Pontryagin maximum principle and its applications

(Lecture 8)

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Plan of previous lecture

1. Proof of the geometric statement of PMP with fixed terminal time
2. Geometric statement of PMP for free time
3. PMP for optimal control problems
4. PMP with transversality conditions

Plan of this lecture

1. PMP with mixed boundary conditions
2. Sub-Riemannian geometry
3. Dido's problem
4. Sub-Riemannian problem on the group of Euclidean motions of a plane

Pontryagin Maximum Principle with transversality conditions

Theorem 1

Let $\widetilde{u}(t)$, $t \in [0, t_1]$, *be an optimal control in the problem*

$$\dot{q} = f_u(q), \qquad q \in M, \quad u \in U \subset \mathbb{R}^m, \tag{1}$$

$$q(0) \in N_0, \qquad q(t_1) \in N_1, \tag{2}$$

$$t_1 > 0 \text{ fixed}, \tag{3}$$

$$J(u) = \int_0^{t_1} \varphi(q(t), u(t))\, dt \to \min . \tag{4}$$

Define a family of Hamiltonians:

$$h_u^\nu(\lambda) = \langle \lambda, f_u(q) \rangle + \nu \varphi(q, u), \qquad \lambda \in T_q^* M,\ q \in M, \quad \nu \in \mathbb{R},\ u \in U.$$

Then there exists a Lipschitzian curve $\lambda_t \in T^*_{\widetilde{q}(t)} M$, $t \in [0, t_1]$, *and a number* $\nu \in \mathbb{R}$ *such that:*

$$\dot{\lambda}_t = \overrightarrow{h^{\nu}_{\widetilde{u}(t)}}(\lambda_t), \tag{5}$$

$$h^{\nu}_{\widetilde{u}(t)}(\lambda_t) = \max_{u \in U} h^{\nu}_u(\lambda_t), \tag{6}$$

$$(\lambda_t, \nu) \neq (0, 0), \qquad t \in [0, t_1], \tag{7}$$

$$\nu \leq 0, \tag{8}$$

$$\lambda_0 \perp T_{\widetilde{q}(0)} N_0, \qquad \lambda_{t_1} \perp T_{\widetilde{q}(t_1)} N_1. \tag{9}$$

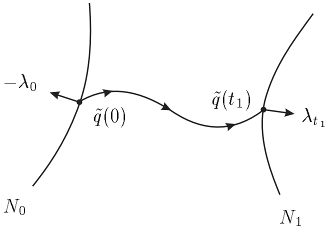


Figure: Transversality conditions (9)

Optimal control problem with mixed boundary conditions

- Consider an optimal control problem of the form:

$$\dot{q} = f_u(q), \qquad q \in M, \quad u \in U \subset \mathbb{R}^m, \tag{10}$$

$$(q(0), q(t_1)) \in N \subset M \times M, \tag{11}$$

$$t_1 > 0 \text{ fixed}, \tag{12}$$

$$J(u) = \int_0^{t_1} \varphi(q(t), u(t))\, dt \to \min, \tag{13}$$

where N is a smooth immersed submanifold of $M \times M$.

Theorem 2

Let $\widetilde{u}$ be an optimal control in problem (10)–(13). Then there hold all statements of Theorem 1 except its transversality condition (9), which is replaced now by the relation

$$(-\lambda_0, \lambda_{t_1}) \perp T_{(\widetilde{q}(0), \widetilde{q}(t_1))} N. \tag{14}$$

Remarks

(1) We identify

$$T^*_{(q_0,q_1)}(M \times M) \cong T^*_{q_0}M \oplus T^*_{q_1}M,$$

so the transversality condition (14) makes sense.

(2) An important particular case of mixed boundary conditions (11) is the case of periodic trajectories:

$$q(t_1) = q(0). \tag{15}$$

Indeed, then

$$N = \Delta \stackrel{\text{def}}{=} \{(q,q) \mid q \in M\} \subset M \times M, \tag{16}$$

the diagonal of the product $M \times M$. In this case the transversality condition (14) reads

$$\langle(-\lambda_0, \lambda_{t_1}), (v, v)\rangle = -\langle\lambda_0, v\rangle + \langle\lambda_{t_1}, v\rangle = 0, \qquad v \in T_{q(0)}M = T_{q(t_1)}M,$$

i.e., $\lambda_0 = \lambda_{t_1}$. That is, an optimal trajectory in the problem with periodic boundary conditions (15) possesses a periodic Hamiltonian lift (extremal).

Proof of Theorem 2.

- We reduce our problem to the case of separated boundary conditions by introducing an auxiliary problem on $M \times M$:

$$\begin{cases} \dot{x} = 0, \\ \dot{q} = f_u(q), \end{cases} \qquad (x, q) \in M \times M, \quad u \in U,$$

$$(x(0), q(0)) \in \Delta, \qquad (x(t_1), q(t_1)) \in N,$$

(the diagonal Δ is defined in (16) above)

$$J(u) = \int_0^{t_1} \varphi(q(t), u(t))\, dt \to \min .$$

- It is obvious that this problem is equivalent to our problem (10)–(13).
- We apply a version of PMP (Theorem 1) to the auxiliary problem.
- The Hamiltonian is the same as for the initial problem:

$$h_u^\nu(\eta, \lambda) = h_u^\nu(\lambda) = \langle \lambda, f_u(q) \rangle + \nu\varphi(q, u), \qquad (\eta, \lambda) \in T^*M \oplus T^*M.$$

- The corresponding Hamiltonian system is

$$\begin{cases} \dot{\eta}_t = 0, \\ \dot{\lambda}_t = \overrightarrow{h^{\nu}_{\widetilde{u}(t)}}(\lambda_t). \end{cases} \tag{17}$$

- All required statements of PMP obviously follow, we should only check transversality conditions.
- At the initial instant $t = 0$ the first of conditions (9) reads:

$$\langle(\eta_0, \lambda_0), (v, v)\rangle = \langle \eta_0, v\rangle + \langle \lambda_0, v\rangle = 0, \qquad v \in T_{\widetilde{q}(0)}M,$$

i.e., $\eta_0 + \lambda_0 = 0$, or, taking into account the first of equations (17), $\eta_{t_1} = -\lambda_0$.
- And at the terminal instant $t = t_1$:

$$(\eta_{t_1}, \lambda_{t_1}) \perp T_{(\tilde{x}(t_1), \widetilde{q}(t_1))}N,$$

that is,

$$(-\lambda_0, \lambda_{t_1}) \perp T_{(\widetilde{q}(0), \widetilde{q}(t_1))}N,$$

which is the required transversality condition (14). □

Remarks

(1) Needless to say, if the terminal time t_1 is free, then one should add to statements of Theorem 2 the additional equality $h^{\nu}_{\widetilde{u}(t)}(\lambda_t) \equiv 0$.

(2) Pontryagin Maximum Principle withstands further generalizations to wider classes of cost functionals and boundary conditions. After certain modifications of argument, the general scheme provides necessary optimality conditions for more general problems.

Sub-Riemannian geometry

- smooth manifold M,
- distribution $\Delta = \{\Delta_q \subset T_qM \mid q \in M\}$, $\dim \Delta_q \equiv \text{const}$,
- scalar product in Δ:

$$g = \{g_q \text{ — scalar product in } \Delta_q \mid q \in M\}$$

- SR manifold (M, Δ, g), SR structure (Δ, g) on M
- horizontal (admissible) curve $q \in \text{Lip}([0, t_1], M)$:

$$\dot{q}(t) \in \Delta_{q(t)} \text{ for a.e. } t \in [0, t_1],$$

- length $l(q(\cdot)) = \int_0^{t_1} (g(\dot{q}(t), \dot{q}(t))^{1/2}\, dt$,
- SR distance $d(q_0, q_1) = \inf\{l(q(\cdot)) \mid q(\cdot)$ horizontal curve,

$$q(0) = q_0,\ q(t_1) = q_1\},$$

- SR length minimizer $q(t)$, $t \in [0, t_1]$: horizontal curve s.t. $l(q(\cdot)) = d(q(0), q(t_1))$,
- sphere $S_R(q_0) = \{q \in M \mid d(q, q_0) = R\}$,
  ball $B_R(q_0) = \{q \in M \mid d(q, q_0) \leq R\}$,
- geodesic: a horizontal curve whose short arcs are minimizers,
- cut time along the geodesic $q(t)$:

$$t_{\text{cut}}(q(\cdot)) = \sup\{t > 0 \mid q(s),\ s \in [0, t],\ \text{minimizer}\ \},$$

- cut point $q(t_1)$, $t_1 = t_{\text{cut}}(q(\cdot))$,
- cut locus $\text{Cut}_{q_0} = \{q_1 \in M \mid q_1 \text{ cut point for some geodes. } q(\cdot),\ \ q(0) = q_0\}$

- first conjugate time along the geodesic $q(t)$:

$$t^1_{\text{conj}}(q(\cdot)) = \sup\{t > 0 \mid q(s),\ s \in [0, t],\ \text{locally optimal}\ \},$$

- $q(\cdot)$ is locally optimal if $\exists$ neighborhood $O \supset \{q(t)\}$ incl. $q(\cdot)$ is a minimizer on $(O, \Delta|_O, g|_O)$,
- first conjugate point along the geodesic $q(t)$: $q(t_1)$, $t_1 = t^1_{\text{conj}}(q(\cdot))$,
- first caustic:

$$\text{Conj}_{q_0} = \{q_1 \in M \mid\ q_1 \text{ first conjugate pt. for a geodesic } q(\cdot), \quad q(0) = q_0\}.$$

Optimal control problem

- SR manifold (M, Δ, g)
- Orthonormal frame:

$$\Delta_q = \operatorname{span}(X_1(q), \ldots, X_k(q)), \quad g(X_i, X_j) = \delta_{ij}, \; i, j = 1, \ldots, k,$$

- Minimizers $q(t)$ are solutions to the problem

$$\dot{q} = \sum_{i=1}^{k} u_i X_i(q), \qquad q \in M, \quad u_i \in \mathbb{R},$$

$$q(0) = q_0, \qquad q(t_1) = q_1,$$

$$l = \int_0^{t_1} \left(\sum_{i=1}^{k} u_i^2(t) \right)^{1/2} dt \to \min$$

$$\Leftrightarrow \quad J = \frac{1}{2} \int_0^{t_1} \sum_{i=1}^{k} u_i^2(t)\, dt \to \min .$$

Theorem 1 (Rashevskii-Chow)

Let M be connected and for all $q \in M$

$$\mathrm{span}(X_i(q),\ [X_i, X_j](q),\ [[X_i, X_j], X_l](q),\ \dots) = T_q M. \tag{18}$$

Then for $\forall\ q_0, q_1 \in M$ $\exists$ the horizontal curve $q(t)$, $t \in [0, t_1]$, incl. $q(0) = q_0$, $q(t_1) = q_1$.

In what follows, the full rank condition (18) is assumed to be satisfied.

Theorem 2 (Filippov)

A shortest path connecting the points $q_0, q_1 \in M$ exists if one of the conditions is met:

- *q_1 is close enough to q_0,*
- *the balls $B_R(q_0)$ are compact,*
- *(Δ, g) is left invariant on the Lie group M.*

Pontryagin maximum principle

- $h_i(\lambda) = \langle \lambda, X_i(q) \rangle$, $\lambda \in T^*M$.

Theorem 3 (Pontryagin)

If $q(t)$, $t \in [0, t_1]$, is the length minimizer corresponding to control $u(t)$, then $\exists \lambda \in \mathrm{Lip}([0, t_1], T^*M)$, $\lambda(t) \in T^*_{q(t)}M$, such that:

(1) *either* $\dot\lambda(t) = \vec{H}(\lambda(t))$, $H(\lambda) = \frac{1}{2}\sum_{i=1}^k h_i^2(\lambda)$, $u_i(t) = h_i(\lambda(t))$,

(2) *or* $h_1(\lambda(t)) = \cdots = h_k(\lambda(t)) \equiv 0$, $\dot\lambda(t) = \sum_{i=1}^k u_i(t)\vec{h}_i(\lambda(t))$.

(1) $\Rightarrow$ $\lambda(t)$ normal extremal, $q(t)$ normal extremal trajectory,

(2) $\Rightarrow$ $\lambda(t)$ is an abnormal extremal, $q(t)$ is an abnormal extremal trajectory.

Optimality of normal geodesics

- $q(t)$ — normal extremal trajectory $\Rightarrow$
  $q(t)$ — geodesic (strengthened Legendre condition)

- $\lambda(t)$ — normal extremal $\Rightarrow$
  $\lambda(t) = e^{t\vec{H}}(\lambda_0), \quad H(\lambda(t)) \equiv \text{const}$

- $\lambda_0 \in C = \{H(\lambda) \equiv 1/2\} \cap T^*_{q_0}M$

- Exponential mapping $\text{Exp} : C \times \mathbb{R}_+ \to M$, $\text{Exp}(\lambda, t) = q(t) = \pi \circ e^{t\vec{H}}(\ lam)$.

- q_1 — Maxwell point on the geodesic $q(t)$:
  $\exists \widetilde{q}(t) \not\equiv q(t)$, $\widetilde{q}(0) = q(0)$, $\widetilde{q}(t_1) = q(t_1) = q_1$.

Theorem 4

Let $q(t) = \text{Exp}(\lambda, t)$ be a normal geodesic that does not contain abnormal segments. If t_1 is the cut time, then $q(t_1)$ is the first Maxwell point or the first conjugate point.

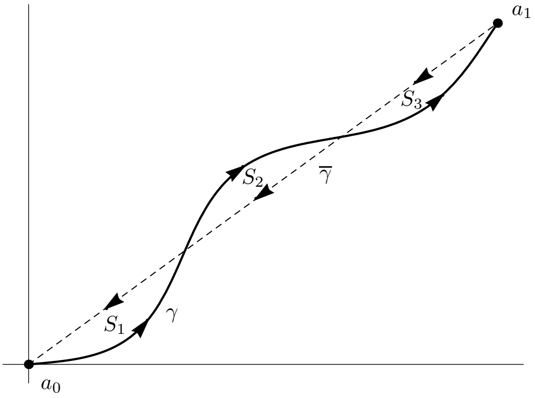
Dido's problem

Given:

- points $a_0, a_1 \in \mathbb{R}^2$
- Lipschitzian curve $\overline{\gamma} \subset \mathbb{R}^2$ connecting a_1 with a_0
- number $S \in \mathbb{R}$.

Find:

- the shortest Lipschitzian curve $\gamma \subset \mathbb{R}^2$ connecting a_0 with a_1 for which the closed curve $\gamma \cup \overline{\gamma}$ bounds a domain in $\mathbb{R}^2$ of the algebraic area S.



Dido's problem

- coordinates x, y in the plane $\mathbb{R}^2$ with the origin a_0. Then $a_0 = (0,0)$, $a_1 = (x_1, y_1)$, $\gamma(t) = (x(t), y(t))$, $t \in [0, t_1]$, $\overline{\gamma}(t) = (\overline{x}(t), \overline{y}(t))$, $t \in [0, \overline{t}_1]$.
- closed curve $\widehat{\gamma} = \gamma \cup \overline{\gamma}$ and a domain bounded by it: $D \subset \mathbb{R}^2$, $\partial D = \widehat{\gamma}$

$$S(D) = \frac{1}{2}\oint_{\widehat{\gamma}} x\,dy - y\,dx = \frac{1}{2}\int_0^{t_1}(x\dot{y} - y\dot{x})\,dt - \overline{I},$$

$$\dot{x}(t) =: u_1(t), \qquad \dot{y}(t) =: u_2(t), \qquad \dot{z}(t) = \frac{1}{2}(xu_2 - yu_1).$$

$$\dot{q} = u_1 X_1(q) + u_2 X_2(q), \qquad u = (u_1, u_2) \in \mathbb{R}^2, \quad q = (x, y, z) \in \mathbb{R}^3,$$

$$X_1 = \frac{\partial}{\partial x} - \frac{y}{2}\frac{\partial}{\partial z}, \qquad X_2 = \frac{\partial}{\partial y} + \frac{x}{2}\frac{\partial}{\partial z},$$

$$q(0) = q_0 = (0,0,0), \qquad q(t_1) = q_1 = (x_1, y_1, z_1),$$

$$l(\gamma) = \int_0^{t_1}\sqrt{\dot{x}^2 + \dot{y}^2}\,dt = \int_0^{t_1}\sqrt{u_1^2 + u_2^2}\,dt \to \min.$$

Dido's problem is stated as the following optimal control problem:

$$\dot{q} = u_1 X_1(q) + u_2 X_2(q), \qquad q \in M = \mathbb{R}^3_{x,y,z}, \quad u = (u_1, u_2) \in \mathbb{R}^2,$$
$$q(0) = q_0 = (0,0,0), \qquad q(t_1) = q_1,$$
$$J = \frac{1}{2}\int_0^{t_1} (u_1^2 + u_2^2)\, dt \to \min,$$
$$X_1 = \frac{\partial}{\partial x} - \frac{y}{2}\frac{\partial}{\partial z}, \quad X_2 = \frac{\partial}{\partial y} + \frac{x}{2}\frac{\partial}{\partial z}.$$

- *Existence of solutions*.
- We have $[X_1, X_2] = X_3 = \frac{\partial}{\partial z}$. The system is symmetric and full-rank, thus it is completely controllable (Rashevskii-Chow Theorem).
- The right-hand side satisfies the bound

$$|u_1 X_1(q) + u_2 X_2(q)| \leq C(1 + |q|), \qquad q \in M, \quad u_1^2 + u_2^2 \leq 1.$$

Thus the Filippov theorem gives existence of optimal controls.

- *Geodesics*.
- Introduce linear on fibers of T^*M Hamiltonians:

$$h_i(\lambda) = \langle \lambda, X_i \rangle, \qquad i = 1, 2, 3, \quad \lambda \in T^*M.$$

- *Abnormal extremals* satisfy the Hamiltonian system $\dot{\lambda} = u_1 \vec{h}_1(\lambda) + u_2 \vec{h}_2(\lambda)$, in coordinates:

$$\begin{aligned} \dot{h}_1 &= -u_2 h_3, \\ \dot{h}_2 &= u_1 h_3, \\ \dot{h}_3 &= 0, \\ \dot{q} &= u_1 X_1 + u_2 X_2, \end{aligned}$$

plus the identities

$$h_1(\lambda_t) = h_2(\lambda_t) \equiv 0.$$

Thus $h_3(\lambda_t) \neq 0$, and the first two equations of the Hamiltonian system yield $u_1(t) = u_2(t) \equiv 0$. So abnormal trajectories are constant.

- *Normal extremals* satisfy the Hamiltonian system $\dot{\lambda} = \vec{H}(\lambda)$ with the Hamiltonian $H = \frac{1}{2}(h_1^2 + h_2^2)$, in coordinates:

$$\dot{h}_1 = -h_2 h_3, \tag{19}$$

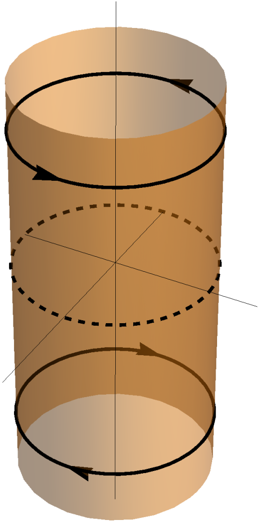
$$\dot{h}_2 = h_1 h_3, \tag{20}$$

$$\dot{h}_3 = 0, \tag{21}$$

$$\dot{q} = h_1 X_1 + h_2 X_2. \tag{22}$$

- The subsystem of the Hamiltonian system for the adjoint variables h_1, h_2, h_3 (the *vertical subsystem*) (19)–(21) has integrals H and h_3. Moreover, in the plane $\{h_3 = 0\}$ the vertical subsystem stays fixed. Thus at the level surface $\{H = 1/2\}$ it has the flow shown in the next slide: rotations in the circles $\{H = 1/2, \ h_3 = \text{const} \neq 0\}$ and fixed points in the circle $\{H = 1/2, \ h_3 = 0\}$.

The flow of the vertical subsystem of the Hamiltonian system of PMP



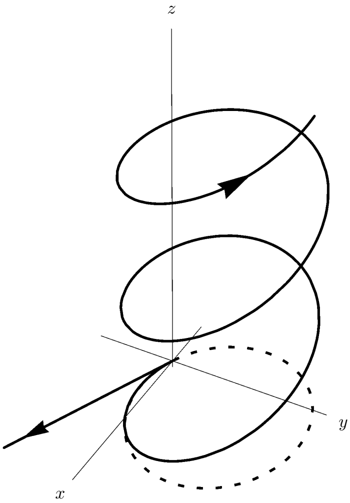
- On the level surface $\{H = \frac{1}{2}\}$, we introduce the polar coordinate θ:

$$h_1 = \cos\theta, \quad h_2 = \sin\theta.$$

Arclength parametrized minimizers satisfy the normal Hamiltonian system

$$\begin{aligned}
\dot{\theta} &= h_3, \\
\dot{h}_3 &= 0, \\
\dot{x} &= \cos\theta, \\
\dot{y} &= \sin\theta, \\
\dot{z} &= -\frac{y}{2}\cos\theta + \frac{x}{2}\sin\theta, \\
(x, y, z)(0) &= (0, 0, 0).
\end{aligned}$$

Geodesics

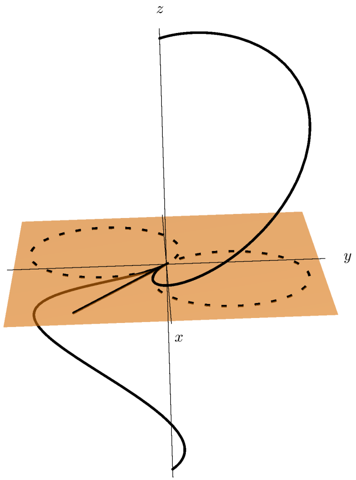


Optimality of geodesics

- Straight lines (case $h_3 = 0$) minimize the Euclidean distance in $\mathbb{R}^2_{x,y}$, thus they are optimal on any segment $t \in [0, t_1]$, $t_1 > 0$.
- Helices (case $h_3 \neq 0$) are not optimal after the first intersection with the z-axis at $t = \frac{2\pi}{|h_3|}$ since these intersections are Maxwell points.
- If $t_1 = \frac{2\pi}{|h_3|}$, then there is a continuum of helices $q(t)$, $t \in [0, t_1]$, coming to the same point $q(t_1)$ at the z-axis; they are obtained one from another by rotations around this axis, thus they all are optimal.
- A part of an optimal arc is optimal, thus the helices are optimal also for $t \in [0, t_1]$, $t_1 \in (0, \frac{2\pi}{|h_3|})$.
- Summing up, the cut time along a geodesic $\mathrm{Exp}(\lambda, t)$ is

$$t_{\mathrm{cut}}(\lambda) = \begin{cases} \frac{2\pi}{|h_3|} & \text{for } h_3 \neq 0, \\ +\infty & \text{for } h_3 = 0. \end{cases} \tag{23}$$

Optimal geodesics



Cut locus and caustic

In Dido's problem the *cut locus*

$$\mathrm{Cut} = \{\mathrm{Exp}(\lambda, t_{\mathrm{cut}}(\lambda)) \mid \lambda \in C\}$$

and the first *caustic*

$$\mathrm{Conj}^1 = \left\{\mathrm{Exp}(\lambda, t^1_{\mathrm{conj}}(\lambda)) \mid \lambda \in C\right\}$$

coincide one with another:

$$\mathrm{Cut} = \mathrm{Conj}^1 = \{(0, 0, z) \in \mathbb{R}^3 \mid z \neq 0\}.$$

Sub-Riemannian distance

Let us describe the *SR distance* $d_0(q) = d(q_0, q)$, $q = (x, y, z) \in \mathbb{R}^3$:

- if $z = 0$, then $d_0(q) = \sqrt{x^2 + y^2}$,
- if $z \neq 0$, $x^2 + y^2 = 0$, then $d_0(q) = 2\sqrt{\pi |z|}$,
- if $z \neq 0$, $x^2 + y^2 \neq 0$, then the distance is determined by the conditions

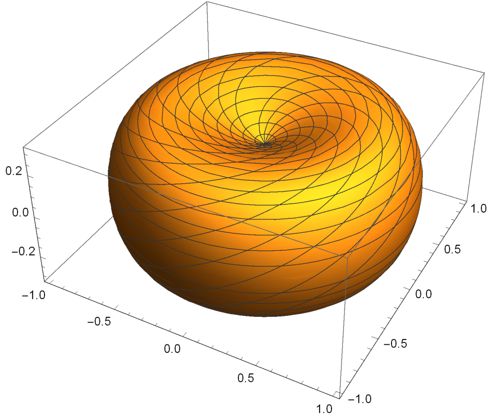
$$d_0(q) = \frac{p}{\sin p}\sqrt{x^2 + y^2},$$
$$\frac{2p - \sin 2p}{8 \sin^2 p} = \frac{|z|}{x^2 + y^2}, \qquad p \in (0, \pi).$$

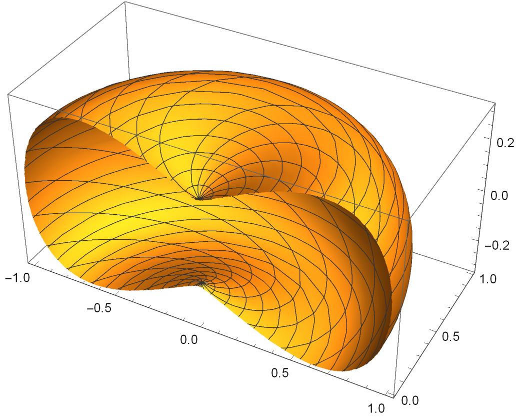
Sub-Riemannian spheres

- The unit *sub-Riemannian sphere* $S = \{q \in \mathbb{R}^3 \mid d_0(q) = 1\}$ is a surface of revolution around the axis z in the form of an apple, see figures at the next slide.
- It has two singular conical points $z = \pm\frac{1}{4\pi}$, $x^2 + y^2 = 0$.
- The remaining spheres $S_R = \{q \in \mathbb{R}^3 \mid d_0(q) = R\}$ are obtained from S by virtue of *dilations*:

$$\delta_s \; : \; (x, y, z) \mapsto (e^s x, e^s y, e^{2s} z), \qquad s \in \mathbb{R},$$
$$S_R = \delta_s(S), \qquad s = \ln R.$$

Sub-Riemannian spheres





Sub-Riemannian problem on the group of Euclidean motions of the plane

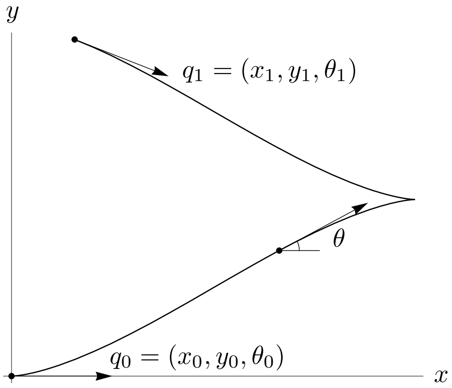
$$\mathrm{SE}(2) = \left\{ \begin{pmatrix} \cos\theta & -\sin\theta & x \\ \sin\theta & \cos\theta & y \\ 0 & 0 & 1 \end{pmatrix} \mid (x,y) \in \mathbb{R}^2,\ \theta \in S^1 \right\}$$

$$X_1(q) = \cos\theta \frac{\partial}{\partial x} + \sin\theta \frac{\partial}{\partial y}, \qquad X_2(q) = \frac{\partial}{\partial \theta}.$$

$$M = \mathrm{SE}(2), \qquad \Delta = \mathrm{span}(X_1, X_2), \qquad g(X_i, X_j) = \delta_{ij}.$$

$$\dot q = u_1 X_1 + u_2 X_2, \qquad q = (x, y, \theta) \in \mathrm{SE}(2), \quad (u_1, u_2) \in \mathbb{R}^2,$$
$$q(0) = q_0 = \mathrm{Id} = (0,0,0), \qquad q(t_1) = q_1,$$
$$l = \int_0^{t_1} \sqrt{u_1^2 + u_2^2}\, dt \to \min.$$

Problem on optimal motion of a mobile robot on a plane



$$I = \int_0^{t_1} \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{\theta}^2}\, dt \to \min$$

Pontryagin maximum principle

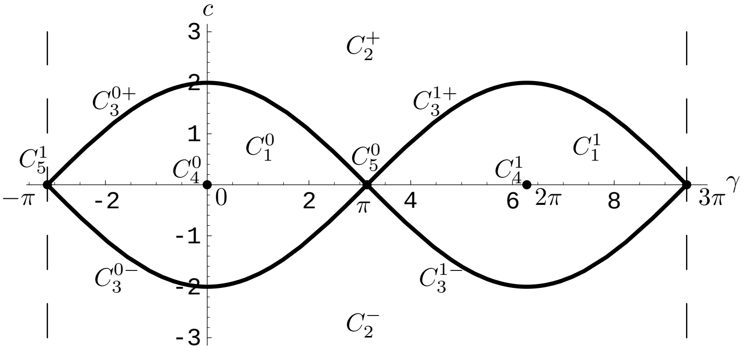
- Abnormal extremal trajectories are constant.
- Normal extremals:

$$\dot{\gamma} = c, \quad \dot{c} = -\sin\gamma, \qquad (\gamma, c) \in C \cong (2S^1_\gamma) \times \mathbb{R}_c,$$
$$\dot{x} = \sin\frac{\gamma}{2}\cos\theta, \quad \dot{y} = \sin\frac{\gamma}{2}\sin\theta, \quad \dot{\theta} = -\cos\frac{\gamma}{2}.$$

- Energy integral $E = \frac{c^2}{2} - \cos\gamma \in [-1, +\infty)$
- $\gamma(t)$, $c(t)$, $q(t)$: parameterization by Jacobi functions sn, cn, dn, E.

Partition of the phase cylinder of the pendulum $C = \cup_{i=1}^{5} C_i$

- $C_1 = \{\lambda \in C \mid E \in (-1, 1)\} \quad \Rightarrow \quad$ pendulum oscillations,
- $C_2 = \{\lambda \in C \mid E \in (1, +\infty)\} \quad \Rightarrow \quad$ rotation of the pendulum,
- $C_3 = \{\lambda \in C \mid E = 1, \ c \neq 0\} \quad \Rightarrow \quad$ critical motion,
- $C_4 = \{\lambda \in C \mid E = -1\} \quad \Rightarrow \quad$ stable equilibrium,
- $C_5 = \{\lambda \in C \mid E = 1, \ c = 0\} \quad \Rightarrow \quad$ unst. equilibrium

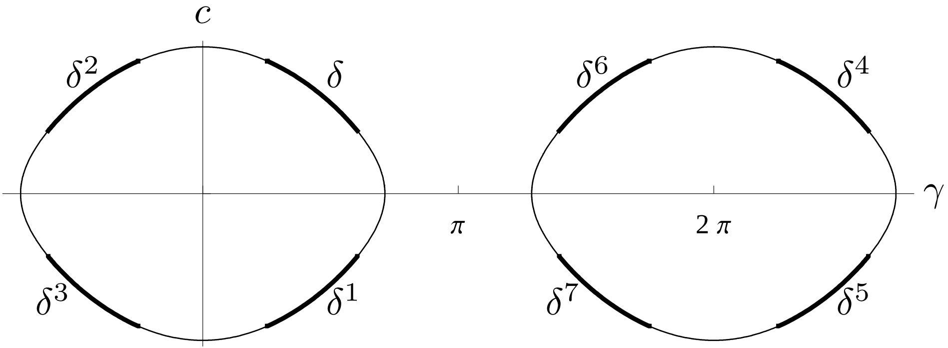


Reflections ε^i in the phase cylinder of a pendulum $\ddot{\gamma} = -\sin\gamma$

- $\varepsilon^i \; : \; C \to C, \quad \varepsilon^i_* \vec{H}_v = \pm \vec{H}_v, \quad \vec{H}_v = c\frac{\partial}{\partial \gamma} - \sin\gamma \frac{\partial}{\partial c} \in \operatorname{Vec} C,$
- Group of symmetries of parallelepiped

$$G = \{\mathrm{Id}, \varepsilon^1, \dots, \varepsilon^7\} = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2.$$

- Action of reflections $\varepsilon^i \; : \; \delta \mapsto \delta^i$ on the pendulum trajectory:



The first Maxwell time corresponding to symmetries

Symmetries of exponential mapping:

$$\mathrm{Exp} \circ \varepsilon^i(\lambda, t) = \varepsilon^i \circ \mathrm{Exp}(\lambda, t), \qquad (\lambda, t) \in C \times \mathbb{R}_+, \quad eps^i \in G.$$

$$t_{\mathrm{Max}}(\lambda) = \min\{t > 0 \mid \exists \varepsilon^i \in G \ : \ \varepsilon^i(\lambda, t) \neq (\lambda, t), \quad Exp \circ \varepsilon^i(\lambda, t) = \mathrm{Exp}(\lambda, t)\}$$

Theorem 5

- $E = -1 \quad \Rightarrow \quad t_{\mathrm{Max}}(\lambda) = \pi,$
- $E \in (-1, 1) \quad \Rightarrow \quad t_{\mathrm{Max}}(\lambda) = 2K(k),\ k = \sqrt{(E+1)/2},$
- $E = 1 \quad \Rightarrow \quad t_{\mathrm{Max}}(\lambda) = +\infty,$
- $E > 1 \quad \Rightarrow \quad t_{\mathrm{Max}}(\lambda) = 2kp_1(k),\ k = \sqrt{2/(E+1)},$

$p_1(k) = \min\{p > 0 \mid \mathrm{cn}(p, k)(\mathrm{E}(p, k) - p) - \mathrm{dn}(p, k)\,\mathrm{sn}(p, k) = 0\}.$

Estimates of the first conjugate time

Theorem 6

- $E \in [-1, 1] \quad \Rightarrow \quad t^1_{\text{conj}}(\lambda) = +\infty$,
- $E > 1 \quad \Rightarrow \quad t^1_{\text{conj}}(\lambda) \in [t_{\text{Max}}(\lambda), 4kK]$,
- $\forall \lambda \in C \quad t^1_{\text{conj}}(\lambda) \geq t_{\text{Max}}(\lambda)$.

Proof method:
Homotopic invariance of the Maslov index (number of conjugate points)

Global structure of exponential mapping

- $\mathrm{Exp} : C \times \mathbb{R}_+ = N \to M$: non-optimal. geodes. for $t > t_{\mathrm{Max}}(\lambda)$,
- $\widehat{N} = \{(\lambda, t) \in C \times \mathbb{R}_+ \mid t \leq \mathrm{t}(\lambda)\}, \qquad \widehat{M} = M \setminus \{q_0\}$,
  $\mathrm{Exp} : \widehat{N} \to \widehat{M}$ surjective, not injective (Maxwell pts),
- $\widetilde{M} = \{q \in M \mid \varepsilon^i(q) \neq q\} =$
  $= \{q \in M \mid \sin\theta \neq 0, \ R_i(q) \neq 0\} = \cup_{i=1}^8 M_i$,
  $\widetilde{N} = \mathrm{Exp}^{-1}(\widetilde{M}) =$
  $= \{(\lambda, t) \in N \mid t < t_{\mathrm{Max}}(\lambda), \ \sin(\gamma_{t/2}/2) \neq 0\} = \cup_{i=1}^8 D_i$,
  $\mathrm{Exp} : \widetilde{N} \to \widetilde{M}$: no Maxwell point neither conjugate points.

Theorem 7

$\mathrm{Exp} : D_i \to M_i$ *is a diffeomorphism,* $i = 1, \dots, 8$.
$\mathrm{Exp} : \widetilde{N} \to \widetilde{M}$ *is a diffeomorphism.*

Diffeomorphic stratifications and cut locus

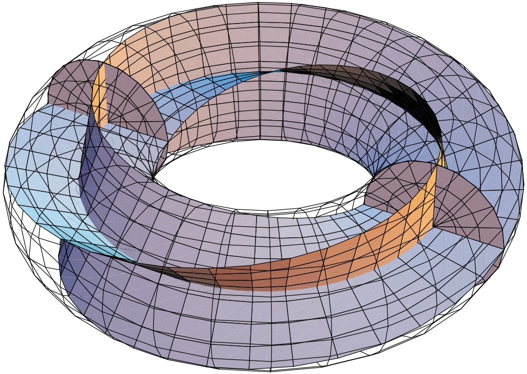
- $\mathrm{Cut}, \mathrm{Max} \subset M' = \widehat{M} \setminus \widetilde{M} = \{q \in M \mid \sin\theta R_1(q) R_2(q) = 0\}$,
- $N' = \widehat{N} \setminus \widetilde{N}$,
- $\mathrm{Exp} : N' \to M'$,
- Stratifications: $N' = \sqcup_{i=1}^{58} N'_i$, $M' = \cup_{i=1}^{58} M'_i$,
- $\mathrm{Exp} : N'_i \to M'_i$ is a diffeomorphism, $i = 1, \dots, 58$
- $\mathrm{Max} = \cup\{M'_i \mid \exists M'_j = M'_i,\ j \neq i\}$,
- $\mathrm{Cut} = \mathrm{Max} \cup (\mathrm{Cut} \cap \mathrm{Conj})$,
- $\mathrm{Cut} = \mathrm{Cut}_{\mathrm{loc}} \cup \mathrm{Cut}_{\mathrm{glob}}$,
- $\mathrm{Cut}_{\mathrm{glob}} = \{q \in M \mid \theta = \pi\}$, $d(q_0, \mathrm{Cut}_{\mathrm{glob}}) = \pi$,
- $\mathrm{Cut}_{\mathrm{loc}} \subset \{R_2 = 0\}$, $\mathrm{cl}(\mathrm{Cut}_{\mathrm{loc}}) \ni q_0$,
- $\mathrm{Cut}_{\mathrm{loc}} = \{q \in M \mid \theta \in (-\pi, \pi),\ R_2 = 0,\ |R_1| > R_1^1(|\theta|)\}$,

  $R_1 = y\cos\frac{\theta}{2} - x\sin\frac{\theta}{2}, \qquad R_2 = x\cos\frac{\theta}{2} + x\sin\frac{\theta}{2}$,

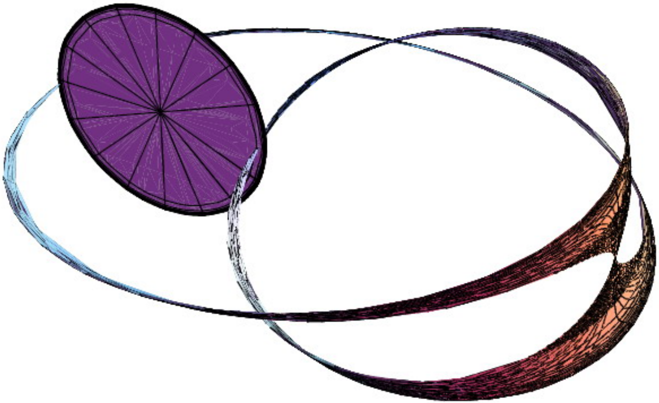
  $R_1^1(\theta) = 2(p_1(k) - \mathrm{E}(p_1(k), k))$,

  $k = k_1(\theta)$ is an inverse function to $\theta = k\,\mathrm{sn}(p_1(k), k)$.

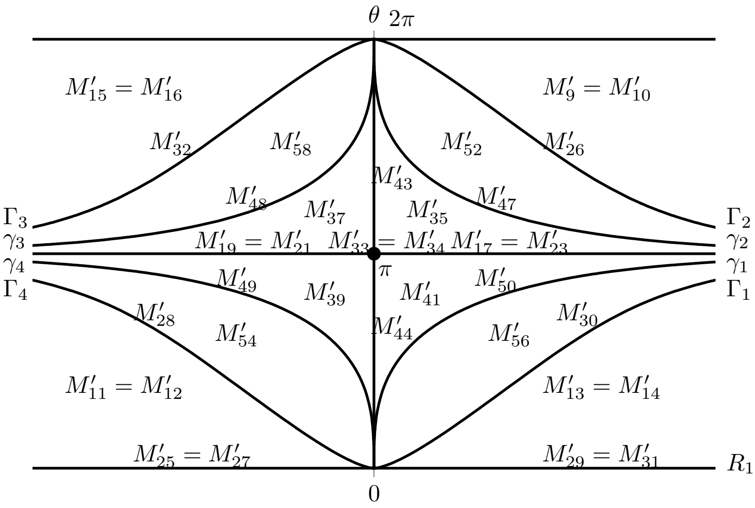
The set $M' \supset \mathrm{Cut} \supset \mathrm{Max}$



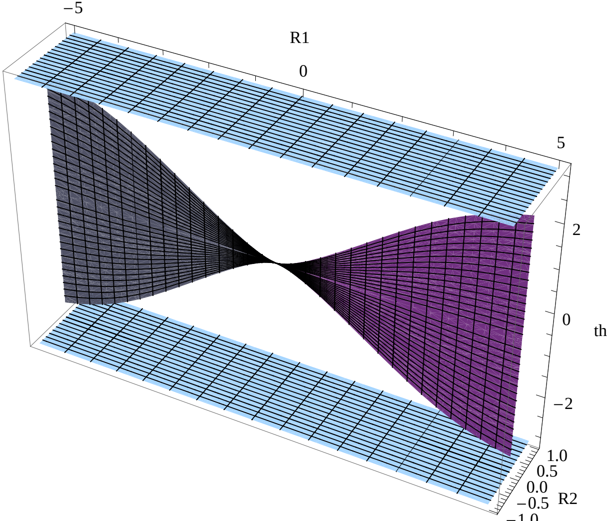
Cut locus: global structure



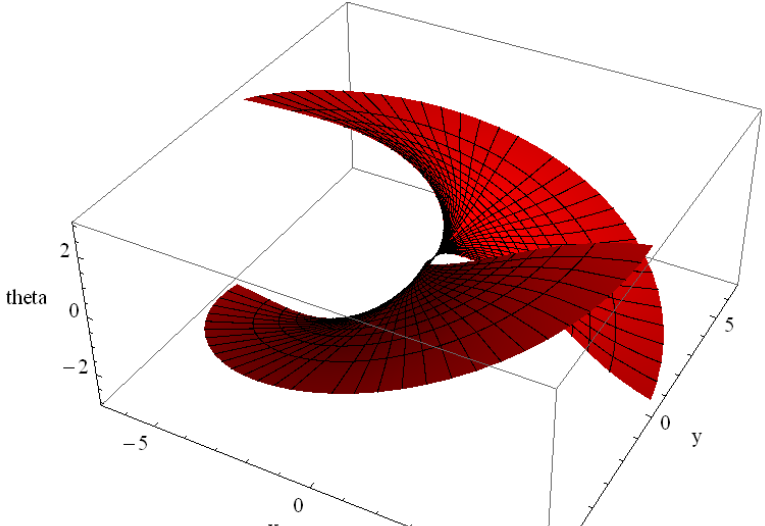
Stratification of the Möbius strip $R_2(q) = 0$



Cut locus in rectifying coordinates (R_1, R_2, θ)



Local component of the cut locus in the original coordinates (x, y, θ)



Optimal synthesis

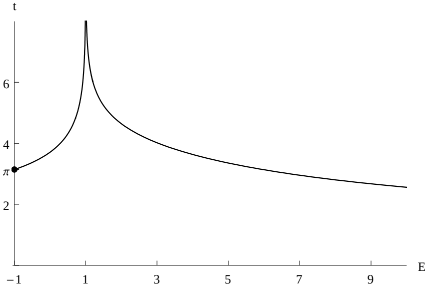
$$q(0) = q_0 = (0, 0, 0), \; q(t_1) = q_1$$

- $q_1 \in \widehat{M} = M \setminus \{q_0\}$
- $\mathrm{Exp} : \widehat{N} \to \widehat{M}$ surjective
- $\mathrm{Exp}^{-1}(q) = \begin{cases} \{(\lambda, t)\}, & \text{if } q \in \widehat{M} \setminus \mathrm{Max}, \\ \{(\lambda', t) \neq (\lambda'', t)\}, & \text{if } q \in \mathrm{Max} \end{cases}$
- $\mathrm{Exp}^{-1}(q_1) = (\lambda, t), \qquad \lambda = (\gamma, c) \in (2S^1) \times \mathbb{R}, \; t > 0$
- $\ddot{\gamma}_s = -\sin \gamma_s, \; (\gamma_0, \dot{\gamma}_0) = (\gamma, c), \; s \in [0, t]$
- $u_1(q_1) = -\sin(\gamma_t/2), \; u_2(q_1) = \cos(\gamma_t/2)$
- the optimal synthesis $q_1 \mapsto (u_1, u_2)$ is two-valued on Max, single-valued on $\widehat{M} \setminus \mathrm{Max}$.

Cut time

Theorem 8

- $t_{\text{cut}}(\lambda) = t_{\text{Max}}(\lambda), \quad \lambda \in C,$
- $t_{\text{cut}} \circ \varepsilon^i = t_{\text{cut}}, \quad \varepsilon^i \in G,$
- $\vec{H}_v t_{\text{cut}} = 0,$
- $t_{\text{cut}} \ : \ C \to (0, +\infty]$ *is continuous,* $\quad t_{\text{cut}}|_{E \neq \pm 1}$ *is smooth.*



Sub-Riemannian spheres

- $R \in (0, \pi) \quad \Rightarrow \quad S_R \cong S^2,$
- $R = \pi \quad \Rightarrow \quad S_R \cong S^2/\{N = S\},$
- $R > \pi \quad \Rightarrow \quad S_R \cong T^2.$

Singularities of the spheres:

$$S_R \cap \mathrm{Cut} = (S_R \cap \mathrm{Max}) \cup (S_R \cap \mathrm{Cut} \cap \mathrm{Conj}).$$