Orbit theorem
(Lecture 3)

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2. Seeing the Traces:

By the stream and under the trees, scattered are the traces of the lost;
The sweet-scented grasses are growing thick — did he find the way?
However remote over the hills and far away the beast may wander,
His nose reaches the heavens and none can conceal it.

Pu-ming, "The Ten Oxherding Pictures"

Reminder: Plan of the previous lecture

1. Lie groups, Lie algebras, and left-invariant optimal control problems
2. Controllability of linear systems
3. Local controllability of nonlinear systems
4. Statement of the Orbit theorem

Plan of this lecture

1. Proof of the Orbit theorem.
2. Corollaries of the Orbit theorem:
    - Orbit and Lie algebra of the system
    - Rashevskii–Chow theorem,
    - Lie algebra rank condition,
    - Frobenius theorem.

Statement of the Orbit theorem

Theorem (*Orbit theorem*, Nagano–Sussmann)

Let $\mathcal{F} \subset \mathrm{Vec}(M)$*, and let* $q_0 \in M$*.*

(1) *The orbit* $\mathcal{O}_{q_0}$ *is a connected immersed submanifold of* M*.*

(2) *For any* $q \in \mathcal{O}_{q_0}$

$$T_q\mathcal{O}_{q_0} = \mathrm{span}(\mathcal{P}_*\mathcal{F})(q) = \mathrm{span}\{(P_*V)(q) \mid P \in \mathcal{P}, \quad V \in \mathcal{F}\},$$
$$\mathcal{P} = \{e^{t_N f_N} \circ \cdots \circ e^{t_1 f_1} \mid t_i \in \mathbb{R}, \quad f_i \in \mathcal{F}, \quad N \in \mathbb{N}\}.$$

Proof of the Orbit theorem: 1/7

Proof.

- Introduce a vector space important in the sequel

$$\Pi_q = \operatorname{span}(\mathcal{P}_* \mathcal{F})(q) \subset T_q M, \qquad q \in M,$$

  this is a candidate tangent space to the orbit $\mathcal{O}_{q_0}$.
- *1)* We prove that for all $q \in \mathcal{O}_{q_0}$ we have $\dim \Pi_q = \dim \Pi_{q_0}$.
- Choose any point $q \in \mathcal{O}_{q_0}$, then $q = Q(q_0)$, $Q \in \mathcal{P}$. Let us show that $Q_*^{-1}(\Pi_q) \subset \Pi_{q_0}$.
- Choose any element $(P_* f)(q) \in \Pi_q$, $P \in \mathcal{P}$, $f \in \mathcal{F}$. Then

$$\begin{aligned} Q_*^{-1}[(P_* f)(q)] &= (Q_*^{-1} \circ P_* f)(Q^{-1}(q)) \\ &= [(Q^{-1} \circ P)_* f](q_0) \in (\mathcal{P}_* \mathcal{F})(q_0) \subset \Pi_{q_0}. \end{aligned}$$

  Thus $Q_*^{-1}(\Pi_q) \subset \Pi_{q_0}$, whence $\dim \Pi_q \leq \dim \Pi_{q_0}$. Interchanging in this arguments q and q_0, we get $\dim \Pi_{q_0} \leq \dim \Pi_q$.
- Finally we have $\dim \Pi_q = \dim \Pi_{q_0}$, $q \in \mathcal{O}_{q_0}$.

Proof of the Orbit theorem: 2/7

- *2)* For any point $q \in M$ denote $m = \dim \Pi_q$, and choose such vector fields $V_1, \dots, V_m \in \mathcal{P}_* \mathcal{F}$ that $\Pi_q = \text{span}(V_1(q), \dots, V_m(q))$.
- Further, define a mapping

$$G_q : (t_1, \dots, t_m) \mapsto e^{t_m V_m} \circ \cdots \circ e^{t_1 V_1}(q), \qquad \mathbb{R}^m \to M.$$

- We have $\frac{\partial G_q}{\partial t_i}(0) = V_i(q)$, thus the vectors $\frac{\partial G_q}{\partial t_1}(0), \dots, \frac{\partial G_q}{\partial t_m}(0)$ are linearly independent.
- Consequently, the restriction of G_q to a sufficiently small neighbourhood W_0 of the origin in $\mathbb{R}^m$ is a submersion.
- *3)* The image $G_q(W_0)$ is an (embedded) submanifold of M, may be, for a smaller neighbourhood W_0.

Proof of the Orbit theorem: 3/7

- *4)* We show that $G_q(W_0) \subset \mathcal{O}_q$.
- We have $G_q(W_0) = \{e^{t_m V_m} \circ \cdots \circ e^{t_1 V_1}(q) \mid t \in W_0\}$.
- Since $V_1 = P_* f$, $P \in \mathcal{P}$, $f \in \mathcal{F}$, we get

$$e^{t_1 V_1}(q) = e^{t_1 P_* f}(q) = P \circ e^{t_1 f} \circ P^{-1}(q) \in \mathcal{O}_q.$$

- We conclude similarly that $e^{t_2 V_2} \circ e^{t_1 V_1}(q) \in \mathcal{O}_q$ etc. Finally we have $G_q(t) \in \mathcal{O}_q$, $t \in W_0$.

Proof of the Orbit theorem: 4/7

- *5)* We show that $G_{q*}(T_t\mathbb{R}^m) = \Pi_{G_q(t)}$, $t \in W_0$. We have $\dim G_{q*}(T_t\mathbb{R}^m) = m = \dim \Pi_{G_q(t)}$, thus it suffices to prove the inclusion $\frac{\partial G_q}{\partial t_i}(t) \in \Pi_{G_q(t)}, \qquad t \in W_0.$
- Let us compute this partial derivative:

$$\frac{\partial G_q}{\partial t_i} = \frac{\partial}{\partial t_i} e^{t_m V_m} \circ \cdots \circ e^{t_i V_i} \circ \cdots \circ e^{t_1 V_1}(q)$$

denote $R = e^{t_m V_m} \circ \cdots \circ e^{t_{i+1} V_{i+1}}$, $q' = e^{t_{i-1} V_{i-1}} \circ \cdots \circ e^{t_1 V_1}(q)$,

$$\begin{aligned} &= \frac{\partial}{\partial t_i} R \circ e^{t_i V_i}(q') = R_* V_i(e^{t_i V_i}(q')) \\ &= (R_* V_i)[R \circ e^{t_i V_i} \circ \cdots \circ e^{t_1 V_1}(q)] \\ &= (R_* V_i)(G_q(t)) \in (\mathcal{P}_* \mathcal{F})(G_q(t)) \subset \Pi_{G_q(t)}. \end{aligned}$$

- Thus $G_{q*}(T_t\mathbb{R}^m) = \Pi_{G_q(t)}$, i.e., the space $\Pi_{G_q(t)}$ is a tangent space to the smooth manifold $G_q(W_0)$ at the point $G_q(t)$.

Proof of the Orbit theorem: 5/7

- *6)* We prove that the sets $G_q(W_0)$ form a base of a ("strong") topology on M.
- *6a)* It is obvious that any point $q \in M$ is contained in the set $G_q(W_0)$.
- *6b)* Let us show that for any point $\widehat{q} \in G_q(W_0) \cap G_{\widetilde{q}}(\widetilde{W_0})$ there exists a set $G_{\widehat{q}}(\widehat{W_0}) \subset G_q(W_0) \cap G_{\widetilde{q}}(\widetilde{W_0})$.
- Take any point $\widehat{q} \in G_q(W_0) \cap G_{\widetilde{q}}(\widetilde{W_0})$ and consider $G_{\widehat{q}}(t) = e^{t_m \widehat{V}_m} \circ \cdots \circ e^{t_1 \widehat{V}_1}(\widehat{q})$.
- For any point $q' \in G_q(W_0)$ we have $\widehat{V}_1(q') \in (\mathcal{P}_* \mathcal{F})(q') \subset \Pi_{q'}$. But $G_q(W_0)$ is a submanifold with the tangent space $T_{q'} G_q(W_0) = \Pi_{q'}$. The vector field $\widehat{V}_1$ is tangent to this submanifold, thus $e^{t_1 \widehat{V}_1}(\widehat{q}) \in G_q(W_0)$ for small $|t_1|$. We conclude similarly that $e^{t_2 \widehat{V}_2} \circ e^{t_1 \widehat{V}_1}(\widehat{q}) \in G_q(W_0)$ for small $|t_1|$, $|t_2|$ etc. Finally we get

$$G_{\widehat{q}}(t) \in G_q(W_0) \text{ for small } |t|.$$

- Similarly $G_{\widehat{q}}(t) \in G_{\widetilde{q}}(\widetilde{W_0})$ for small $|t|$. Thus $G_{\widehat{q}}(\widehat{W_0}) \subset G_q(W_0) \cap G_{\widetilde{q}}(\widetilde{W_0})$ for some neighbourhood $\widehat{W_0}$, and property 6b) is proved.

Proof of the Orbit theorem: 6/7

- It follows from properties 6a) and 6b) that the sets $G_q(W_0)$ form a base of topology on the set M. Denote the corresponding topological space as $M^{\mathcal{F}}$.
- *7)* We show that for any $q_0 \in M$ the orbit $\mathcal{O}_{q_0}$ is connected, open and closed in the space $M^{\mathcal{F}}$.
- The mappings $t_i \mapsto e^{t_i f_i}(q)$ are continuous in $M^{\mathcal{F}}$, thus $\mathcal{O}_{q_0}$ is connected.
- Any point $q \in \mathcal{O}_{q_0}$ is contained in the neighbourhood $G_q(W_0) \subset \mathcal{O}_q = \mathcal{O}_{q_0}$, thus the orbit is open in $M^{\mathcal{F}}$.
- Finally, any orbit is a complement in M to orbits with which it does not intersect. Thus any orbit is closed in $M^{\mathcal{F}}$.
- So any orbit $\mathcal{O}_{q_0}$ is a connected component of the topological space $M^{\mathcal{F}}$.

Proof of the Orbit theorem: 7/7

- *8)* Introduce a smooth structure on $\mathcal{O}_{q_0}$ as follows:
  - the sets $G_q(W_0)$ are called coordinate neighbourhoods
  - the mappings $G_q^{-1} : G_q(W_0) \to W_0$ are called coordinate mappings.
- It is easy to see that these coordinate neighbourhoods and mappings agree: for any intersecting neighbourhoods $G_q(W_0)$ and $G_{\widetilde{q}}(\widetilde{W_0})$ the composition

$$G_{\widetilde{q}} \circ G_q \; : \; G_q^{-1}(G_q(W_0) \cap G_{\widetilde{q}}(\widetilde{W_0})) \to G_{\widetilde{q}}^{-1}(G_q(W_0) \cap G_{\widetilde{q}}(\widetilde{W_0}))$$

  is a diffeomorphism.
- Thus the orbit $\mathcal{O}_{q_0}$ is a smooth manifold.
- Moreover, $\mathcal{O}_{q_0} \subset M$ is an immersed submanifold of dimension $m = \dim \Pi_{q_0}$.
- *9)* It follows from item 5) above that the smooth manifold $\mathcal{O}_{q_0}$ has a tangent space

$$T_q\mathcal{O}_{q_0} = \Pi_q = \operatorname{span}(\mathcal{P}_*\mathcal{F})(q), \qquad q \in \mathcal{O}_{q_0}.$$

- The Orbit theorem is proved.

Statement of the Orbit theorem

Theorem (*Orbit theorem*, Nagano–Sussmann)

Let $\mathcal{F} \subset \mathrm{Vec}(M)$, and let $q_0 \in M$.

(1) *The orbit $\mathcal{O}_{q_0}$ is a connected immersed submanifold of M.*

(2) *For any $q \in \mathcal{O}_{q_0}$*

$$T_q\mathcal{O}_{q_0} = \mathrm{span}(\mathcal{P}_*\mathcal{F})(q) = \mathrm{span}\{(P_*V)(q) \mid P \in \mathcal{P}, \quad V \in \mathcal{F}\},$$
$$\mathcal{P} = \{e^{t_N f_N} \circ \cdots \circ e^{t_1 f_1} \mid t_i \in \mathbb{R}, \quad f_i \in \mathcal{F}, \quad N \in \mathbb{N}\}.$$

Corollary: Orbit and Lie algebra of the system

Corollary

For any $q_0 \in M$ *and any* $q \in \mathcal{O}_{q_0}$ *we have* $\mathrm{Lie}_q(\mathcal{F}) \subset T_q\mathcal{O}_{q_0}$*, where*

$$\mathrm{Lie}_q(\mathcal{F}) = \mathrm{span}\{[f_N, [\dots, [f_2, f_1] \dots]](q) \mid f_i \in \mathcal{F},\ N \in \mathbb{N}\} \subset T_q M.$$

- *Proof.* Let $q_0 \in M$, $q \in \mathcal{O}_{q_0}$.
- Take any $f \in \mathcal{F}$. Then $\varphi(t) = e^{tf}(q) \in \mathcal{O}_{q_0}$, thus $\dot{\varphi}(0) = f(q) \in T_q\mathcal{O}_{q_0}$. It follows that $\mathcal{F}(q) \subset T_q\mathcal{O}_{q_0}$.
- Further, take any $f_1, f_2 \in \mathcal{F}$, then $\varphi(t) = e^{-tf_2} \circ e^{-tf_1} \circ e^{tf_2} \circ e^{tf_1}(q) \in \mathcal{O}_{q_0}$. Thus

  $$\left.\frac{d}{dt}\right|_{t=0} \varphi(\sqrt{t}) = [f_1, f_2](q) \in T_q\mathcal{O}_{q_0}.$$

  It follows that $[\mathcal{F}, \mathcal{F}](q) \subset T_q\mathcal{O}_{q_0}$.
- We prove similarly that $[[\mathcal{F}, \mathcal{F}], \mathcal{F}](q) \subset T_q\mathcal{O}_{q_0}$, and by induction that $\mathrm{Lie}_q(\mathcal{F}) \subset T_q\mathcal{O}_{q_0}$. □

Analytic and non-analytic cases

- In the analytic case the inclusion $\mathrm{Lie}_q(\mathcal{F}) \subset T_q\mathcal{O}_{q_0}$ turns into an equality.

Proposition

Let M and $\mathcal{F}$ be real-analytic. Then for any $q_0 \in M$ and any $q \in \mathcal{O}_{q_0}$

$$\mathrm{Lie}_q(\mathcal{F}) = T_q\mathcal{O}_{q_0}.$$

- But in a smooth non-analytic case the inclusion $\mathrm{Lie}_q(\mathcal{F}) \subset T_q\mathcal{O}_{q_0}$ may become strict.
- Example: Orbit of non-analytic system.
  - let $M = \mathbb{R}^2_{x,y}$, $\mathcal{F} = \{f_1, f_2\}$, $f_1 = \frac{\partial}{\partial x}$, $f_2 = a(x)\frac{\partial}{\partial y}$, where $a \in C^\infty(\mathbb{R})$, $a(x) = 0$ for $x \leq 0$, $a(x) > 0$ for $x > 0$.
  - It is easy to see that $\mathcal{O}_q = \mathbb{R}^2$ for any $q = (x, y) \in \mathbb{R}^2$.
  - Although, for $x \leq 0$ we have

$$\mathrm{Lie}_q(\mathcal{F}) = \mathrm{span}(f_1(q)) \neq T_q\mathcal{O}_q.$$

Corollary: Rashevskii–Chow theorem

- A system $\mathcal{F} \subset \text{Vec}(M)$ is called *completely nonholonomic* (*full-rank*, *bracket-generating*) if $\text{Lie}_q(\mathcal{F}) = T_q M \quad \forall q \in M$.

Theorem (Rashevskii–Chow)

If $\mathcal{F} \subset \text{Vec}(M)$ is full-rank and M is connected, then $\mathcal{O}_q = M \; \forall \; q \in M$.

Proof.

- Take any $q \in M$ and any $q_1 \in \mathcal{O}_q$.
- We have $T_{q_1}\mathcal{O}_q \supset \text{Lie}_{q_1}(\mathcal{F}) = T_{q_1}M$, thus $\dim \mathcal{O}_q = \dim M$, i.e., $\mathcal{O}_q$ is open in M.
- On the other hand, any orbit is closed as a complement to the union of all other orbits.
- Thus any orbit is a connected component of M. Since M is connected, each orbit coincides with M.

□

Corollary: Lie algebra rank condition

Corollary (Lie algebra rank condition, LARC)

If a manifold M is connected, and a system $\mathcal{F} \subset \mathrm{Vec}(M)$ is symmetric and completely nonholonomic, then it is controllable on M.

Distributions

- A *distribution* on a smooth manifold M is a smooth mapping

$$\Delta \colon q \mapsto \Delta_q \subset T_q M, \quad q \in M,$$

  where the vector subspaces Δ_q have the same dimension called the *rank* of Δ.
- An immersed submanifold $N \subset M$ is called an *integral manifold* of a distribution Δ if

$$\forall q \in N \quad T_q N = \Delta_q.$$

- A distribution Δ on M is called *integrable* if for any point $q \in M$ there exists an integral manifold $N_q \ni q$.
- Denote by

$$\bar{\Delta} = \{f \in \mathrm{Vec}(M) \mid f(q) \in \Delta_q \quad \forall q \in M\}$$

  the set of vector fields tangent to Δ.
- A distribution Δ is called *holonomic* if $[\bar{\Delta}, \bar{\Delta}] \subset \bar{\Delta}$.

Corollary: Frobenius theorem

Theorem (Frobenius)

A distribution is integrable iff it is holonomic.

Proof:

- *Necessity*. Take any $f, g \in \bar{\Delta}$. Let $q \in M$, and let $N_q \ni q$ be the integral manifold of Δ through q.
- Then
$$\varphi(t) = e^{-tg} \circ e^{-tf} \circ e^{tg} \circ e^{tf}(q) \in N_q,$$
thus
$$\left.\frac{d}{dt}\right|_{t=0} \varphi(\sqrt{t}) = [f, g](q) \in T_q N_q = \Delta_q.$$
- So $[f, g] \in \bar{\Delta}$, and the inclusion $[\bar{\Delta}, \bar{\Delta}] \subset \bar{\Delta}$ follows.

Frobenius theorem

- *Sufficiency*. We consider only the analytic case.
- We have
$$[\bar{\Delta}, \bar{\Delta}] \subset \bar{\Delta}, \qquad [[\bar{\Delta}, \bar{\Delta}], \bar{\Delta}] \subset [\bar{\Delta}, \bar{\Delta}] \subset \bar{\Delta}.$$
- Inductively $\mathrm{Lie}_q(\bar{\Delta}) \subset \bar{\Delta}_q = \Delta_q$.
- The reverse inclusion is obvious, thus $\mathrm{Lie}_q(\bar{\Delta}) = \Delta_q$, $q \in M$.
Denote $N_q = \mathcal{O}_q(\bar{\Delta})$ and prove that N_q is an integral manifold of Δ:
$$T_{q'} N_q = T_{q'}(\mathcal{O}_q(\bar{\Delta})) = \mathrm{Lie}_{q'}(\bar{\Delta}) = \Delta_{q'}, \quad q' \in N_q.$$
- So $N_q \ni q$ is the integral manifold of Δ, and Δ is integrable. $\square$

Corollary: Frobenius condition

- Consider a *local frame* of Δ:

$$\Delta_q = \operatorname{span}(f_1(q), \dots, f_k(q)), \quad q \in S \subset M, \quad f_1, \dots, f_k \in \operatorname{Vec}(S), \quad k = \dim \Delta_q,$$

where S is an open subset of M.

- Then the inclusion $[\bar{\Delta}, \bar{\Delta}] \subset \bar{\Delta}$ takes the form

$$[f_i, f_j](q) = \sum_{l=1}^{k} c_{ij}^l(q) f_l(q), \qquad q \in S, \quad c_{ij}^l \in C^\infty(S).$$

- This equality is called the *Frobenius condition*.

Example:
The sub-Riemannian problem on the group of motions of the plane

- The control system has the following form:

$$\mathcal{F} = \{u_1 f_1 + u_2 f_2 \mid (u_1, u_2) \in \mathbb{R}^2\} \subset \mathrm{Vec}(\mathbb{R}^2 \times S^1),$$

$$f_1 = \cos\theta \frac{\partial}{\partial x} + \sin\theta \frac{\partial}{\partial y}, \qquad f_2 = \frac{\partial}{\partial \theta}.$$

- The system is symmetric: $\mathcal{F} = -\mathcal{F}$.
- Compute its Lie algebra:

$$[f_1, f_2] = \sin\theta \frac{\partial}{\partial x} - \cos\theta \frac{\partial}{\partial y} =: f_3,$$

$$\mathrm{Lie}_q(\mathcal{F}) = \mathrm{span}(f_1(q), f_2(q), f_3(q)) = T_q(\mathbb{R}^2 \times S^1).$$

- The system $\mathcal{F}$ is completely nonholonomic, thus controllable.