

# Applications of geometric control theory (*Lecture 11*)

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«*Elements of geometric control theory*»

Lecture course in Dept. of Mathematics and Mechanics

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## Reminder: Plan of the previous lecture

1. Sub- Lorentzian problem on the Heisenberg group

## Plan of this lecture

1. Applications to metric geometry (metric lines in sub-Riemannian Carnot groups),
2. Applications to PDEs (kernel of sub-Riemannian heat equation),
3. Applications to models of vision and image processing,
4. Applications to robotics,
5. Some open problems,
6. Further reading.

## Metric lines

- Let  $\Gamma : \mathbb{R} \rightarrow M$  be a geodesic in a sub-Riemannian manifold  $M$ .
- $\Gamma$  is called a *metric line* if it is an isometry, i.e., if any of the following conditions hold:
  - for any  $a, b \in \mathbb{R}$  the restriction  $\Gamma|_{[a,b]}$  is a sub-Riemannian minimizer,
  - for any  $a, b \in I$  the sub-Riemannian distance  $d(\Gamma(a), \Gamma(b)) = |a - b|$ .
- Important question in the theory of metric groups: characterization of metric lines in sub-Riemannian Carnot groups

## Carnot groups

- A *stratification* of a Lie algebra  $\mathfrak{g}$  is a direct-sum decomposition into vector subspaces for some  $s \in \mathbb{N}$

$$\mathfrak{g} = V_1 \oplus \cdots \oplus V_s, \quad [V_1, V_k] = V_{k+1}, \quad k = 1, \dots, s, \quad (1)$$

$$V_s \neq \{0\}, \quad V_{s+1} = \{0\}. \quad (2)$$

- A *Carnot algebra*  $\mathfrak{g}$  is a Lie algebra that has a stratification.
- The integer  $s$  in (1)–(2) is called the *step* of the Carnot algebra  $\mathfrak{g}$ . The subspaces  $V_k$ ,  $1 \leq k \leq s$  are called *layers* of degree  $k$ . The dimension of the first layer  $V_1$  is called the *rank* of the Carnot algebra  $\mathfrak{g}$ . The first layer  $V_1$  generates  $\mathfrak{g}$  as a Lie algebra. Any Carnot algebra is nilpotent.
- A *Carnot group* is a connected and simply connected Lie group whose Lie algebra is a Carnot algebra.
- If the first layer  $V_1$  is endowed with a scalar product, then the left translations of this layer and the scalar product define a left-invariant sub-Riemannian structure on the corresponding Carnot group  $G$ . In this case we call  $G$  a *sub-Riemannian Carnot group*.

## Metric lines in Carnot groups

### Theorem 1

*If a geodesic  $g : \mathbb{R} \rightarrow G$  in a sub-Riemannian Carnot group  $G$  projects to the first layer  $V_1$  to a straight line, then  $g$  is a metric line.*

### Theorem 2

*If  $G$  is a sub-Riemannian Carnot group of step  $s \leq 2$ , then any metric line in  $G$  projects to the first layer  $V_1$  to a straight line.*

### Theorem 3

*If  $G$  is a sub-Riemannian free Carnot group of step  $s > 2$ , then there exists a metric line in  $G$  that projects to the first layer  $V_1$  to a curve distinct from straight line:*

## Sub-Riemannian heat equation

- Let  $M$  be an oriented sub-Riemannian manifold with an orthonormal frame  $X_1, \dots, X_m \in \text{Vec}(M)$ ,  $\dim M = n \geq m$ .
- Let  $\omega$  be a non-vanishing  $n$ -form on  $M$  (*volume form*) and let  $\text{div}_\omega$  be the corresponding divergence of vector field.
- The *horizontal gradient* of a function  $\varphi \in C^\infty(M)$  is the vector field  $\text{grad}_H(\varphi) = \sum_{i=1}^m X_i(\varphi)X_i \in \text{Vec}(M)$ .
- The *sub-Riemannian Laplacian* is the operator  $\Delta_H\varphi = \text{div}_\omega(\text{grad}_H(\varphi))$ .
- The *sub-Riemannian heat equation* is

$$\frac{\partial}{\partial t}\varphi(q, t) = \Delta_H\varphi(q, t).$$

- This equation describes a diffusion process with a density  $\varphi(q, t)$  depending on the position  $q \in M$  and time  $t$  subject to conditions:
  - the diffusion is performed in the direction of the distribution  $\text{span}(X_1, \dots, X_m)$  in which  $\varphi$  is decreasing fastest,
  - the diffusing quantity (heat, mass, ...) satisfies a conservation law.

## Existence of solutions to the heat equation

### Theorem 4

Assume that the distribution  $\text{span}(X_1, \dots, X_m)$  is full-rank, i.e.,  $\text{Lie}(X_1, \dots, X_m)(q) = T_q M$  for any  $q \in M$ . Then the sub-Riemannian heat equation is **hypoelliptic**, i.e., any its solution  $\varphi : \Omega \subset M \times \mathbb{R} \rightarrow \mathbb{R}$  belongs to  $C^\infty(\Omega)$ .

### Theorem 5

Assume that the distribution  $\text{span}(X_1, \dots, X_m)$  is full-rank, and let any points  $q_0, q_1 \in M$  be connected by a sub-Riemannian length minimizer. Then the Cauchy problem

$$\begin{aligned} \frac{\partial}{\partial t} \varphi(q, t) &= \Delta_H \varphi(q, t), & (q, t) \in M \times [0, +\infty) \\ \varphi(q, 0) &= \varphi_0(q) \in L^2(M) \end{aligned}$$

has a unique solution in  $L^2(M \times [0, +\infty))$ . Moreover, such a solution belongs to  $C^\infty(M \times (0, +\infty))$ .

## The heat equation on the Heisenberg group

- Consider the sub-Riemannian structure on  $\mathbb{R}^3$  with an orthonormal frame  $X_1 = \frac{\partial}{\partial x} - \frac{y}{2} \frac{\partial}{\partial z}$ ,  $X_2 = \frac{\partial}{\partial y} + \frac{x}{2} \frac{\partial}{\partial z}$ .
- Then the sub-Riemannian heat equation is

$$\frac{\partial}{\partial t} \varphi(x, y, z, t) = \Delta_H(\varphi) = ((X_1)^2 + (X_2)^2) \varphi(x, y, z, t).$$

- The *heat kernel* is a function  $K_t(q, \bar{q})$  such that the solution to the Cauchy problem

$$\begin{aligned} \frac{\partial}{\partial t} \varphi(q, t) &= \Delta_H \varphi(q, t), & (q, t) \in \mathbb{R}^3 \times [0, +\infty) \\ \varphi(q, 0) &= \varphi_0(q) \in L^2(\mathbb{R}^3, dh) \end{aligned}$$

can be expressed as

$$\varphi(q, t) = \int_{\mathbb{R}^3} K_t(q, \bar{q}) \varphi_0(\bar{q}) dh(\bar{q}), \quad t > 0,$$

where  $dh$  is the Haar measure on the Heisenberg group.

## The Gaveau-Hulanicki fundamental solution for the Heisenberg group

### Theorem 6

*The heat kernel for the sub-Riemannian structure on the Heisenberg group is given by the formula*

$$K_t(q, \bar{q}) = P_t(q^{-1} \cdot \bar{q}), \quad q, \bar{q} \in \mathbb{R}^3,$$

where

$$P_t(x, y, z) = \frac{1}{(2\pi t)^2} \int_{\mathbb{R}} \frac{2\tau}{\sinh(2\tau)} \exp\left(-\frac{\tau(x^2 + y^2)}{2t \tanh(2\tau)}\right) \cos\left(\frac{2z\tau}{t}\right) d\tau, \quad t > 0.$$

Notice that  $P_t(q)$  represents the evolution at time  $t$  of an initial condition that at time zero is concentrated at the origin (the Dirac delta  $\delta_0$ ):

$$P_t(q) = K_t(q, 0) = \int_{\mathbb{R}^3} K_t(q, \bar{q}) \delta_0(\bar{q}) dh(\bar{q}).$$

Problem statement:  
Optimal motion of a mobile robot in the plane

$$q(0) = q_0, \quad q(t_1) = q_1, \quad I = \int_0^{t_1} \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{\theta}^2} dt \rightarrow \min$$

## Optimal control problem

$$\dot{x} = u \cos \theta, \quad \dot{y} = u \sin \theta, \quad \dot{\theta} = v,$$

$$(x, y) \in \mathbb{R}^2, \quad \theta \in S^1 = \mathbb{R}/(2\pi \mathbb{Z}),$$

$$q = (x, y, \theta) \in M = \mathbb{R}^2 \times S^1,$$

$$(u, v) \in \mathbb{R}^2,$$

$$q(0) = q_0, \quad q(t_1) = q_1,$$

$$I = \int_0^{t_1} \sqrt{u^2 + v^2} dt \rightarrow \min.$$

## Extremal trajectories: generic cases

## Extremal trajectories: special cases

## Cut time and cut points

$$t_{\text{cut}}(\lambda) = t(\lambda) = \begin{cases} t_{\varepsilon^5}^1 = 2K(k) = T/2, & \lambda \in C_1, \\ t_{\varepsilon^2}^1 = 2kp_1^1(k) \in (T, 2T), & \lambda \in C_2, \\ +\infty, & \lambda \in C_3 \cup C_5, \\ t_{\varepsilon^5}^1 = \pi = T/2, & \lambda \in C_4 \end{cases}$$
$$p = p_1^1(k) \quad : \quad \text{cn}(p, k)(E(p, k) - p) - \text{dn}(p, k) \text{sn}(p, k) = 0$$

# Cut locus

## Optimal solutions

$$x_1 \neq 0, \quad y_1 = 0, \quad \theta_1 = 0$$

$$x_1 = 0, \quad y_1 = 0, \quad \theta_1 \neq 0$$

## Optimal solutions

$$x_1 \neq 0, \quad y_1 = 0, \quad \theta_1 = \pi$$

## Optimal solutions

$$x_1 = 0, \quad y_1 \neq 0, \quad \theta_1 = 0$$

## Optimal solutions

Generic boundary conditions:

systems of equations in Jacobi's functions  $\Rightarrow$

$\Rightarrow$  software (MATHEMATICA).

Sub-Riemannian caustic  $\{\text{Exp}(\lambda, t) \mid \lambda \in N, t = t_{\text{conj}}^1(\lambda)\}$

## Sub-Riemannian spheres

- $d(q_0, q_1) = \inf\{l(q(\cdot)) \mid q(0) = q_0, q(t_1) = q_1\},$
- $S_R = \{q \in M \mid d(q_0, q) = R\},$
- $R = 0 \quad \Rightarrow \quad S_R = \{q_0\},$
- $R \in (0, \pi) \quad \Rightarrow \quad S_R \cong S^2,$
- $R = \pi \quad \Rightarrow \quad S_R \cong S^2 / \{N = S\},$
- $R > \pi \quad \Rightarrow \quad S_R \cong T^2.$

Global structure of sub-Riemannian spheres:

$$R < \pi, \quad R = \pi, \quad R > \pi$$

## Application: Anthropomorphic restoration of curves

## Neurogeometry and sub-Riemannian problem on $\mathbb{R}^2 \times \mathbb{R}P^1$

- J.Petitot, The neurogeometry of pinwheels as a sub-Riemannian contact structure, *J. Physiology - Paris* 97 (2003), 265–309.
- J.Petitot, *Neurogeometrie de la vision — Modeles mathematiques et physiques des architectures fonctionnelles*, 2008, Editions de l'Ecole Polytechnique.

$$\begin{aligned} \dot{x} &= u \cos \theta, & \dot{y} &= u \sin \theta, & \dot{\theta} &= v, \\ q &= (x, y, \theta), & (x, y) &\in \mathbb{R}^2, & \theta &\in \mathbb{R}P^1 = \mathbb{R}/(\pi \mathbb{Z}), \\ (u, v) &\in \mathbb{R}^2, \\ q(0) &= q_0, & q(t_1) &= q_1, \\ l &= \int_0^{t_1} \sqrt{u^2 + v^2} dt \rightarrow \min. \end{aligned}$$

Optimal solution for the problem on  $\mathbb{R}^2 \times \mathbb{R}P^1$

## Restored curves

## Smooth and non-smooth arcs

## Initial family of curves

## Restored family of curves

## Restored family of curves

# Sub-Riemannian geodesics on Lie groups in image processing

## Geodesics on $SE(2)$ for flat images

## Comparison with classical methods

## Geodesics on $SO(3)$ for spherical images

## Elimination of cusp points: geodesics on PTR(2)

## Geodesics on $SE(3)$ for 3D images

# Modelling and explanation of visual illusions

## Some open problems

1. Smoothness of sub-Riemannian length minimizers — solved!
2. Sard's property in sub-Riemannian geometry
3. Singularities of sub-Riemannian spheres near abnormal minimizers
4. Structure of exponential mapping and sub-Riemannian spheres for generic Martinet, Engel, and Cartan structures
5. Optimal synthesis for 3D left-invariant sub-Riemannian structures
6. Metric lines in sub-Riemannian Carnot groups of step  $> 2$

## Further reading

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3. R. Montgomery, *A tour of subriemannian geometries, their geodesics and applications*, Amer. Math. Soc., 2002
4. A.A. Agrachev, Yu.L. Sachkov, *Control theory from the geometric viewpoint*, Berlin Heidelberg New York Tokyo. Springer-Verlag. 2004
5. A.A. Agrachev, *Some open problems*, Geometric Control Theory and Sub-Riemannian Geometry, pp. 1–13. Springer INdAM Series, vol. 5. Springer, 2014.
6. A. A. Agrachev, Topics in sub-Riemannian geometry, *Russian Math. Surveys*, 71:6 (2016), 989–1019

## Further reading

7. A. Agrachev, D. Barilari, U. Boscain, *A Comprehensive Introduction to sub-Riemannian Geometry from Hamiltonian viewpoint*, Cambridge Studies in Advanced Mathematics, Cambridge Univ. Press, 2019
8. Yu. L. Sachkov, *Introduction to geometric control theory* (in Russian), Moscow, URSS, 2021. English translation: *Introduction to geometric control*, Springer, 2022.
9. Yu. L. Sachkov, Left-invariant optimal control problems on Lie groups: classifications and problems integrable in elementary functions, *Russian mathematical surveys*, 77:1(463) (2022), 109–176
10. Yu. L. Sachkov, Left-invariant optimal control problems on Lie groups: problems integrable by elliptic functions, *Russian math. surveys*, 78:1(469) (2023), 67–166

*Thank you for your attention!*