Sub-Lorentzian problem on the Heisenberg group

(Lecture 10)

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Plan of previous lecture

1. Euler elastic problem

Plan of this lecture

1. Sub-Riemannian geometry
2. Sub-Lorentzian geometry
3. Left-invariant sub-Lorentzian structure on the Heisenberg group
4. Previously obtained results by M. Grochowski
5. Pontryagin maximum principle, parameterization of extremal trajectories, exponential mapping
6. Exponential mapping is a diffeomorphism, its inverse
7. Optimality of extremal trajectories, optimal synthesis
8. Sub-Lorentzian distance: explicit formula, symmetries
9. Sub-Lorentzian spheres of positive and zero radii
10. Discussion and questions

Sub-Riemannian geometry

- Smooth manifold M,
- vector distribution $\Delta = \{\Delta_q \subset T_qM \mid q \in M\}$, $\dim \Delta_q \equiv \text{const}$,
- inner product in Δ:

$$g = \{g_q \text{ — inner product in } \Delta_q \mid q \in M\}$$

- sub-Riemannian structure (Δ, g) on M
- horizontal curve $q \in \text{Lip}([0, t_1], M)$:

$$\dot{q}(t) \in \Delta_{q(t)} \text{ a.e. } t \in [0, t_1],$$

- sub-Riemannian length $l(q(\cdot)) = \int_0^{t_1} (g(\dot{q}(t), \dot{q}(t))^{1/2}\, dt$,
- sub-Riemannian (Carnot-Carathéodory) distance
  $d(q_0, q_1) = \inf\{l(q(\cdot)) \mid q(\cdot) \text{ horiz. curve}, \ q(0) = q_0, \ q(t_1) = q_1\}$,

Sub-Riemannian geometry

- sub-Riemannian minimizer $q(t)$, $t \in [0, t_1]$: horizontal curve s.t. $l(q(\cdot)) = d(q(0), q(t_1))$,
- sub-Riemannian sphere $S_R(q_0) = \{q \in M \mid d(q, q_0) = R\}$, sub-Riemannian ball $B_R(q_0) = \{q \in M \mid d(q, q_0) \leq R\}$,
- geodesic: horizontal curve whose small arcs are minimizers,
- cut time along a geodesic $q(t)$:

$$t_{\text{cut}}(q(\cdot)) = \sup\{t > 0 \mid q(s),\ s \in [0, t],\ \text{minimizer}\ \},$$

- cut point $q(t_1)$, $t_1 = t_{\text{cut}}(q(\cdot))$,
- cut locus $\text{Cut}_{q_0} = \{q_1 \in M \mid q_1 \text{ cut point for some geod. } q(\cdot), \quad q(0) = q_0\}$

Example: SR geometry on the Heisenberg group

- $M = \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \mid (a, b, c) \in \mathbb{R}^3 \right\}$
- $X_1 = \frac{\partial}{\partial x} - \frac{y}{2}\frac{\partial}{\partial z}, \quad X_2 = \frac{\partial}{\partial y} + \frac{x}{2}\frac{\partial}{\partial z}, \; x = a, \; y = b, \; z = c - ab/2$
- $\Delta_q = \mathrm{span}(X_1(q), X_2(q)), \quad g(X_i, X_j) = \delta_{ij}$

Sub-Lorentzian geometry

- Smooth manifold M,
- vector distribution $\Delta = \{\Delta_q \subset T_q M \mid q \in M\}$, $\dim \Delta_q \equiv \text{const}$,
- Lorentzian metric (nondegenerate quadratic form of index 1) in Δ:

$$g = \{g_q \text{ — Lorentzian metric in } \Delta_q \mid q \in M\}$$

- sub-Lorentzian (SL) structure (Δ, g) on M
- horizontal vector: $v \in \Delta_q$,
- horizontal vector v is called:
  - timelike if $g(v) < 0$
  - spacelike if $g(v) > 0$ or $v = 0$,
  - lightlike if $g(v) = 0$ and $v \neq 0$,
  - nonspacelike if $g(v) \leq 0$
- Lipschitzian curve in M is called timelike if it has timelike velocity vector a.e.,
- spacelike, lightlike and nonspacelike curves are defined similarly.

Lightlike cone for $g = dx^2 + dy^2 - dz^2$

Sub-Lorentzian geometry

- A time orientation X is an arbitrary timelike vector field in M.
- A nonspacelike vector $v \in \Delta_q$ is future directed if $g(v, X(q)) < 0$, and past directed if $g(v, X(q)) > 0$.
- A future directed timelike curve $q(t)$, $t \in [0, t_1]$, is called arclength parametrized if $g(\dot{q}(t), \dot{q}(t)) \equiv -1$.
- Any future directed timelike curve can be parametrized by arclength, similarly to the arclength parametrization of a horizontal curve in sub-Riemannian geometry.
- The length of a nonspacelike curve $\gamma \in \mathrm{Lip}([0, t_1], M)$ is

$$l(\gamma) = \int_0^{t_1} |g(\dot{\gamma}, \dot{\gamma})|^{1/2} dt.$$

- For points $q_1, q_2 \in M$ denote by $\Omega_{q_1 q_2}$ the set of all future directed nonspacelike curves in M that connect q_1 to q_2.
- In the case $\Omega_{q_1 q_2} \neq \emptyset$ denote the sub-Lorentzian distance from the point q_1 to the point q_2 as

$$d(q_1, q_2) = \sup\{l(\gamma) \mid \gamma \in \Omega_{q_1 q_2}\}. \qquad (1)$$

- A future directed nonspacelike curve γ is called a SL length maximizer if it realizes the supremum in (1) between its endpoints $\gamma(0) = q_1$, $\gamma(t_1) = q_2$.
- The causal future of a point $q_0 \in M$ is the set $J^+(q_0)$ of points $q_1 \in M$ for which there exists a future directed nonspacelike curve γ that connects q_0 and q_1.
- The chronological future $I^+(q_0)$ of a point $q_0 \in M$ is defined similarly via future directed timelike curves γ.
- Let $q_0 \in M$, $q_1 \in J^+(q_0)$. The search for SL length maximizers that connect q_0 with q_1 reduces to the search for future directed nonspacelike curves γ that solve the problem

$$l(\gamma) \to \max, \qquad \gamma(0) = q_0, \quad \gamma(t_1) = q_1. \tag{2}$$

Sub-Lorentzian geometry

- Vector fields $X_1, \dots, X_k \in \mathrm{Vec}(M)$ form an orthonormal frame for (Δ, g) if

$$\Delta_q = \mathrm{span}(X_1(q), \dots, X_k(q)), \qquad q \in M,$$
$$g_q(X_1, X_1) = -1, \qquad g_q(X_i, X_i) = 1, \quad i = 2, \dots, k,$$
$$g_q(X_i, X_j) = 0, \quad i \neq j.$$

- Assume that time orientation is defined by a timelike vector field $X \in \mathrm{Vec}(M)$ for which $g(X, X_1) < 0$ (e.g., $X = X_1$). Then the SL problem for the SL structure with the orthonormal frame $X_1, \dots, X_k$ is stated as follows:

$$\dot{q} = \sum_{i=1}^{k} u_i X_i(q), \qquad q \in M,$$
$$u \in U = \left\{ (u_1, \dots, u_k) \in \mathbb{R}^k \mid u_1 \geq \sqrt{u_2^2 + \dots + u_k^2} \right\},$$
$$q(0) = q_0, \quad q(t_1) = q_1, \qquad l(q(\cdot)) = \int_0^{t_1} \sqrt{u_1^2 - u_2^2 - \dots - u_k^2}\, dt \to \max.$$

Sub-Lorentzian geometry

- The SL length is preserved under monotone Lipschitzian time reparametrizations $t(s)$, $s \in [0, s_1]$. Thus if $q(t)$, $t \in [0, t_1]$, is a sub-Lorentzian length maximizer, then so is any its reparametrization $q(t(s))$, $s \in [0, s_1]$.
- In this lecture we choose primarily the following parametrization of trajectories: the arclength parametrization ($u_1^2 - u_2^2 - \cdots - u_k^2 \equiv 1$) for timelike trajectories, and the parametrization with $u_1(t) \equiv 1$ for future directed lightlike trajectories.

Statement of the SL problem on the Heisenberg group

- The Heisenberg group is the space $M \simeq \mathbb{R}^3_{x,y,z}$ with the product rule

$$(x_1, y_1, z_1) \cdot (x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2 + (x_1 y_2 - x_2 y_1)/2).$$

- It is a three-dimensional nilpotent Lie group with a left-invariant frame

$$X_1 = \frac{\partial}{\partial x} - \frac{y}{2}\frac{\partial}{\partial z}, \qquad X_2 = \frac{\partial}{\partial y} + \frac{x}{2}\frac{\partial}{\partial z}, \qquad X_3 = \frac{\partial}{\partial z}, \tag{3}$$

with the only nonzero Lie bracket $[X_1, X_2] = X_3$.

- Consider the left-invariant SL problem on the Heisenberg group M defined by the orthonormal frame (X_1, X_2), with the time orientation X_1:

$$\dot{q} = u_1 X_1 + u_2 X_2, \qquad q \in M, \tag{4}$$

$$u \in U = \{(u_1, u_2) \in \mathbb{R}^2 \mid u_1 \geq |u_2|\}, \tag{5}$$

$$q(0) = q_0 = \mathrm{Id} = (0, 0, 0), \quad q(t_1) = q_1, \tag{6}$$

$$l(q(\cdot)) = \int_0^{t_1} \sqrt{u_1^2 - u_2^2}\, dt \to \max. \tag{7}$$

Reduced SL problem on the Heisenberg group

- Reduced sub-Lorentzian problem

$$\dot{q} = u_1 X_1 + u_2 X_2, \qquad q \in M, \tag{8}$$

$$u \in \operatorname{int} U = \{(u_1, u_2) \in \mathbb{R}^2 \mid u_1 > |u_2|\}, \tag{9}$$

$$q(0) = q_0 = \mathrm{Id} = (0,0,0), \quad q(t_1) = q_1, \tag{10}$$

$$l(q(\cdot)) = \int_0^{t_1} \sqrt{u_1^2 - u_2^2}\, dt \to \max. \tag{11}$$

- In the full problem (4)–(7) admissible trajectories $q(\cdot)$ are future directed nonspacelike ones, while in the reduced problem (8)–(11) admissible trajectories $q(\cdot)$ are only future directed timelike ones.
- Passing to arclength-parametrized future directed timelike trajectories:

$$\dot{q} = u_1 X_1 + u_2 X_2, \qquad q \in M, \qquad u_1^2 - u_2^2 = 1, \qquad u_1 > 0, \tag{12}$$

$$q(0) = q_0 = \mathrm{Id} = (0,0,0), \quad q(t_1) = q_1, \tag{13}$$

$$t_1 \to \max. \tag{14}$$

Previously obtained results by M. Grochowski

(1) Sub-Lorentzian extremal trajectories were parametrized by hyperbolic and linear functions: were obtained formulas equivalent to our formulas (17), (18).

(2) It was proved that there exists a domain in M containing $q_0 = \mathrm{Id}$ in its boundary at which the sub-Lorentzian distance $d(q_0, q)$ is smooth.

(3) The attainable sets of the sub-Lorentzian structure from the point $q_0 = \mathrm{Id}$ were computed: the chronological future of the point q_0

$$I^+(q_0) = \{(x, y, z) \in M \mid -x^2 + y^2 + 4|z| < 0, \ x > 0\},$$

and the causal future of the point q_0

$$J^+(q_0) = \{(x, y, z) \in M \mid -x^2 + y^2 + 4|z| \leq 0, \ x \geq 0\}. \qquad (15)$$

In the standard language of control theory, $I^+(q_0)$ is the attainable set of the reduced system (8), (9) from the point q_0 for arbitrary positive time. Thus the attainable set of the reduced system (8), (9) from the point q_0 for arbitrary nonnegative time is

$$\mathcal{A} = I^+(q_0) \cup \{q_0\}.$$

Previously obtained results by M. Grochowski

(3) The attainable set of the full system (4), (5) from the point q_0 for arbitrary nonnegative time is $\mathrm{cl}(\mathcal{A}) = J^+(q_0)$.

(4) The attainable set $\mathcal{A}$ was also computed by H. Abels and E.B. Vinberg, they called its boundary as the Heisenberg beak. See the set $\partial\mathcal{A}$ below, and its views from the y- and z-axes in the next slide.

Views of the Heisenberg beak

Previously obtained results by M. Grochowski

(5) The lower bound of the sub-Lorentzian distance

$$\sqrt{x^2 - y^2 - 4|z|} \le d(q_0, q), \qquad q = (x, y, z) \in J^+(q_0),$$

was proved. It was also noted that an upper bound

$$d(q_0, q) \le C\sqrt{x^2 - y^2 - 4|z|}$$

does not hold for any constant $C \in \mathbb{R}$.

(6) It was proved that there exist non-Hamiltonian maximizers, i.e., maximizers that are not projections of the Hamiltonian vector field $\vec{H}$, $H = \frac{1}{2}(h_2^2 - h_1^2)$, related to the problem.

Pontryagin maximum principle

- Denote points of the cotangent bundle T^*M as λ. Introduce linear on fibers of T^*M Hamiltonians $h_i(\lambda) = \langle \lambda, X_i \rangle$, $i = 1, 2, 3$.
- Define the Hamiltonian of the Pontryagin maximum principle (PMP) for the sub-Lorentzian problem (4)–(7)

$$h_u^\nu(\lambda) = u_1 h_1(\lambda) + u_2 h_2(\lambda) - \nu\sqrt{u_1^2 - u_2^2}, \qquad \lambda \in T^*M, \quad u \in U, \quad \nu \in \mathbb{R}.$$

- It follows from PMP that if $u(t)$, $t \in [0, t_1]$, is an optimal control in problem (4)–(7), and $q(t)$, $t \in [0, t_1]$, is the corresponding optimal trajectory, then there exists a curve $\lambda_\cdot \in \mathrm{Lip}([0, t_1], T^*M)$, $\pi(\lambda_t) = q(t)$, and a number $\nu \in \{0, -1\}$ for which there hold the conditions for a.e. $t \in [0, t_1]$:
  1. the Hamiltonian system $\dot{\lambda}_t = \vec{h}^\nu_{u(t)}(\lambda_t)$,
  2. the maximality condition $h^\nu_{u(t)}(\lambda_t) = \max_{v \in U} h^\nu_v(\lambda_t) \equiv 0$,
  3. the nontriviality condition $(\nu, \lambda_t) \neq (0, 0)$.

Abnormal case

Theorem 1

In the abnormal case $\nu = 0$ there exist $\tau_1, \tau_2 \geq 0$ such that:

(1) $h_3(\lambda_t) \equiv \text{const} > 0$:

$$\begin{aligned} t \in (0, \tau_1) \quad &\Rightarrow \quad h_1(\lambda_t) = h_2(\lambda_t) < 0, \qquad u_1(t) = -u_2(t), \\ t \in (\tau_1, \tau_1 + \tau_2) \quad &\Rightarrow \quad h_1(\lambda_t) = -h_2(\lambda_t) < 0, \qquad u_1(t) = u_2(t). \end{aligned}$$

(2) $h_3(\lambda_t) \equiv \text{const} < 0$:

$$\begin{aligned} t \in (0, \tau_1) \quad &\Rightarrow \quad h_1(\lambda_t) = -h_2(\lambda_t) < 0, \qquad u_1(t) = u_2(t), \\ t \in (\tau_1, \tau_1 + \tau_2) \quad &\Rightarrow \quad h_1(\lambda_t) = h_2(\lambda_t) < 0, \qquad u_1(t) = -u_2(t). \end{aligned}$$

(3) $h_3(\lambda_t) \equiv 0$:

$$\begin{aligned} &(h_1, h_2)(\lambda_t) \equiv \text{const} \neq (0,0), \qquad h_1(\lambda_t) \equiv -|h_2(\lambda_t)|, \\ &u(t) \equiv \text{const}, \qquad u_1(t) \equiv \pm u_2(t), \qquad \pm = -\operatorname{sgn}(h_1 h_2(\lambda_t)). \end{aligned}$$

Normal case

- In the normal case ($\nu = -1$) extremals exist only for $h_1 \le -|h_2|$.
- In the case $h_1 = -|h_2|$ normal controls and extremal trajectories coincide with the abnormal ones.
- And in the domain $\{\lambda \in T^*M \mid h_1 < -|h_2|\}$ extremals are reparametrizations of trajectories of the Hamiltonian vector field $\vec{H}$ with the Hamiltonian $H = \frac{1}{2}(h_2^2 - h_1^2)$.
- In the arclength parametrization, the extremal controls are

$$(u_1, u_2)(t) = (-h_1(\lambda_t), h_2(\lambda_t)), \qquad (16)$$

and the extremals satisfy the Hamiltonian ODE $\dot{\lambda} = \vec{H}(\lambda)$ and belong to the level surface $\{H(\lambda) = \frac{1}{2}\}$, in coordinates:

$$\begin{aligned} \dot{h}_1 &= -h_2 h_3, \qquad \dot{h}_2 = -h_1 h_3, \qquad \dot{h}_3 = 0, \\ \dot{q} &= \cosh\psi\, X_1 + \sinh\psi\, X_2, \\ h_1 &= -\cosh\psi, \qquad h_2 = \sinh\psi, \qquad \psi \in \mathbb{R}. \end{aligned}$$

Parametrization of normal trajectories

- If $h_3 = 0$, then

$$x = t \cosh \psi, \quad y = t \sinh \psi, \quad z = 0. \tag{17}$$

- If $c := h_3 \neq 0$, then

$$x = \frac{\sinh(\psi + ct) - \sinh \psi}{c}, \quad y = \frac{\cosh(\psi + ct) - \cosh \psi}{c}, \quad z = \frac{\sinh(ct) - ct}{2c^2}. \tag{18}$$

Theorem 2

Normal controls and trajectories either coincide with abnormal ones (in the case $h_1(\lambda_t) = -|h_2(\lambda_t)|$), or can be arclength parametrized to get controls (16) *and future directed timelike trajectories* (17) *if $c = 0$, or* (18) *if $c \neq 0$.*
In particular, each normal extremal can be parameterized so that $H(\lambda_t) \equiv \text{const} \in \{0, \frac{1}{2}\}$.

Exponential mapping

- Normal trajectories are either nonstrictly normal (i.e., simultaneously normal and abnormal) in the case $H = 0$, or strictly normal (i.e., normal but not abnormal) in the case $H = \frac{1}{2}$.
- Strictly normal arclength-parametrized trajectories are described by the exponential mapping

$$\mathrm{Exp} : N \to \widetilde{\mathcal{A}}, \qquad (\lambda, t) \mapsto q(t) = \pi \circ e^{t\vec{H}}(\lambda), \tag{19}$$

$$N = C \times \mathbb{R}_+, \qquad \mathbb{R}_+ = (0, +\infty), \qquad C = T^*_{\mathrm{Id}} M \cap H^{-1}\left(\frac{1}{2}\right) \simeq \mathbb{R}^2_{\psi,c},$$

$$\widetilde{\mathcal{A}} = \operatorname{int} \mathcal{A} = I^+(q_0)$$

given explicitly by formulas (17), (18).

Projections of strictly normal trajectories

- Projections of strictly normal (future directed timelike) trajectories to the plane (x, y) are:
  - either rays $y = kx$, $x \geq 0$, $k \in (-1, 1)$ (for $c = 0$),
  - or arcs of hyperbolas with asymptotes $x = \pm y > 0$ (for $c \neq 0$).

Projections of nonstrictly normal trajectories

- Projections of nonstrictly normal trajectories to the plane (x, y) are broken lines with one or two edges parallel to the rays $x = \pm y > 0$.
- Projections of all extremal trajectories (as well as of all admissible trajectories) to the plane (x, y) are contained in the angle $\{(x, y) \in \mathbb{R}^2 \mid x \geq |y|\}$, which is the projection of the attainable set $J^+(q_0)$ to this plane.

Symplectic foliation

- The Hamiltonian $H = \frac{1}{2}(h_2^2 - h_1^2)$ is preserved on each extremal.
- On the other hand, since the problem is left-invariant, the extremals respect the symplectic foliation on the dual of the Heisenberg Lie algebra $T^*_{\mathrm{Id}}M = \{(h_1, h_2, h_3)\}$ consisting of 2-dimensional symplectic leaves $\{h_3 = \text{const} \neq 0\}$ and 0-dimensional leaves $\{h_3 = 0, \ (h_1, h_2) = \text{const}\}$.
- Thus projections of extremals to $T^*_{\mathrm{Id}}M = \{(h_1, h_2, h_3)\}$ belong to intersections of the level surfaces $\left\{H = \text{const} \in \left\{0, \frac{1}{2}\right\}\right\}$ with the symplectic leaves:
  - branches of hyperbolas $h_1^2 - h_2^2 = 1$, $h_1 < 0$, $h_3 \neq 0$,
  - points $(h_1, h_2) = \text{const}$, $H \in \left\{0, \frac{1}{2}\right\}$, $h_1 \leq -|h_2|$, $h_3 = 0$,
  - angles $h_1 = -|h_2|$, $h_3 \neq 0$.

  See figs in the next slide.

Vertical part of the geodesic flow on $T^*_{\mathrm{Id}}M = \{(h_1, h_2, h_3)\}$

Hamiltonian and non-Hamiltonian extremal trajectories

- In the terminology of M.Grochowski, strictly normal extremal trajectories $q(t) = \pi \circ e^{t\vec{H}}(\lambda)$, $\lambda \in C$, are Hamiltonian since they are projections of trajectories of the Hamiltonian vector field $\vec{H}$.
- Nonstrictly normal extremal trajectories given by items (1), (2) of Th. *1* are non-Hamiltonian, e.g., the broken curves

$$\begin{cases} e^{t(X_1+X_2)}, & t \in [0, \tau_1], \\ e^{(t-\tau_1)(X_1-X_2)} \circ e^{\tau_1(X_1+X_2)}, & t \in [\tau_1, \tau_2], \end{cases} \qquad (20)$$

and

$$\begin{cases} e^{t(X_1-X_2)}, & t \in [0, \tau_1], \\ e^{(t-\tau_1)(X_1+X_2)} \circ e^{\tau_1(X_1-X_2)}, & t \in [\tau_1, \tau_2], \end{cases} \qquad (21)$$

for $0 < \tau_1 < \tau_2$.
- Although, each smooth arc of the broken trajectories (20), (21) is a reparametrization of projection of a trajectory of the Hamiltonian vector field $\vec{H}$ contained in a face of the angle $\{(h_1, h_2, h_3) \in T^*_{\mathrm{Id}}M \mid h_1 = -|h_2|\}$.

Inversion of the exponential mapping

Theorem 3

The exponential mapping $\operatorname{Exp} : N \to \widetilde{\mathcal{A}}$ *is a real-analytic diffeomorphism. The inverse mapping* $\operatorname{Exp}^{-1} : \widetilde{\mathcal{A}} \to N$, $(x, y, z) \mapsto (\psi, c, t)$, *is given by the following formulas:*

$$z = 0 \quad \Rightarrow \quad \psi = \operatorname{artanh} \frac{y}{x}, \quad c = 0, \quad t = \sqrt{x^2 - y^2}, \tag{22}$$

$$z \neq 0 \quad \Rightarrow \quad \psi = \operatorname{artanh} \frac{y}{x} - p, \quad c = (\operatorname{sgn} z)\sqrt{\frac{\sinh 2p - 2p}{2z}}, \quad t = \frac{2p}{c}, \tag{23}$$

where $p = \beta\left(\frac{z}{x^2 - y^2}\right)$, *and* $\beta : (-\frac{1}{4}, \frac{1}{4}) \to \mathbb{R}$ *is the inverse function to the diffeomorphism*

$$\alpha : \mathbb{R} \to \left(-\frac{1}{4}, \frac{1}{4}\right), \qquad \alpha(p) = \frac{\sinh 2p - 2p}{8 \sinh^2 p}.$$

See plots of the functions $\alpha(p)$ and $\beta(z)$ in the next slide.

Plots of the functions $\alpha(p)$ and $\beta(z)$

Lagrangian manifolds

- Let M be a smooth manifold, then the cotangent bundle T^*M bears the Liouville 1-form $s = pdq \in \Lambda^1(T^*M)$ and the symplectic 2-form $\sigma = ds = dp \wedge dq \in \Lambda^2(T^*M)$.
- A submanifold $\mathcal{L} \subset T^*M$ is called a Lagrangian manifold if $\dim \mathcal{L} = \dim M$ and $\sigma|_{\mathcal{L}} = 0$.
- Consider an optimal control problem

$$\dot{q} = f(q, u), \qquad q \in M, \quad u \in U,$$
$$q(t_0) = q_0, \qquad q(t_1) = q_1,$$
$$J[q(\cdot)] = \int_{t_0}^{t_1} \varphi(q, u)\, dt \to \min, \qquad t_0 \text{ is fixed}, \quad t_1 \text{ is free}.$$

- Let $g_u(\lambda) = \langle \lambda, f(q, u) \rangle - \varphi(q, u)$, $\lambda \in T^*M$, $q = \pi(\lambda)$, $u \in U$, be the normal Hamiltonian of PMP.
- Suppose that the maximized normal Hamiltonian $G(\lambda) = \max_{u \in U} g_u(\lambda)$ is smooth in an open domain $O \subset T^*M$, and let the v. field $\vec{G} \in \mathrm{Vec}(O)$ be complete.

Sufficient optimality condition

Theorem 4

- *Let* $\mathcal{L} \subset G^{-1}(0) \cap O$ *be a Lagrangian submanifold such that the form* $s|_{\mathcal{L}}$ *is exact.*
- *Let the projection* $\pi : \mathcal{L} \to \pi(\mathcal{L})$ *be a diffeomorphism on a domain in* M.
- *Consider an extremal* $\widetilde{\lambda}_t = e^{t\vec{G}}(\lambda_0)$, $t \in [t_0, t_1]$, *contained in* $\mathcal{L}$, *and the corresponding extremal trajectory* $\widetilde{q}(t) = \pi(\widetilde{\lambda}_t)$.
- *Consider also any other trajectory* $q(t) \in \pi(\mathcal{L})$, $t \in [t_0, \tau]$, *such that* $q(t_0) = \widetilde{q}(t_0)$, $q(\tau) = \widetilde{q}(t_1)$.
- *Then* $J[\widetilde{q}(\cdot)] < J[q(\cdot)]$.

Optimality in the reduced SL problem

- For the reduced SL problem the maximized Hamiltonian $G = 1 - \sqrt{h_1^2 - h_2^2}$ is smooth on the domain $O = \{\lambda \in T^*M \mid h_1 < -|h_2|\}$, and the Hamiltonian vector field $\vec{G} \in \mathrm{Vec}(O)$ is complete
- In the domain O the Hamiltonian vector fields $\vec{G}$ and $\vec{H}$ have the same trajectories up to a monotone time reparametrization; moreover, on the level surface $\left\{H = \frac{1}{2}\right\} = \{G = 0\}$ they just coincide between themselves.
- Define the set

$$\mathcal{L} = \left\{ e^{t\vec{G}}(\lambda_0) \mid \lambda_0 \in C,\ t > 0 \right\}. \tag{24}$$

Lemma 5

$\mathcal{L} \subset T^*M$ is a Lagrangian manifold such that $s|_{\mathcal{L}}$ is exact.

Theorem 6

For any point $q_1 \in \operatorname{int} \mathcal{A} = I^+(q_0)$ the strictly normal trajectory $q(t) = \mathrm{Exp}(\lambda, t)$, $t \in [0, t_1]$, is the unique optimal trajectory of the reduced SL problem connecting q_0 with q_1, where $(\lambda, t_1) = \mathrm{Exp}^{-1}(q_1) \in N$.

The cost function for the equivalent reduced SL problem

Denote

$$
\begin{aligned}
\widetilde{d}(q_1) &= \sup\{ l(q(\cdot)) \mid \text{ traj. } q(\cdot) \text{ of (8)–(11)}, \; q(0) = q_0, \; q(t_1) = q_1 \} \\
&= \sup\{ t_1 > 0 \mid \exists \text{ traj. } q(\cdot) \text{ of (12)–(14) s.t. } q(0) = q_0, \; q(t_1) = q_1 \},
\end{aligned}
$$

where $q_1 \in \operatorname{int} \mathcal{A} = I^+(q_0)$.

Theorem 7

Let $q = (x, y, z) \in I^+(q_0)$. Then

$$
\widetilde{d}(q) = \sqrt{x^2 - y^2} \cdot \frac{p}{\sinh p}, \qquad p = \beta\left(\frac{z}{x^2 - y^2}\right). \tag{25}
$$

The function $\widetilde{d} : I^+(q_0) \to \mathbb{R}_+$ is real-analytic.

Optimality in the full SL problem

Theorem 8

Let $q_1 \in \operatorname{int} A = I^+(q_0)$. Then the SL length maximizers for the full problem are reparametrizations of the corresponding SL length maximizers for the reduced problem described above.

In particular, $d|_{I^+(q_0)} = \widetilde{d}$.

Theorem 9

Let $q_1 = (x_1, y_1, z_1) \in \partial A = J^+(q_0) \setminus I^+(q_0)$, $q_1 \neq q_0$. Then an optimal trajectory in the full SL problem is a future directed lightlike piecewise smooth trajectory with one or two subarcs generated by the vector fields $X_1 \pm X_2$.

Length maximizers in the full SL problem

Corollary 10

For any $q_1 \in J^+(q_0)$, $q_1 \neq q_0$, *there is a unique, up to reparametrization, SL length minimizer in the full problem that connects* q_0 *and* q_1*:*

- *if* $q_1 \in \operatorname{int} \mathcal{A} = I^+(q_0)$, *then* $q(\cdot)$ *is a future directed timelike strictly normal trajectory.*
- *if* $q_1 \in \partial \mathcal{A} = J^+(q) \setminus I^+(q_0)$, *then* $q(\cdot)$ *is a future directed lightlike nonstrictly normal trajectory.*

Corollary 11

Any SL length maximizer of the full problem problem of positive length is timelike and strictly normal.

- The broken trajectories described above are optimal in the SL problem, while in SR problems trajectories with angle points cannot be optimal.
- Moreover, these broken trajectories are normal and nonsmooth, which is also impossible in SR geometry.

Denote $d(q) := d(q_0, q)$, $q \in J^+(q_0)$.

Theorem 12

Let $q = (x, y, z) \in J^+(q_0)$. Then

$$d(q) = \sqrt{x^2 - y^2} \cdot \frac{p}{\sinh p}, \qquad p = \beta\left(\frac{z}{x^2 - y^2}\right). \tag{26}$$

In particular:

(1) $z = 0 \quad \Leftrightarrow \quad d(q) = \sqrt{x^2 - y^2}$,

(2) $q \in J^+(q_0) \setminus I^+(q_0) \quad \Leftrightarrow \quad d(q) = 0$.

Plot of $d|_{z=0} = \sqrt{x^2 - y^2}$

Plot of $d|_{y=0}$

Plot of $d|_{x=1}$

Regularity of the sub-Lorentzian distance

Theorem 13

(1) *The function* $d(\cdot)$ *is continuous on* $J^+(q_0)$ *and real-analytic on* $I^+(q_0)$.

(2) *The function* $d(\cdot)$ *is not Lipschitz near points* $q = (x, y, z)$ *with* $x = |y| > 0$, $z = 0$.

Remark 1

The sub-Lorentzian distance $d : J^+(q_0) \to [0, +\infty)$ is not uniformly continuous since the same holds for its restriction $d|_{z=0} = \sqrt{x^2 - y^2}$ on the angle $\{x \geq |y|\}$.

Bounds of the sub-Lorentzian distance

Theorem 14

(1) *The ratio* $\dfrac{\sqrt{x^2 - y^2 - 4|z|}}{d(q)}$ *takes any values in the segment* $[0, 1]$ *for* $q = (x, y, z) \in J^+(q_0)$.

(2) *For any* $q = (x, y, z) \in J^+(q_0)$ *there holds the bound* $d(q) \leq \sqrt{x^2 - y^2}$, *moreover, the ratio* $\dfrac{d(q)}{\sqrt{x^2 - y^2}}$ *takes any values in the segment* $[0, 1]$.

Bounds of the sub-Lorentzian distance

The two-sided bound

$$\sqrt{x^2 - y^2 - 4|z|} \leq d(q) \leq \sqrt{x^2 - y^2}, \qquad q \in J^+(q_0), \tag{27}$$

is visualized in figure below, which shows plots of the surfaces (from below to top):

$$\sqrt{x^2 - y^2} = 1, \qquad d(q) = 1, \qquad \sqrt{x^2 - y^2 - 4|z|} = 1, \qquad q \in J^+(q_0).$$

Theorem 15

(1) *The hyperbolic rotations* $X_0 = y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$ *and reflections* $\varepsilon^1 : (x, y, z) \mapsto (x, -y, z)$, $\varepsilon^2 : (x, y, z) \mapsto (x, y, -z)$ *preserve* $d(\cdot)$.

(2) *The dilations* $Y = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + 2z\frac{\partial}{\partial z}$ *stretch* $d(\cdot)$:

$$d(e^{sY}(q)) = e^s d(q), \qquad s \in \mathbb{R}, \quad q \in J^+(q_0).$$

The unit sub-Lorentzian sphere

$$S = \{\mathrm{Exp}(\lambda, 1) \mid \lambda \in C\}$$

Theorem 16

(1) *The unit SL sphere S is a regular real-analytic manifold diffeomorphic to $\mathbb{R}^2$.*

(2) *Let $q = \mathrm{Exp}(\psi, c, 1) \in S$, $(\psi, c) \in C$, then the tangent space*

$$T_q S = \left\{ v = \sum_{i=1}^{3} v_i X_i(q) \mid -v_1 \cosh(\psi + c) + v_2 \sinh(\psi + c) + v_3 c = 0 \right\}. \quad (28)$$

(3) *S is the graph of the function $x = \sqrt{y^2 + f(z)}$, where $f(z) = e \circ k(z)$, $e(w) = \frac{\sinh^2 w}{w^2}$, $k(z) = b(z)/2$, $b = a^{-1}$, $a(c) = \frac{\sinh c - c}{2c^2}$.*

(4) *The function $f(z)$ is real-analytic, even, strictly convex, unboundedly and strictly increasing for $z \geq 0$. This function has a Taylor decomposition $f(z) = 1 + 12z^2 + O(z^4)$ as $z \to 0$ and an asymptote $4|z|$ as $z \to \infty$.*

The unit sub-Lorentzian sphere

(5) The function $f(z)$ satisfies the bounds

$$4|z| < f(z) < 4|z| + 1, \qquad z \neq 0. \tag{29}$$

(6) A section of the sphere S by a plane $\{z = \text{const}\}$ is a branch of the hyperbola $x^2 - y^2 = f(z)$, $x > 0$. A section of the sphere S by a plane $\{x = \text{const} > 1\}$ is a strictly convex curve $y^2 + f(z) = x^2$ diffeomorphic to S^1.

(7) The sub-Lorentzian distance from the point q_0 to a point $q = (x, y, z) \in \widetilde{\mathcal{A}}$ may be expressed as $d(q) = R$, where $x^2 - y^2 = R^2 f(z/R^2)$.

(8) The sub-Lorentzian ball $B = \{q \in M \mid d(q) \leq 1\}$ has infinite volume in the coordinates x, y, z.

The unit sub-Lorentzian sphere

The unit sub-Lorentzian sphere

Sub-Lorentzian sphere of zero radius

$$S(0) = \{q \in M \mid d(q) = 0\}.$$

Theorem 17

(1) $S(0) = J^+(q_0) \setminus I^+(q_0) = \partial J^+(q_0) = \partial I^+(q_0) = \partial \mathcal{A}$.

(2) $S(0)$ *is the graph of a continuous function* $x = \Phi(y, z) := \sqrt{y^2 + 4|z|}$, *thus a 2-dimensional topological manifold.*

(3) *The function* $\Phi(y, z)$ *is even in* y *and* z, *real-analytic for* $z \neq 0$, *Lipschitz near* $z = 0$, $y \neq 0$, *and Hölder with constant* $\frac{1}{2}$, *non-Lipschitz near* $(y, z) = (0, 0)$.

(4) $S(0)$ *is filled by broken lightlike trajectories with one or two edges, and is parametrized by them as follows:*

$$S(0) = \left\{ e^{\tau_2(X_1 - X_2)} e^{\tau_1(X_1 + X_2)} = (\tau_1 + \tau_2, \tau_1 - \tau_2, -\tau_1\tau_2) \mid \tau_i \geq 0 \right\} \cup \left\{ e^{\tau_2(X_1 + X_2)} e^{\tau_1(X_1 - X_2)} = (\tau_1 + \tau_2, \tau_2 - \tau_1, \tau_1\tau_2) \mid \tau_i \geq 0 \right\}.$$

Sub-Lorentzian sphere of zero radius

(5) The flows of the vector fields Y, X_0 preserve $S(0)$. Moreover, the symmetries Y, X_0 provide a regular parametrization of

$$S(0) \cap \{\operatorname{sgn} z = \pm 1\} = \left\{ e^{sY} \circ e^{rX_0}(q_\pm) \mid r, s > 0 \right\}, \tag{30}$$

where $q_\pm = (x_\pm, y_\pm, z_\pm)$ is any point in $S(0) \cap \{\operatorname{sgn} z = \pm 1\}$.

(6) $S(0) = \left\{ 16z^2 = (x^2 - y^2)^2, \ x^2 - y^2 \geq 0, \ x \geq 0 \right\}$ is a semi-algebraic set.

(7) The zero-radius sphere is a Whitney stratified set with the stratification

$$\begin{aligned} S(0) = {} & \big(S(0) \cap \{z > 0\}\big) \cup \big(S(0) \cap \{z < 0\}\big) \\ & \cup \big(S(0) \cap \{z = 0, \ y > 0\}\big) \cup \big(S(0) \cap \{z = 0, \ y < 0\}\big) \cup \{q_0\}. \end{aligned}$$

(8) Intersection of the sphere $S(0)$ with a plane $\{z = \text{const} \neq 0\}$ is a branch of a hyperbola $\{x^2 - y^2 = 4|z|, \ x > 0, z = \text{const}\}$, intersection with a plane $\{z = 0\}$ is an angle $\{x = |y|, z = 0\}$, intersection with a plane $\{y = kx\}$, $k \in (-1, 1)$, is a union of two half-parabolas $\{4z = \pm(1 - k^2)x^2, \ x \geq 0, \ y = kx\}$, and intersection with a plane $\{y = \pm x\}$ is a ray $\{y = \pm x, \ z = 0\}$.

Conclusion

The results described in this lecture for the SL problem on the Heisenberg group differ drastically from the known results for the SR problem on the same group:

1. The SL problem is not completely controllable.
2. Filippov's existence theorem for optimal controls cannot be immediately applied to the SL problem.
3. In the SL problem all extremal trajectories are infinitely optimal, thus the cut locus and the conjugate locus for them are empty.
4. The SL length maximizers coming to the zero-radius sphere are nonsmooth (concatenations of two smooth arcs forming a corner, nonstrictly normal extremal trajectories).
5. SL spheres and SL distance are real-analytic if $d > 0$.

It would be interesting to understand which of these properties persist for more general SL problems (e.g., for left-invariant problems on Carnot groups).