

# Examples and statements of control problems (*Lecture 1*)

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«Geometric control theory, sub-Riemannian geometry, and their applications»

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## Plan of lecture

1. Examples of optimal control problems
2. Statements of the main problems of this course:
  - 2.1 controllability problem,
  - 2.2 optimal control problem.
3. Smooth manifolds and vector fields.

## Examples of optimal control problems:

### 1. Stopping a train

Given:

- material point of mass  $m > 0$  with coordinate  $x \in \mathbb{R}$
- force  $F$  bounded by the absolute value by  $F_{\max} > 0$
- initial position  $x_0$  and initial velocity  $\dot{x}_0$  of the material point

Find:

- force  $F$  that steers the point to the origin with zero velocity, for a minimal time.

$$\dot{x}_1 = x_2, \quad (x_1, x_2) \in \mathbb{R}^2,$$

$$\dot{x}_2 = u, \quad |u| \leq 1,$$

$$(x_1, x_2)(0) = (x_0, \dot{x}_0), \quad (x_1, x_2)(t_1) = (0, 0),$$

$$t_1 \rightarrow \min.$$

## 2. Control of linear oscillator

Given:

- pendulum that performs small oscillations under the action of a force bounded by the absolute value

Find:

- force that steers the pendulum from an arbitrary position and velocity to the stable equilibrium for a minimum time.

$$\begin{aligned}\dot{x}_1 &= x_2, & x &= (x_1, x_2) \in \mathbb{R}^2, \\ \dot{x}_2 &= -x_1 + u, & |u| &\leq 1, \\ x(0) &= x^0, & x(t_1) &= 0, \\ t_1 &\rightarrow \min.\end{aligned}$$

### 3. The Markov–Dubins car

Given:

- model of a car given by a unit vector attached at a point  $(x, y) \in \mathbb{R}^2$ , with orientation  $\theta \in S^1$
- The car moves forward with the unit velocity and can simultaneously rotate with an angular velocity  $|\dot{\theta}| \leq 1$
- an initial and a terminal state of the car

Find:

- angular velocity in such a way that the time of motion is as minimum as possible.

$$\dot{x} = \cos \theta, \quad q = (x, y, \theta) \in \mathbb{R}_{x,y}^2 \times S_\theta^1 = M,$$

$$\dot{y} = \sin \theta, \quad |u| \leq 1,$$

$$\dot{\theta} = u,$$

$$q(0) = q_0, \quad q(t_1) = q_1,$$

$$t_1 \rightarrow \min .$$

## 4. The sub-Riemannian problem on the group of motions of the plane

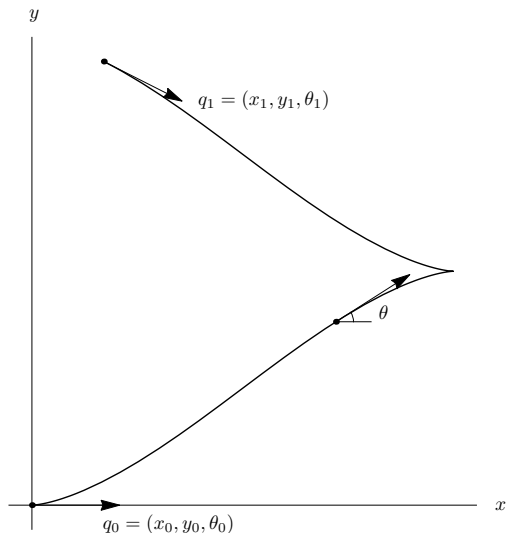
Given:

- model of a car in the plane that can move forward or backward with an arbitrary linear velocity and simultaneously rotate with an arbitrary angular velocity
- state of the car is given by its position in the plane and orientation angle
- an initial and a terminal state of the car

Find:

- motion of the car from a given initial state to a given terminal state, so that the length of the path in the space of positions and orientations is as minimum as possible.

## 4. The sub-Riemannian problem on the group of motions of the plane



$$\dot{x} = u \cos \theta, \quad q = (x, y, \theta) \in \mathbb{R}^2 \times S^1,$$

$$\dot{y} = u \sin \theta, \quad (u, v) \in \mathbb{R}^2,$$

$$\dot{\theta} = v,$$

$$q(0) = q_0, \quad q(t_1) = q_1,$$

$$l = \int_0^{t_1} \sqrt{u^2 + v^2} dt \rightarrow \min.$$

## 5. Euler elasticae

Given:

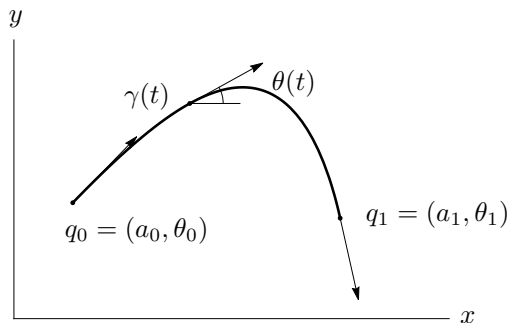
- uniform elastic rod of length  $l$  in the plane
- the rod has fixed endpoints and tangents at endpoints

Find:

- the profile of the rod.



## 5. Euler elasticae



$$\dot{x} = \cos \theta, \quad q = (x, y, \theta) \in \mathbb{R}^2 \times S^1,$$

$$\dot{y} = \sin \theta, \quad u \in \mathbb{R},$$

$$\dot{\theta} = u,$$

$$q(0) = q_0, \quad q(t_1) = q_1,$$

$t_1 = l$  is the length of the rod,

$$J = \frac{1}{2} \int_0^{t_1} u^2 dt \rightarrow \min.$$

## 6. The plate-ball problem

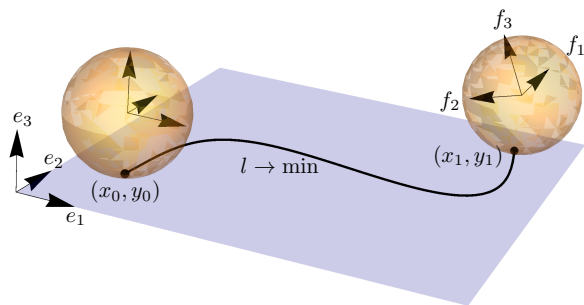
Given:

- uniform sphere roll without slipping or twisting on a horizontal plane
- imagine: the sphere rolls between two horizontal planes, a fixed lower one and a moving upper one
- absence of slipping: the contact point of the sphere with the plane has zero instantaneous velocity
- absence of twisting means that the angular velocity vector of the sphere is horizontal
- admissible motions are obtained by horizontal motions of the upper plane
- initial and terminal states of the sphere.

Find:

- roll the sphere so that the length of the curve in the plane traced by the contact point was the minimum possible.

## 6. The plate-ball problem



$$\dot{x} = u, \quad \dot{y} = v, \quad (u, v) \in \mathbb{R}^2,$$

$$\dot{R} = R \begin{pmatrix} 0 & 0 & -u \\ 0 & 0 & -v \\ u & v & 0 \end{pmatrix},$$

$$q = (x, y, R) \in \mathbb{R}^2 \times \text{SO}(3),$$

$$q(0) = q_0, \quad q(t_1) = q_1,$$

$$l = \int_0^{t_1} \sqrt{u^2 + v^2} dt \rightarrow \min.$$

## 7. Anthropomorphic curve reconstruction

Given:

- greyscale image as a set of isophotes (level lines of brightness)
- image corrupted in some domain.

Find:

- anthropomorphic reconstruction of the image.

## 7. Anthropomorphic curve reconstruction

- D. Hubel and T. Wiesel (1981 Nobel Prize): a human brain stores curves not as sequences of planar points  $(x_i, y_i)$ , but as sequences of positions and orientations  $(x_i, y_i, \theta_i)$
- model of the primary visual cortex V1 of the human brain by J. Petitot, G. Citti and A. Sarti: corrupted curves of images are reconstructed according to a variational principle
- human brain lifts images  $(x(t), y(t))$  from the plane to the space of positions and orientations  $(x(t), y(t), \theta(t))$ .

$$\dot{x} = u \cos \theta, \quad \dot{y} = u \sin \theta, \quad \dot{\theta} = v,$$

$$J = \int_0^{t_1} (u^2 + v^2) dt \rightarrow \min \quad \Leftrightarrow \quad l = \int_0^{t_1} \sqrt{u^2 + v^2} dt \rightarrow \min,$$

- Optimal trajectories for the sub-Riemannian problem on the group of motions of the plane solve the problem of anthropomorphic curve reconstruction!

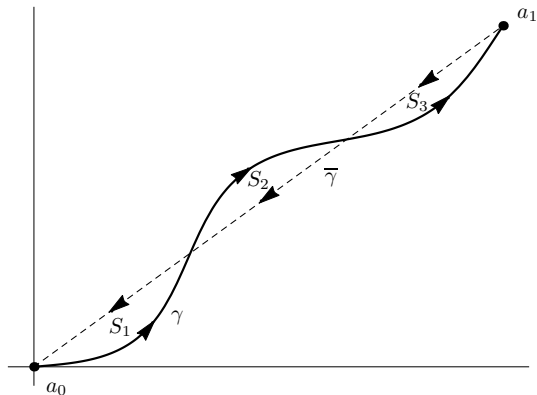
## 8. Dido's problem

Given:

- points  $a_0, a_1 \in \mathbb{R}^2$
- Lipschitzian curve  $\bar{\gamma} \subset \mathbb{R}^2$  connecting  $a_1$  with  $a_0$
- number  $S \in \mathbb{R}$ .

Find:

- the shortest Lipschitzian curve  $\gamma \subset \mathbb{R}^2$  connecting  $a_0$  with  $a_1$  for which the closed curve  $\gamma \cup \bar{\gamma}$  bounds a domain in  $\mathbb{R}^2$  of the algebraic area  $S$ .



## 8. Dido's problem

- coordinates  $x, y$  in the plane  $\mathbb{R}^2$  with the origin  $a_0$ . Then  $a_0 = (0, 0)$ ,  $a_1 = (x_1, y_1)$ ,  $\gamma(t) = (x(t), y(t))$ ,  $t \in [0, t_1]$ ,  $\bar{\gamma}(t) = (\bar{x}(t), \bar{y}(t))$ ,  $t \in [0, \bar{t}_1]$ .
- closed curve  $\hat{\gamma} = \gamma \cup \bar{\gamma}$  and a domain bounded by it:  $D \subset \mathbb{R}^2$ ,  $\partial D = \hat{\gamma}$

$$S(D) = \frac{1}{2} \oint_{\hat{\gamma}} x dy - y dx = \frac{1}{2} \int_0^{t_1} (x\dot{y} - y\dot{x}) dt - \bar{I},$$

$$\dot{x}(t) =: u_1(t), \quad \dot{y}(t) =: u_2(t), \quad \dot{z}(t) = \frac{1}{2}(xu_2 - yu_1).$$

$$\dot{q} = u_1 X_1(q) + u_2 X_2(q), \quad u = (u_1, u_2) \in \mathbb{R}^2, \quad q = (x, y, z) \in \mathbb{R}^3,$$

$$X_1 = \frac{\partial}{\partial x} - \frac{y}{2} \frac{\partial}{\partial z}, \quad X_2 = \frac{\partial}{\partial y} + \frac{x}{2} \frac{\partial}{\partial z},$$

$$q(0) = q_0 = (0, 0, 0), \quad q(t_1) = q_1 = (x_1, y_1, z_1),$$

$$l(\gamma) = \int_0^{t_1} \sqrt{\dot{x}^2 + \dot{y}^2} dt = \int_0^{t_1} \sqrt{u_1^2 + u_2^2} dt \rightarrow \min.$$

## Dynamical systems and control systems

- *smooth dynamical system*, or an *ordinary differential equation*:

$$\dot{q} = f(q), \quad q \in M$$

- deterministic: given an initial condition  $q(0) = q_0$  and a time  $t > 0$ , there exists a unique solution  $q(t)$
- *control system*:

$$\dot{q} = f(q, u), \quad q \in M, \quad u \in U. \quad (1)$$

- *control* function  $u = u(t) \in U \Rightarrow$  a nonautonomous ODE

$$\dot{q} = f(q, u(t)). \quad (2)$$

- Together with an initial condition  $q(0) = q_0$ , ODE (2) determines a unique solution — a *trajectory*  $q_u(t)$ .
- Regularity assumptions for controls  $u(\cdot)$ : piecewise constant,  $L^\infty$ , ...



## Attainable sets

- *Attainable set* of control system (1) from a point  $q_0$  for arbitrary times:

$$\mathcal{A}_{q_0} = \{q_u(t) \mid q_u(0) = q_0, \quad u \in L^\infty([0, t], U), \quad t \geq 0\}.$$

- attainable set from the point  $q_0$  for a time  $t_1 \geq 0$ :

$$\mathcal{A}_{q_0}(t_1) = \{q_u(t_1) \mid q_u(0) = q_0, \quad u \in L^\infty([0, t_1], U)\},$$

- attainable set from the point  $q_0$  for times not greater than  $t_1 \geq 0$ :

$$\mathcal{A}_{q_0}(\leq t_1) = \bigcup_{t=0}^{t_1} \mathcal{A}_{q_0}(t).$$

## Controllability problem

A control system (1) is called:

- *globally (completely) controllable* if  $\mathcal{A}_{q_0} = M$  for any  $q_0 \in M$
  - *globally controllable from a point*  $q_0 \in M$  if  $\mathcal{A}_{q_0} = M$
  - *locally controllable at*  $q_0$  if  $q_0 \in \text{int } \mathcal{A}_{q_0}$
  - *small time locally controllable (STLC) at*  $q_0$  if  $q_0 \in \text{int } \mathcal{A}_{q_0}(\leq t_1)$  for any  $t_1 > 0$
- 
- Local controllability problem: necessary conditions and sufficient conditions of STLC for arbitrary dimension of the state space  $M$ , but local controllability tests are available only for the case  $\dim M = 2$ .
  - Global controllability problem: global controllability conditions only for very symmetric systems (linear systems, left-invariant systems on Lie groups).

## Optimal control problem

- Let  $q_1 \in \mathcal{A}_{q_0}(t_1)$ . Then typically  $q_0, q_1$  are connected by continuum of trajectories
- *Cost functional* to be minimized:

$$J = \int_0^{t_1} \varphi(q, u) dt.$$

- *Optimal control problem:*

$$\dot{q} = f(q, u), \quad q \in M, \quad u \in U,$$

$$q(0) = q_0, \quad q(t_1) = q_1,$$

$$J = \int_0^{t_1} \varphi(q, u) dt \rightarrow \min .$$

- Other important mathematical control problems: equivalence, stabilization, observability, etc., which we do not touch upon.

## Smooth manifolds

A smooth  $k$ -dimensional *submanifold*  $M \subset \mathbb{R}^n$  is usually defined by one of the following equivalences:

(a) implicitly by a system of regular equations:

$$\begin{aligned} f_1(x) = \cdots = f_{n-k}(x) = 0, \quad x \in \mathbb{R}^n, \\ \text{rank} \left( \frac{\partial f_1}{\partial x}, \dots, \frac{\partial f_{n-k}}{\partial x} \right) = n - k, \end{aligned}$$

(b) or by a regular parametrization:

$$\begin{aligned} x = \Phi(y), \quad y \in \mathbb{R}^k, \quad x \in \mathbb{R}^n, \\ \text{rank} \frac{\partial \Phi}{\partial y} = k. \end{aligned}$$

An abstract smooth *manifold*  $M$  (not embedded into  $\mathbb{R}^n$ ) is defined via a system of charts (local coordinates) that mutually agree.

## Tangent vectors and spaces

A mapping between smooth manifolds is called *smooth* if it is smooth (of class  $C^\infty$ ) in local coordinates.

The *tangent space* to a smooth submanifold  $M \subset \mathbb{R}^n$  at a point  $x \in M$  is defined as follows for the two definitions above of a submanifold:

(a)  $T_x M = \text{Ker } \frac{\partial f}{\partial x}(x),$

(b)  $T_x M = \text{Im } \frac{\partial \Phi}{\partial y}(y), x = \Phi(y).$

Given a smooth mapping  $F : M \rightarrow N$  between smooth manifolds, for any  $q \in M$  the *differential*  $F_{*q} : T_q M \rightarrow T_{F(q)} N$  is defined as follows:

$$F_{*q} v = \left. \frac{d}{dt} \right|_{t=0} F(\gamma(t)),$$

where  $\gamma : (-\varepsilon, \varepsilon) \rightarrow M$  is a smooth curve such that  $\gamma(0) = q$ ,  $\dot{\gamma}(0) = v$ .

## Smooth vector fields and their commutativity

- A smooth *vector field* on a manifold  $M$  is a smooth mapping  $M \ni q \mapsto V(q) \in T_q M$ . Notation:  $V \in \text{Vec}(M)$ .
- A *trajectory of a vector field*  $V$  through a point  $q_0 \in M$  is a solution to the Cauchy problem  $\dot{q}(t) = V(q(t))$ ,  $q(0) = q_0$ .
- Suppose that a trajectory  $q(t)$  exists for all times  $t \in \mathbb{R}$ , then we denote  $e^{tV}(q_0) := q(t)$ . The one-parameter group of diffeomorphisms  $e^{tV} : M \rightarrow M$  is the *flow* of the vector field  $V$ .
- We say that vector fields  $V$  and  $W$  *commute* if their flows commute:

$$e^{tV} \circ e^{sW} = e^{sW} \circ e^{tV}, \quad t, s \in \mathbb{R}.$$

- In the general case vector fields  $V$  and  $W$  do not commute and the curve

$$\varphi(t) = e^{-tW} \circ e^{-tV} \circ e^{tW} \circ e^{tV}(q_0)$$

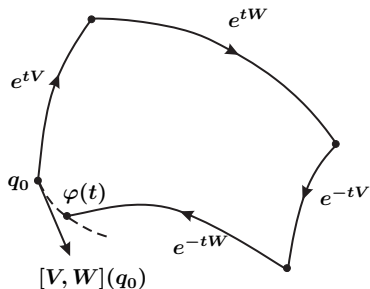
satisfies the inequality  $\varphi(t) \neq q_0$ ,  $t \in \mathbb{R}$ .

- The leading nontrivial term of the Taylor expansion of  $\varphi(t)$ ,  $t \rightarrow 0$ , is taken as the measure of noncommutativity of vector fields  $V$  and  $W$ .

## Lie brackets of vector fields

- The *commutator* (*Lie bracket*) of the vector fields  $V, W$  at the point  $q_0$  is defined as  $[V, W](q_0) := \frac{1}{2}\ddot{\varphi}(0)$ , so that

$$\varphi(t) = q_0 + t^2[V, W](q_0) + o(t^2), \quad t \rightarrow 0.$$



- In local coordinates  $[V, W] = \frac{\partial W}{\partial x} V - \frac{\partial V}{\partial x} W$ .

## Example: Car in the plane

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{pmatrix} = u \begin{pmatrix} \cos \theta \\ \sin \theta \\ 0 \end{pmatrix} + v \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad V = \cos \theta \frac{\partial}{\partial x} + \sin \theta \frac{\partial}{\partial y}, \quad W = \frac{\partial}{\partial \theta}.$$

$$[V, W] = \frac{\partial W}{\partial q} V - \frac{\partial V}{\partial q} W = 0 \cdot V - \begin{pmatrix} 0 & 0 & -\sin \theta \\ 0 & 0 & \cos \theta \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \sin \theta \\ -\cos \theta \\ 0 \end{pmatrix}.$$

Another way of computing Lie brackets, via commutator of differential operators:

$$\begin{aligned} [V, W] &= V \circ W - W \circ V = \left( \cos \theta \frac{\partial}{\partial x} + \sin \theta \frac{\partial}{\partial y} \right) \frac{\partial}{\partial \theta} - \frac{\partial}{\partial \theta} \left( \cos \theta \frac{\partial}{\partial x} + \sin \theta \frac{\partial}{\partial y} \right) \\ &= \sin \theta \frac{\partial}{\partial x} - \cos \theta \frac{\partial}{\partial y}. \end{aligned}$$



## Example: Car in the plane

- Notice the visual meaning of the vector fields  $V$ ,  $W$ ,  $[V, W]$  for the car in the plane:
  - $V$  generates the motion forward
  - $W$  generates rotations of the car
  - $[V, W]$  generates motion of the car in the direction perpendicular to its orientation, thus physically forbidden.
- Choosing alternating motions of the car:

forward  $\rightarrow$  rotation counter-clockwise  $\rightarrow$  backward  $\rightarrow$  rotation clockwise,

we can move the car infinitesimally in the forbidden direction. So the Lie bracket  $[V, W]$  is generated by a car during parking manoeuvres in a limited space.