1) \subset M$. We show that $\mathrm{int}\,\mathcal{A}_q \cap W(q_1) \neq \varnothing$.
- There exists $f_1 \in \mathcal{F}$ such that $f_1(q_1) \neq 0$, otherwise $\mathcal{F}(q_1) = \{0\} = \mathrm{Lie}_{q_1}(\mathcal{F}) = T_{q_1}M$, a contradiction. Consider the following set for a small $\varepsilon_1 > 0$:

$$N_1 = \{e^{t_1 f_1}(q_1) \mid 0 < t_1 < \varepsilon_1\} \subset W(q_1) \cap \mathcal{A}_q.$$

- N_1 is a smooth 1-dimensional manifold. If $\dim M = 1$, then N_1 is open, thus $N_1 \subset \mathrm{int}\,\mathcal{A}_q$, so $\mathrm{int}\,\mathcal{A}_q \cap W(q_1) \neq \varnothing$. Since the neighbourhood $W(q_1)$ is arbitrary, $q_1 \in \mathrm{cl}(\mathrm{int}\,\mathcal{A}_q)$.

Proof of Krener's theorem: 2/2

- Let $\dim M > 1$. There exist $q_2 = e^{t_1^1 f_1}(q_1) \in N_1 \cap W(q_1)$ and $f_2 \in \mathcal{F}$ such that $f_2(q_2) \notin T_{q_2} N_1$. Otherwise $\dim \mathcal{F}(q_2) = \dim \mathrm{Lie}_{q_2}(\mathcal{F}) = \dim T_{q_2} M = 1$ for any $q_2 \in N_2 \cap W$, and $\dim M = 1$.
- Consider the following set for a small ε_2:

$$N_2 = \{e^{t_2 f_2} \circ e^{t_1 f_1}(q_1) \mid t_1^1 < t_1 < t_1^1 + \varepsilon_2,\ 0 < t_2 < \varepsilon_2\} \subset W(q_1) \cap \mathcal{A}_q.$$

- N_2 is a smooth 2-dimensional manifold.
- If $\dim M = 2$, then N_2 is open, thus $N_2 \subset \operatorname{int} \mathcal{A}_q \cap W(q_1) \neq \varnothing$ and $q_1 \in \mathrm{cl}(\operatorname{int} \mathcal{A}_q)$.
- If $\dim M > 2$, we proceed by induction. □

A control system $\mathcal{F} \subset \mathrm{Vec}(M)$ is called *accessible* at a point $q \in M$ if $\operatorname{int} \mathcal{A}_q \neq \emptyset$. In the analytic case the accessibility property is equivalent to the full-rank condition.

Example: Stopping a train

- The control system has the form

$$\dot{x} = f_1(x) + u f_2(x), \qquad x = (x_1, x_2) \in \mathbb{R}^2, \quad |u| \leq 1,$$
$$f_1 = x_2 \frac{\partial}{\partial x_1}, \qquad f_2 = \frac{\partial}{\partial x_2}.$$

- We have $[f_1, f_2] = -\frac{\partial}{\partial x_1}$, whence the system $\mathcal{F} = \{f_1 + u f_2 \mid u \in [-1, 1]\}$ is full-rank: $\mathrm{Lie}_x(\mathcal{F}) = \mathrm{span}\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right)(x) = T_x\mathbb{R}^2 \qquad \forall x \in \mathbb{R}^2.$
- Thus

$$\mathcal{O}_x = \mathbb{R}^2 \qquad \forall x \in \mathbb{R}^2.$$

- In order to find the attainable sets, we compute trajectories of the system with a constant control $u \neq 0$: they are the parabolas

$$\frac{x_2^2}{2} = u x_1 + C.$$

- Now it is visually obvious that the system is controllable.

Example: Markov–Dubins car (1/2)

- The control system has the form

$$\dot{q} = f_1(q) + u f_2(q), \qquad q = (x, y, \theta) \in M = \mathbb{R}^2 \times S^1, \quad |u| \leq 1,$$

$$f_1 = \cos\theta \frac{\partial}{\partial x} + \sin\theta \frac{\partial}{\partial y}, \qquad f_2 = \frac{\partial}{\partial \theta}.$$

- We have

$$[f_1, f_2] = \sin\theta \frac{\partial}{\partial x} - \cos\theta \frac{\partial}{\partial y} =: f_3.$$

- Thus the system $\mathcal{F} = \{f_1 + u f_2 \mid u \in [-1, 1]\}$ is full-rank:

$$\mathrm{Lie}_q(\mathcal{F}) = \mathrm{span}(f_1(q), f_2(q), f_3(q)) = T_q M \qquad \forall q \in M,$$

consequently,

$$\mathcal{O}_q = M \qquad \forall q \in M.$$

- In order to describe the attainable sets, we replace the initial system $\mathcal{F}$ by a restricted system $\mathcal{F}_1 = \{f_1 \pm f_2\} \subset \mathcal{F}$ and prove that $\mathcal{F}_1$ is controllable (then $\mathcal{F}$ is controllable as well).

Example: Markov–Dubins car (2/2)

- Trajectories of the restricted system $\dot{x} = \cos\theta$, $\dot{y} = \sin\theta$, $\dot{\theta} = \pm 1$, have the form

$$\theta = \theta_0 \pm t, \qquad x = x_0 \pm (\sin(\theta_0 \pm t) - \sin\theta_0), \qquad y = y_0 \pm (\cos\theta_0 - \cos(\theta_0 \pm t)).$$

- These trajectories are periodic: $e^{(t+2\pi n)(f_1 \pm f_2)} = e^{t(f_1 \pm f_2)}, \qquad t \in \mathbb{R}, \quad n \in \mathbb{Z}$. So a shift along the fields $f_1 \pm f_2$ in the negative time can be obtained as a shift in the positive time.
- Consequently, if we introduce the system $\mathcal{F}_2 = \{f_1 \pm f_2, \ -f_1 \pm f_2\}$, then we get

$$\mathcal{A}_q(\mathcal{F}_2) = \mathcal{A}_q(\mathcal{F}_1), \qquad q \in M.$$

- But the system $\mathcal{F}_2$ is symmetric and full-rank, thus $\mathcal{A}_q(\mathcal{F}_2) = \mathcal{O}_q(\mathcal{F}_2) = M$, whence

$$\mathcal{A}_q(\mathcal{F}) = \mathcal{A}_q(\mathcal{F}_1) = M \text{ for all } q \in M.$$

  That is, the Markov–Dubins car is completely controllable in the space $\mathbb{R}^2 \times S^1$.

Statement of optimal control problem

- We consider the following *optimal control problem*:

$$\dot{q} = f(q, u), \qquad q \in M, \quad u \in U \subset \mathbb{R}^m, \tag{1}$$

$$q(0) = q_0, \qquad q(t_1) = q_1, \tag{2}$$

$$J[u] = \int_0^{t_1} \varphi(q, u)\, dt \to \min, \tag{3}$$

$$t_1 \text{ fixed or free.}$$

- A solution $q(t)$, $t \in [0, t_1]$, to this problem is said to be *(globally) optimal*.
- The following assumptions are made for the dynamics $f(q, u)$:
  - the mapping $q \mapsto f(q, u)$ is smooth for any $u \in U$,
  - the mapping $(q, u) \mapsto f(q, u)$ is continuous for any $q \in M$, $u \in \mathrm{cl}(U)$,
  - the mapping $(q, u) \mapsto \frac{\partial f}{\partial q}(q, u)$ is continuous for any $q \in M$, $u \in \mathrm{cl}(U)$.
- The same assumptions are made for the function $\varphi(q, u)$ that determines the cost functional J.
- Admissible control is $u \in L^\infty([0, t_1], U)$.

Reduction to the study of attainable sets

- In order to include the functional J into dynamics of the system, introduce a new variable equal to the running value of the cost functional along a trajectory $q_u(t)$:

$$y(t) = \int_0^t \varphi(q, u)\, dt.$$

- Respectively, we introduce an extended state $\widehat{q} = \begin{pmatrix} y \\ q \end{pmatrix} \in \mathbb{R} \times M$ that satisfies an *extended control system*

$$\frac{d\widehat{q}}{dt} = \begin{pmatrix} \dot{y} \\ \dot{q} \end{pmatrix} = \begin{pmatrix} \varphi(q, u) \\ f(q, u) \end{pmatrix} =: \widehat{f}(\widehat{q}, u).$$

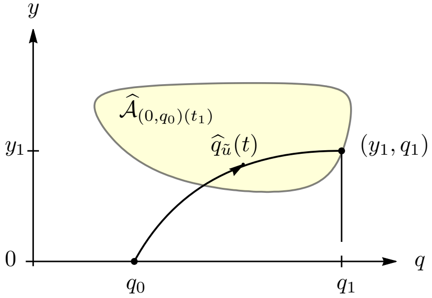
- The boundary conditions for this system are

$$\widehat{q}(0) = \begin{pmatrix} 0 \\ q_0 \end{pmatrix}, \qquad \widehat{q}(t_1) = \begin{pmatrix} J \\ q_1 \end{pmatrix}.$$

Reduction to the study of attainable sets

- A trajectory $q_{\tilde{u}}(t)$ is optimal for the optimal control problem with fixed time t_1 if and only if the corresponding trajectory $\widehat{q}_{\tilde{u}}(t)$ of the extended system comes to a point (y_1, q_1) of the attainable set $\widehat{\mathcal{A}}_{(0,q_0)}(t_1)$ such that

$$\widehat{\mathcal{A}}_{(0,q_0)}(t_1) \cap \{(y, q_1) \mid y < y_1\} = \emptyset.$$



- For the problem with free terminal time an analogous condition is written for the attainable set $\widehat{\mathcal{A}}_{(0,q_0)}$.

Filippov's theorem

Corollary

If the attainable set $\widehat{\mathcal{A}}_{(0,q_0)}(t_1)$ *is compact and* $q_1 \in \mathcal{A}_{q_0}(t_1)$*, then the optimal control problem* (1)–(3) *with fixed time* t_1 *has a solution.*

Theorem (Filippov)

Suppose that control system (1) *satisfies the hypotheses:*

(1) *the set* U *is compact*

(2) *the set* $f(q, U)$ *is convex for all* $q \in M$

(3) *there exists a compact set* $K \subset M$ *such that for all* $q \in M \backslash K$, $u \in U$ *there holds the equality* $f(q, u) = 0$.

Then the attainable sets $\mathcal{A}_{q_0}(t)$, $\mathcal{A}_{q_0}(\leq t)$ *are compact for any* $q_0 \in M$, $t > 0$.

Existence of optimal controls in optimal control problem

Corollary

Let the optimal control problem (1)–(3) *satisfy the hypotheses:*

(1) *the set* U *is compact*

(2) *the set* $\left\{ \begin{pmatrix} \varphi(q,u) \\ f(q,u) \end{pmatrix} \mid u \in U \right\}$ *is convex for all* $q \in M$

(3) *there exists a compact set* $K \subset \mathbb{R} \times M$ *such that* $\widehat{\mathcal{A}}_{(0,q_0)}(t_1) \subset K$

(4) $q_1 \in \mathcal{A}_{q_0}(t_1)$.

Then the problem (1)–(3) *with fixed time* t_1 *has a solution.*

Proof of the existence conditions for optimal control problem

- *Proof*. There exists a compact set $K' \subset \mathbb{R} \times M$ such that $K \subset \operatorname{int} K'$. Take a function $a \in C^\infty(\mathbb{R} \times M)$ such that

$$a|_K \equiv 1, \qquad a|_{(\mathbb{R}\times M)\setminus K'} \equiv 0.$$

- Consider a new extended control system:

$$\frac{d\widehat{q}}{dt} = a(\widehat{q})\widehat{f}(\widehat{q}, u), \qquad \widehat{q} \in \mathbb{R} \times M, \quad u \in U.$$

- This system has compact attainable sets for time t_1 , which coincide with the corresponding attainable sets of the extended system.
- Then optimal control problem (1)–(3) has a solution (by Filippov's theorem). □

Existence of solutions to time-optimal problem

Now consider a *time-optimal problem*

$$\dot{q} = f(q, u), \qquad q \in M, \quad u \in U \subset \mathbb{R}^m, \tag{4}$$

$$q(0) = q_0, \qquad q(t_1) = q_1, \tag{5}$$

$$t_1 \to \min. \tag{6}$$

Corollary

Let the following conditions hold:

(1) *the set U is compact*

(2) *the set $f(q, U)$ is convex for all $q \in M$*

(3) *there exist $t_1 > 0$ and a compact set $K \subset M$ such that*

$$q_1 \in \mathcal{A}_{q_0}(\leq t_1) \subset K.$$

Then time-optimal problem (4)–(6) *has a solution.*

Elements of symplectic geometry

- Let M be an n-dimensional smooth manifold. Then the disjoint union of its tangent spaces $TM = \bigsqcup_{q \in M} T_q M = \{(q, v) \mid q \in M,\ v \in T_q M\}$ is called its *tangent bundle*.
- If $(q_1, \dots, q_n)$ are local coordinates on M, then any tangent vector $v \in T_q M$ has a decomposition $v = \sum_{i=1}^n v_i \frac{\partial}{\partial q_i}$. So $(q_1, \dots, q_n;\ v_1, \dots, v_n)$ are local coordinates on TM, which is thus a $2n$-dimensional smooth manifold.
- For any point $q \in M$, the dual space $(T_q M)^* = T_q^* M$ is called the *cotangent space* to M at q. Thus $T_q^* M$ consists of linear forms on $T_q M$. The disjoint union $T^* M = \bigsqcup_{q \in M} T_q^* M = \{(q, p) \mid q \in M,\ p \in T_q^* M\}$ is called the *cotangent bundle*.
- If $(q_1, \dots, q_n)$ are local coordinates on M, then any covector $\lambda \in T^* M$ has a decomposition $\lambda = \sum_{i=1}^n p_i \, dq_i$. Thus $(q_1, \dots, q_n;\ p_1, \dots, p_n)$ are local coordinates on $T^* M$ called the *canonical coordinates*. So $T^* M$ is a smooth $2n$-dimensional manifold.
- The *canonical projection* is the mapping $\pi \colon T^* M \to M, \quad T_q^* M \ni \lambda \mapsto q \in M$.