Pontryagin maximum principle - 2

(Lecture 7)

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«Elements of geometric control theory»

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Plan of previous lecture

1. Hamiltonian vector fields
2. Linear on fibers Hamiltonians
3. Pontryagin Maximum Principle: the geometric statement and discussion

Plan of this lecture

1. Proof of the geometric statement of PMP with fixed terminal time
2. Geometric statement of PMP for free time
3. PMP for optimal control problems
4. PMP with transversality conditions

Optimal control problem

- Consider an optimal control problem for a control system

$$\dot{q} = f_u(q), \qquad q \in M, \quad u \in U \subset \mathbb{R}^m, \tag{1}$$

with the initial condition

$$q(0) = q_0. \tag{2}$$

- Define the following family of Hamiltonians:

$$h_u(\lambda) = \langle \lambda, f_u(q) \rangle, \qquad \lambda \in T_q^* M, \; q \in M, \; u \in U.$$

- In terms of the previous lecture,

$$h_u(\lambda) = f_u^*(\lambda).$$

- Fix an arbitrary instant $t_1 > 0$.
- In Lecture 1 we reduced the optimal control problem to the study of boundary of attainable sets.

The geometric statement of PMP with fixed terminal time

Theorem 1 (PMP)

Let $\widetilde{u}(t)$, $t \in [0, t_1]$, be an admissible control and $\widetilde{q}(t) = q_{\widetilde{u}}(t)$ the corresponding solution of Cauchy problem (1), (2). *If $\widetilde{q}(t_1) \in \partial \mathcal{A}_{q_0}(t_1)$, then there exists a Lipschitzian curve in the cotangent bundle*

$$\lambda_t \in T^*_{\widetilde{q}(t)} M, \qquad 0 \leq t \leq t_1,$$

such that

$$\lambda_t \neq 0, \tag{3}$$

$$\dot{\lambda}_t = \vec{h}_{\widetilde{u}(t)}(\lambda_t), \tag{4}$$

$$h_{\widetilde{u}(t)}(\lambda_t) = \max_{u \in U} h_u(\lambda_t) \tag{5}$$

for almost all $t \in [0, t_1]$.

Proof of the geometric statement of PMP with fixed terminal time

- We start from two auxiliary lemmas.
- Denote the positive orthant in $\mathbb{R}^m$ as

$$\mathbb{R}^m_+ = \{(x_1, \dots, x_m) \in \mathbb{R}^m \mid x_i \geq 0, \ i = 1, \dots, m\}.$$

Lemma 2

Let a vector-function $F : \mathbb{R}^m \to \mathbb{R}^n$ *be Lipschitzian,* $F(0) = 0$*, and differentiable at* 0*:*

$$\exists \ F'_0 = \left. \frac{d\,F}{d\,x} \right|_0 .$$

Assume that

$$F'_0(\mathbb{R}^m_+) = \mathbb{R}^n.$$

Then for any neighborhood of the origin $O_0 \subset \mathbb{R}^m$

$$0 \in \operatorname{int} F(O_0 \cap \mathbb{R}^m_+).$$

Remark 1

The statement of this lemma holds if the orthant $\mathbb{R}^m_+$ is replaced by an arbitrary convex cone $C \subset \mathbb{R}^m$. In this case the proof given below works without any changes.

Proof of Lemma 2.

- Choose points $y_0, \dots, y_n \in \mathbb{R}^n$ that generate an n-dimensional simplex centered at the origin: $\dfrac{1}{n+1}\displaystyle\sum_{i=0}^{n} y_i = 0.$
- Since the mapping $F_0' : \mathbb{R}^m_+ \to \mathbb{R}^n$ is surjective and the positive orthant $\mathbb{R}^m_+$ is a convex cone, it is easy to show that restriction to the interior $F_0'|_{\operatorname{int}\mathbb{R}^m_+}$ is also surjective:

$$\exists\ v_i \in \operatorname{int}\mathbb{R}^m_+ \quad \text{such that} \quad F_0' v_i = y_i, \quad i = 0, \dots, n.$$

- The points $y_0, \dots, y_n$ are affinely independent in $\mathbb{R}^n$, thus their preimages $v_0, \dots, v_n$ are also affinely independent in $\mathbb{R}^m$.

- The mean

$$v = \frac{1}{n+1} \sum_{i=0}^{n} v_i$$

belongs to $\operatorname{int} \mathbb{R}^m_+$ and satisfies the equality

$$F_0' v = 0.$$

- Further, the subspace

$$W = \operatorname{span}\{v_i - v \mid i = 0, \ldots, n\} \subset \mathbb{R}^m$$

is n-dimensional.
- Since $v \in \operatorname{int} \mathbb{R}^m_+$, we can find an n-dimensional ball $B_\delta \subset W$ of a sufficiently small radius δ centered at the origin such that

$$v + B_\delta \subset \operatorname{int} \mathbb{R}^m_+.$$

- Since $F_0'(v_i - v) = F_0' v_i$, then $F_0' W = \mathbb{R}^n$, i.e., the linear mapping $F_0' : W \to \mathbb{R}^n$ is invertible.

- Consider the following family of mappings:

$$
\begin{aligned}
&G_\alpha \,:\, B_\delta \to \mathbb{R}^n, \qquad \alpha \in [0, \alpha_0),\\
&G_\alpha(w) = \frac{1}{\alpha} F(\alpha(v+w)), \qquad \alpha > 0,\\
&G_0(w) = F_0' w.
\end{aligned}
$$

- By the hypotheses of this lemma,

$$F(x) = F_0' x + o(x), \qquad x \in \mathbb{R}^m,\ x \to 0,$$

thus

$$G_\alpha(w) = \frac{1}{\alpha}(F_0'(\alpha(v+w)) + o(\alpha(v+w))) = F_0' w + o(1),\ \alpha \to 0,\ w \in B_\delta. \quad (6)$$

- Since the mapping F is Lipschitzian, all mappings G_α are Lipschitzian with a common constant.
- Thus the family G_α is equicontinuous. Equality (6) means that uniformly in $w \in B_\delta$ we have $G_\alpha \to G_0, \qquad \alpha \to 0.$

- So the continuous mapping $G_\alpha \circ G_0^{-1} : G_0(B_\delta) \to \mathbb{R}^n$ is uniformly close to the identity mapping, hence the difference $\mathrm{Id} - G_\alpha \circ G_0^{-1}$ is uniformly close to the zero mapping.
- For any $\tilde{x} \in \mathbb{R}^n$ sufficiently close to the origin, the continuous mapping

$$\mathrm{Id} - G_\alpha \circ G_0^{-1} + \tilde{x}$$

  transforms the set $G_0(B_\delta)$ into itself.
- By Brower's fixed point theorem, this mapping has a fixed point $x \in G_0(B_\delta)$:

$$x - G_\alpha \circ G_0^{-1}(x) + \tilde{x} = x,$$

  i.e.,

$$G_\alpha \circ G_0^{-1}(x) = \tilde{x}.$$

- It follows that $\mathrm{int}\, G_\alpha(B_\delta) \ni 0$, consequently, $\mathrm{int}\, F(\alpha(v + B_\delta)) \ni 0$ for small $\alpha > 0$. Thus $\mathrm{int}\, F(O_0 \cap \mathbb{R}^m_+) \ni 0$ for a small neighborhood $O_0 \in \mathbb{R}^m$. □

- Now we start to compute a convex approximation of the attainable set $\mathcal{A}_{q_0}(t_1)$ at the point $q_1 = \widetilde{q}(t_1)$ corresponding to a reference control $\widetilde{u}(\cdot)$.
- Take any admissible control $u(t)$ and express the endpoint of a trajectory via Variations Formula:

$$\begin{aligned} q_u(t_1) &= q_0 \circ \overrightarrow{\exp} \int_0^{t_1} f_{u(\tau)}\, d\tau = q_0 \circ \overrightarrow{\exp} \int_0^{t_1} f_{\widetilde{u}(\tau)} + (f_{u(\tau)} - f_{\widetilde{u}(\tau)})\, d\tau \\ &= q_0 \circ \overrightarrow{\exp} \int_0^{t_1} f_{\widetilde{u}(\tau)}\, d\tau \circ \overrightarrow{\exp} \int_0^{t_1} \left(P_\tau^{t_1}\right)_* (f_{u(\tau)} - f_{\widetilde{u}(\tau)})\, d\tau \\ &= q_1 \circ \overrightarrow{\exp} \int_0^{t_1} \left(P_\tau^{t_1}\right)_* (f_{u(\tau)} - f_{\widetilde{u}(\tau)})\, d\tau. \end{aligned}$$

- Introduce the following vector field depending on two parameters:

$$g_{\tau,u} = \left(P_\tau^{t_1}\right)_* (f_u - f_{\widetilde{u}(\tau)}), \qquad \tau \in [0, t_1], \quad u \in U. \tag{7}$$

- We showed that

$$q_u(t_1) = q_1 \circ \overrightarrow{\exp} \int_0^{t_1} g_{\tau,u(\tau)}\, d\tau. \tag{8}$$

- Notice that $g_{\tau,\widetilde{u}(\tau)} \equiv 0, \qquad \tau \in [0, t_1].$

Lemma 3

Let $\mathcal{T} \subset [0, t_1]$ be the set of Lebesgue points of the control $\widetilde{u}(\cdot)$. If

$$\operatorname{cone}\{g_{\tau,u}(q_1) \mid \tau \in \mathcal{T},\ u \in U\} = T_{q_1}M,$$

then $q_1 \in \operatorname{int} \mathcal{A}_{q_0}(t_1)$.

Remark 2

The set $\operatorname{cone}\{g_{\tau,u}(q_1) \mid \tau \in \mathcal{T},\ u \in U\} \subset T_{q_1}M$ is a local convex approximation of the attainable set $\mathcal{A}_{q_0}(t_1)$ at the point q_1 corresponding to a reference control $\widetilde{u}(\cdot)$.

- Recall that a point $\tau \in [0, t_1]$ is called a *Lebesgue point* of a function $u \in L^1[0, t_1]$ if $\lim\limits_{t \to \tau} \dfrac{1}{|t-\tau|} \displaystyle\int_\tau^t |u(\theta) - u(\tau)|\, d\theta = 0$.
- At Lebesgue points of u, the integral $\displaystyle\int_0^t u(\theta)\, d\theta$ is differentiable and $\dfrac{d}{d\,t} \left(\displaystyle\int_0^t u(\theta)\, d\theta \right) = u(t)$.

- The set of Lebesgue points has the full measure in the domain $[0, t_1]$.

Proof of Lemma 3.

- We can choose vectors

$$g_{\tau_i,u_i}(q_1) \in T_{q_1}M, \qquad \tau_i \in \mathcal{T}, \quad u_i \in U, \quad i = 1, \dots, k,$$

  that generate the whole tangent space as a positive convex cone:

$$\operatorname{cone}\{g_{\tau_i,u_i}(q_1) \mid i = 1, \dots, k\} = T_{q_1}M,$$

  moreover, we can choose points τ_i distinct: $\tau_i \neq \tau_j$, $i \neq j$.
- Indeed, if $\tau_i = \tau_j$ for some $i \neq j$, we can find a sufficiently close Lebesgue point $\tau_j' \neq \tau_j$ such that the difference $g_{\tau_j',u_j}(q_1) - g_{\tau_j,u_j}(q_1)$ is as small as we wish.
- This is possible since for any $\tau \in \mathcal{T}$ and any $\varepsilon > 0$

$$\frac{1}{|t-\tau|}\operatorname{meas}\{t' \in [\tau, t] \mid |u(t') - u(\tau)| \leq \varepsilon\} \to 1 \text{ as } t \to \tau.$$

- We suppose that $\tau_1 < \tau_2 < \dots < \tau_k$.

- We define a family of variations of controls that follow the reference control $\widetilde{u}(\cdot)$ everywhere except neighborhoods of τ_i, and follow u_i near τ_i (such variations are called *needle-like*).
- More precisely, for any $s = (s_1, \ldots, s_k) \in \mathbb{R}^k_+$ consider a control of the form

$$u_s(t) = \begin{cases} u_i, & t \in [\tau_i, \tau_i + s_i], \\ \widetilde{u}(t), & t \notin \cup_{i=1}^k [\tau_i, \tau_i + s_i]. \end{cases} \tag{9}$$

- For small s, the segments $[\tau_i, \tau_i + s_i]$ do not overlap since $\tau_i \neq \tau_j$, $i \neq j$.
- In view of formula (8), the endpoint of the trajectory corresponding to the control constructed is expressed as follows:

$$\begin{aligned} q_{u_s}(t_1) &= q_0 \circ \overrightarrow{\exp} \int_0^{t_1} f_{u_s(t)}\, dt \\ &= q_1 \circ \overrightarrow{\exp} \int_{\tau_1}^{\tau_1+s_1} g_{t,u_1}\, dt \circ \overrightarrow{\exp} \int_{\tau_2}^{\tau_2+s_2} g_{t,u_2}\, dt \circ \cdots \\ &\qquad \circ \overrightarrow{\exp} \int_{\tau_k}^{\tau_k+s_k} g_{t,u_k}\, dt. \end{aligned}$$

- The mapping

$$F \;:\; s = (s_1, \dots, s_k) \mapsto q_{u_s}(t_1)$$

is Lipschitzian, differentiable at $s = 0$, and

$$\left.\frac{\partial F}{\partial s_i}\right|_{s=0} = g_{\tau_i, u_i}(q_1).$$

- By Lemma 2,

$$F(0) = q_1 \in \operatorname{int} F(O_0 \cap \mathbb{R}^k_+)$$

for any neighborhood $O_0 \subset \mathbb{R}^k$.
- But the curve $q_{u_s}(t)$, $t \in [0, t_1]$, is an admissible trajectory for small $s \in \mathbb{R}^k_+$, thus $F(O_0 \cap \mathbb{R}^k_+) \subset \mathcal{A}_{q_0}(t_1)$ and $q_1 \in \operatorname{int} \mathcal{A}_{q_0}(t_1)$.

□

Proof of Theorem 1.

- Let the endpoint of the reference trajectory $q_1 = \widetilde{q}(t_1) \in \partial\mathcal{A}_{q_0}(t_1)$.
- By Lemma 3, the origin $0 \in T_{q_1}M$ belongs to the boundary of the convex set $\mathrm{cone}\{g_{t,u}(q_1) \mid t \in \mathcal{T},\ u \in U\}$, so this set has a hyperplane of support at the origin:

$$\exists\, \lambda_{t_1} \in T^*_{q_1}M, \quad \lambda_{t_1} \neq 0,$$

  such that

$$\langle \lambda_{t_1}, g_{t,u}(q_1)\rangle \leq 0 \qquad \forall \text{ a.e. } t \in [0, t_1], \quad u \in U.$$

- Taking into account that $g_{\tau,u} = \left(P_\tau^{t_1}\right)_* \left(f_u - f_{\widetilde{u}(\tau)}\right)$, we rewrite this inequality as follows:

$$\langle \lambda_{t_1}, \left(P^{t_1}_{t*} f_u\right)(q_1)\rangle \leq \langle \lambda_{t_1}, \left(P^{t_1}_{t*} f_{\widetilde{u}(t)}\right)(q_1)\rangle,$$

  i.e.,

$$\langle \left(P_t^{t_1}\right)^* \lambda_{t_1}, f_u(\widetilde{q}(t))\rangle \leq \langle \left(P_t^{t_1}\right)^* \lambda_{t_1}, f_{\widetilde{u}(t)}(\widetilde{q}(t))\rangle.$$

- The action of the flow $P_t^{t_1}$ on covectors defines the curve in the cotangent bundle:

$$\lambda_t \stackrel{\text{def}}{=} \left(P_t^{t_1}\right)^* \lambda_{t_1} \in T^*_{\widetilde{q}(t)} M, \qquad t \in [0, t_1].$$

- In terms of this covector curve, the inequality above reads

$$\langle \lambda_t, f_u(\widetilde{q}(t)) \rangle \leq \langle \lambda_t, f_{\widetilde{u}(t)}(\widetilde{q}(t)) \rangle.$$

- Thus the maximality condition of PMP holds along the reference trajectory:

$$h_u(\lambda_t) \leq h_{\widetilde{u}(t)}(\lambda_t) \qquad \forall\ u \in U \quad \forall \text{ a.e. } t \in [0, t_1].$$

- The curve λ_t is a trajectory of the nonautonomous Hamiltonian flow with the Hamiltonian function $f^*_{\widetilde{u}(t)} = h_{\widetilde{u}(t)}$:

$$\lambda_t = \lambda_{t_1} \circ \left(\overrightarrow{\exp} \int_t^{t_1} f_{\widetilde{u}(\theta)}\, d\theta \right)^* = \lambda_{t_1} \circ \overrightarrow{\exp} \int_{t_1}^{t} \vec{h}_{\widetilde{u}(\theta)}\, d\theta,$$

thus it satisfies the Hamiltonian equation of PMP

$$\dot{\lambda}_t = \vec{h}_{\widetilde{u}(t)}(\lambda_t).$$

□

Geometric statement of PMP for free time

Theorem 4

Let $\widetilde{u}(\cdot)$ be an admissible control such that $\widetilde{q}(t_1) \in \partial \left(\cup_{|t-t_1|<\varepsilon} \mathcal{A}_{q_0}(t) \right)$ for some $t_1 > 0$ and $\varepsilon \in (0, t_1)$. Then there exists a Lipschitzian curve

$$\lambda_t \in T^*_{\widetilde{q}(t)} M, \quad \lambda_t \neq 0, \qquad 0 \leq t \leq t_1,$$

such that

$$\begin{aligned} \dot{\lambda}_t &= \vec{h}_{\widetilde{u}(t)}(\lambda_t), \\ h_{\widetilde{u}(t)}(\lambda_t) &= \max_{u \in U} h_u(\lambda_t), \\ h_{\widetilde{u}(t)}(\lambda_t) &= 0 \end{aligned} \tag{10}$$

for almost all $t \in [0, t_1]$.

Remark 3

In problems with free time, there appears one more variable, the terminal time t_1. In order to eliminate it, we have one additional condition — equality (10). This condition is indeed scalar since the previous two equalities imply that $h_{\widetilde{u}(t)}(\lambda_t) = \text{const}$.

Proof of Theorem 4.

- We reduce the case of free time to the case of fixed time by extension of the control system via substitution of time. Admissible trajectories of the extended system are reparametrized admissible trajectories of the initial system (the positive direction of time on trajectories is preserved).
- Let a new time be a smooth function

$$\varphi : \mathbb{R} \to \mathbb{R}, \quad \dot{\varphi} > 0.$$

- We find an ODE for a reparametrized trajectory:

$$\frac{d}{d\,t} q_u(\varphi(t)) = \dot{\varphi}(t) f_{u(\varphi(t))}(q_u(\varphi(t))),$$

so the required equation is

$$\dot{q} = \dot{\varphi}(t) f_{u(\varphi(t))}(q).$$

- Now consider along with the initial control system

$$\dot{q} = f_u(q), \qquad u \in U,$$

an extended system of the form

$$\dot{q} = v f_u(q), \qquad u \in U, \ |v - 1| < \delta, \tag{11}$$

where $\delta = \varepsilon / t_1 \in (0, 1)$.

- Admissible controls of the new system are

$$w(t) = (v(t), u(t)),$$

and the reference control corresponding to the control $\widetilde{u}(\cdot)$ of the initial system is

$$\tilde{w}(t) = (1, \widetilde{u}(t)).$$

- It is easy to see that since $\widetilde{q}(t_1) \in \partial \left(\cup_{|t - t_1| < \varepsilon} \mathcal{A}_{q_0}(t) \right)$, then the trajectory of the new system through the point q_0 corresponding to the control $\tilde{w}(\cdot)$ comes at the moment t_1 to the boundary of the attainable set of the new system for time t_1.
- Thus $\tilde{w}(t)$ satisfies PMP with fixed time.

- We apply the geometric statement of PMP for fixed time to the new system (11).
- The Hamiltonian for the new system is $vh_u(\lambda)$.
- Then the maximality condition reads

$$1 \cdot h_{\widetilde{u}(t)}(\lambda_t) = \max_{u \in U,\ |v-1|<\delta} vh_u(\lambda_t).$$

- We take $u = \widetilde{u}(t)$ under the maximum and obtain

$$h_{\widetilde{u}(t)}(\lambda_t) = 0,$$

then we restrict the maximum to the set $v = 1$ and come to

$$h_{\widetilde{u}(t)}(\lambda_t) = \max_{u \in U} h_u(\lambda_t).$$

- The Hamiltonian systems along $\tilde{w}(\cdot)$ and $\widetilde{u}(\cdot)$ coincide one with another, thus the proposition follows.

□

PMP for optimal control problems

- Now we apply PMP in geometric form to optimal control problems, starting from problems with fixed time.
- For a control system

$$\dot{q} = f_u(q), \qquad q \in M, \quad u \in U, \tag{12}$$

with the boundary conditions

$$q(0) = q_0, \quad q(t_1) = q_1, \quad q_0, q_1 \in M \text{ fixed}, \tag{13}$$

$$t_1 > 0 \text{ fixed}, \tag{14}$$

and the cost functional

$$J(u) = \int_0^{t_1} \varphi(q_u(t), u(t))\, dt \tag{15}$$

we consider the optimal control problem

$$J(u) \to \min . \tag{16}$$

- We transform the problem to a geometric one.

- We extend the state space:

$$\widehat{q} = \begin{pmatrix} y \\ q \end{pmatrix} \in \mathbb{R} \times M,$$

define the extended vector field $\widehat{f_u} \in \mathrm{Vec}(\mathbb{R} \times M)$:

$$\widehat{f_u}(q) = \begin{pmatrix} \varphi(q,u) \\ f_u(q) \end{pmatrix},$$

and come to the new control system:

$$\frac{d\,\widehat{q}}{d\,t} = \widehat{f_u}(q) \quad \Leftrightarrow \quad \begin{cases} \dot{y} = \varphi(q,u), \\ \dot{q} = f_u(q) \end{cases} \tag{17}$$

with the boundary conditions

$$\widehat{q}(0) = \widehat{q}_0 = \begin{pmatrix} 0 \\ q_0 \end{pmatrix}, \qquad \widehat{q}(t_1) = \begin{pmatrix} J(u) \\ q_1 \end{pmatrix}.$$

- If a control $\widetilde{u}(\cdot)$ is optimal for problem (12)–(16), then the trajectory $\widehat{q}_{\widetilde{u}}(t)$ of the extended system (17) starting from $\widehat{q}_0$ satisfies the condition

$$\widehat{q}_{\widetilde{u}}(t_1) \in \partial \widehat{\mathcal{A}}_{\widehat{q}_0}(t_1),$$

  where $\widehat{\mathcal{A}}_{\widehat{q}_0}(t_1)$ is the attainable set of system (17) from the point $\widehat{q}_0$ for time t_1.
- So we can apply the geometric statement of PMP.
- But the geometric statement of PMP applied to the extended system (17) does not distinguish minimum and maximum of the cost $J(u)$.
- In order to have conditions valid only for minimum, we introduce a new control parameter v and consider a new system of the form

$$\begin{cases} \dot{y} = \varphi(q, u) + v, \\ \dot{q} = f_u(q), \end{cases} \qquad v \geq 0, \quad u \in U. \tag{18}$$

- Now the trajectory of system (18) corresponding to the controls $\tilde{v}(t) \equiv 0$, $\widetilde{u}(t)$, comes to the boundary of the attainable set of this system at time t_1.

- We apply the geometric statement of PMP to system (18).
- We have

$$T_{(y,q)}(\mathbb{R} \times M) = \mathbb{R} \oplus T_q M,$$
$$T^*_{(y,q)}(\mathbb{R} \times M) = \mathbb{R} \oplus T^*_q M = \{(\nu, \lambda)\}.$$

- The Hamiltonian function for system (18) has the form

$$\widehat{h}_{(v,u)}(\nu, \lambda) = \langle \lambda, f_u \rangle + \nu(\varphi + v),$$

and the Hamiltonian system of PMP is

$$\begin{cases} \dot{\nu} = \frac{\partial \widehat{h}}{\partial y} = 0, \\ \dot{y} = \varphi(q, u) + v, \\ \dot{\lambda} = \vec{h}_{\widetilde{u}(t)}(\nu, \lambda). \end{cases} \tag{19}$$

- Here $\vec{h}_u(\nu, \lambda)$ is the Hamiltonian vector field with the Hamiltonian function $h_u(\nu, \lambda) = \langle \lambda, f_u \rangle + \nu\varphi$.

- The first of equations (19) means that

$$\nu = \text{const}$$

along the reference trajectory.

- The maximality condition has the form

$$\langle \lambda_t, f_{\widetilde{u}(t)} \rangle + \nu\varphi(\widetilde{q}(t), \widetilde{u}(t)) = \max_{u \in U,\ v \geq 0} \left(\langle \lambda_t, f_u \rangle + \nu\varphi(\widetilde{q}(t), u) + \nu v \right).$$

- Since the previous maximum is attained, we have

$$\nu \leq 0,$$

thus we can set $v = 0$ in the right-hand side of the maximality condition:

$$\langle \lambda_t, f_{\widetilde{u}(t)} \rangle + \nu\varphi(\widetilde{q}(t), \widetilde{u}(t)) = \max_{u \in U} \left(\langle \lambda_t, f_u \rangle + \nu\varphi(\widetilde{q}(t), u) \right).$$

- So we proved the PMP for optimal control problems with fixed terminal time.

Theorem 5

Let $\widetilde{u}(t)$*,* $t \in [0, t_1]$*, be an optimal control for problem* (12)–(16)*:*

$$J(\widetilde{u}) = \min\{J(u) \mid q_u(t_1) = q_1\}.$$

Define a Hamiltonian function

$$h_u^\nu(\lambda) = \langle \lambda, f_u \rangle + \nu\varphi(q, u), \qquad \lambda \in T_q^* M, \quad u \in U, \quad \nu \in \mathbb{R}.$$

Then there exists a nontrivial pair:

$$(\nu, \lambda_t) \neq 0, \qquad \nu \in \mathbb{R}, \quad \lambda_t \in T^*_{\widetilde{q}(t)} M,$$

such that the following conditions hold:

$$\begin{aligned} &\dot{\lambda}_t = \vec{h}^\nu_{\widetilde{u}(t)}(\lambda_t), \\ &h^\nu_{\widetilde{u}(t)}(\lambda_t) = \max_{u \in U} h_u^\nu(\lambda_t) \quad \forall \text{ a.e. } t \in [0, t_1], \\ &\nu \leq 0. \end{aligned}$$

Remarks

(1) If we have a maximization problem instead of minimization problem (16), then the preceding inequality for ν should be reversed:

$$\nu \geq 0.$$

(2) For the problem with free time t_1: (12), (13), (15), (16), necessary optimality conditions of PMP are the same as in Theorem 5 plus one additional scalar equality $h^{\nu}_{\tilde{u}(t)}(\lambda_t) \equiv 0$ (exercise).

- There are two distinct possibilities for the constant parameter ν in Theorem 5:

  (a) if $\nu \neq 0$, then the curve λ_t is called a *normal extremal*. Since the pair (ν, λ_t) can be multiplied by any positive number, we can normalize $\nu < 0$ and assume that $\nu = -1$ in the normal case;

  (b) if $\nu = 0$, then λ_t is an *abnormal extremal*.

- So we can always assume that $\nu = -1$ or 0.

$$\dot{q} = f_u(q), \qquad q \in M, \quad u \in U,$$

$$q(0) = q_0, \quad q(t_1) = q_1, \quad q_0, q_1 \text{ fixed}, \qquad t_1 = \int_0^{t_1} 1\, dt \to \min .$$

Corollary 6

Let an admissible control $\widetilde{u}(t)$*,* $t \in [0, t_1]$*, be time-optimal. Define a Hamiltonian function* $h_u(\lambda) = \langle \lambda, f_u \rangle, \qquad \lambda \in T_q^* M, \quad u \in U$*. Then there exists a Lipschitzian curve* $\lambda_t \in T^* M, \quad \lambda_t \neq 0, \qquad t \in [0, t_1]$*, such that the following conditions hold for almost all* $t \in [0, t_1]$*:*

$$\begin{aligned} \dot{\lambda}_t &= \vec{h}_{\widetilde{u}(t)}(\lambda_t), \\ h_{\widetilde{u}(t)}(\lambda_t) &= \max_{u \in U} h_u(\lambda_t), \\ h_{\widetilde{u}(t)}(\lambda_t) &\geq 0. \end{aligned} \tag{20}$$

Proof of Corollary 6.

- Apply PMP for optimal control problems with free terminal time, taking $\varphi \equiv 1$.
- Then the Hamiltonian system and the maximality condition follow.
- Inequality (20) is equivalent to conditions $h_{\widetilde{u}(t)}(\lambda_t) + \nu = 0$ and $\nu \leq 0$.
- The inequality $\lambda_t \neq 0$ is obtained as follows: if $\lambda_t = 0$, then $h_{\widetilde{u}(t)}(\lambda_t) = 0$, thus $\nu = 0$.
- But the pair (ν, λ_t) must be nontrivial, consequently, $\lambda_t \neq 0$.

□

PMP with general boundary conditions

- We prove versions of Pontryagin Maximum Principle for optimal control problems in which boundary points of trajectories belong to prescribed manifolds.
- First consider the following problem:

$$\dot{q} = f_u(q), \qquad q \in M, \quad u \in U \subset \mathbb{R}^m, \tag{21}$$

$$q(0) \in N_0, \qquad q(t_1) \in N_1, \tag{22}$$

$$t_1 > 0 \text{ fixed}, \tag{23}$$

$$J(u) = \int_0^{t_1} \varphi(q(t), u(t))\, dt \to \min. \tag{24}$$

- Here N_0 and N_1 are given immersed submanifolds of the state space M.
- So the boundary points $q(0)$ and $q(t_1)$ are not fixed as before, but should belong to N_0 and N_1 respectively.

- If a trajectory $\widetilde{q}(t)$ is optimal for this problem, then it is optimal as well for the problem with the fixed boundary points $\widetilde{q}(0)$, $\widetilde{q}(t_1)$ considered before.
- Consequently, the statement of Theorem 5 should be satisfied for $\widetilde{q}(t)$.
- But now we need additional conditions that select boundary points $\widetilde{q}(0) \in N_0$ and $\widetilde{q}(t_1) \in N_1$.
- It is reasonable to expect that they should be determined by $(\dim N_0 + \dim N_1)$ scalar equalities.
- Such conditions can easily be formulated in the Hamiltonian framework, they are called *transversality conditions*, see (29) below.

Theorem 7

Let $\widetilde{u}(t)$, $t \in [0, t_1]$, *be an optimal control in problem* (21)–(24). *Define a family of Hamiltonians:*

$$h_u^\nu(\lambda) = \langle \lambda, f_u(q) \rangle + \nu\varphi(q, u), \qquad \lambda \in T_q^*M,\ q \in M, \quad \nu \in \mathbb{R},\ u \in U.$$

Then there exists a Lipschitzian curve $\lambda_t \in T^*_{\widetilde{q}(t)}M$, $t \in [0, t_1]$, *and a number* $\nu \in \mathbb{R}$ *such that:*

$$\dot{\lambda}_t = \overrightarrow{h^\nu_{\widetilde{u}(t)}}(\lambda_t), \tag{25}$$

$$h^\nu_{\widetilde{u}(t)}(\lambda_t) = \max_{u \in U} h^\nu_u(\lambda_t), \tag{26}$$

$$(\lambda_t, \nu) \neq (0, 0), \qquad t \in [0, t_1], \tag{27}$$

$$\nu \leq 0, \tag{28}$$

$$\lambda_0 \perp T_{\widetilde{q}(0)} N_0, \qquad \lambda_{t_1} \perp T_{\widetilde{q}(t_1)} N_1. \tag{29}$$

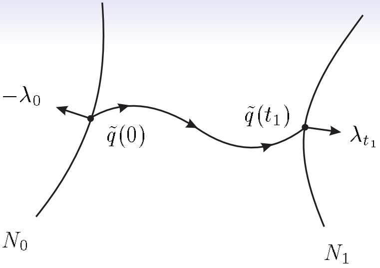

Figure: Transversality conditions (29)

- Any linear functional on a linear space acts naturally on a subspace by restriction, so transversality conditions (29) read respectively as follows:

$$\langle \lambda_0, v \rangle = 0, \quad v \in T_{\tilde{q}(0)} N_0, \qquad \langle \lambda_{t_1}, w \rangle = 0, \quad w \in T_{\tilde{q}(t_1)} N_1.$$

- The problem with free time: (21), (22), (24), is reduced to the case of fixed t_1 as before, so for this problem holds the previous theorem with the additional condition $h^{\nu}_{\tilde{u}(t)}(\lambda_t) \equiv 0$.

- The scheme of proof developed in previous versions of PMP can be applied to much more general problems after appropriate modifications. Now we only indicate how the proofs of these theorems should be changed in order to cover the new boundary conditions $q(0) \in N_0$, $q(t_1) \in N_1$.
- (1) First consider the special case where the initial point is fixed: let $N_0 = \{q_0\}$ for some point $q_0 \in M$.
- As in the proof of PMP for optimal control problems with two-point boundary condition, we introduce an extended system on $\mathbb{R} \times M$:

$$\widehat{q} = \begin{pmatrix} y \\ q \end{pmatrix} \in \mathbb{R} \times M, \qquad \widehat{q}(0) = \widehat{q}_0 = \begin{pmatrix} 0 \\ q_0 \end{pmatrix},$$

$$\widehat{f}_u(q) = \begin{pmatrix} \varphi(q,u) + v \\ f_u(q) \end{pmatrix} \in T_{(y,\widehat{q})}(\mathbb{R} \times M) = \mathbb{R} \times T_q M,$$

$$\frac{d\,\widehat{q}}{d\,t} = \widehat{f}_u(q) \quad \Leftrightarrow \quad \begin{cases} \dot{y} = \varphi(q,u) + v, \\ \dot{q} = f_u(q). \end{cases} \tag{30}$$

- Further, in the case of fixed terminal point $q(t_1)$, the necessary condition for optimality of the trajectory $q_{\widetilde{u}}(t)$ was the following:

$$\widehat{q}_1 \in \partial \widehat{\mathcal{A}}_{\widehat{q}_0}(t_1). \tag{31}$$

  Here $\widehat{\mathcal{A}}$ is the attainable set of the extended system (30) and $\widehat{q}_1 = \widehat{q}_{\widetilde{u}}(t_1)$.
- Now, when the target manifold N_1 is not a point, we should modify the argument. In a sense, we reduce the target manifold to a point defining it locally by an equation $\Phi = 0$.
- Choose a submersion

$$\Phi \, : \, O_{q_{\widetilde{u}(t_1)}} \to \mathbb{R}^p, \qquad p = \dim M - \dim N_1,$$

  of a small neighborhood $O_{q_{\widetilde{u}(t_1)}} \subset M$, so that

$$\Phi^{-1}(0) = N_1 \cap O_{q_{\widetilde{u}(t_1)}}.$$

- Further, extend the submersion: define the mapping

$$\widehat{\Phi} : \mathbb{R} \times O_{q_{\widetilde{u}(t_1)}} \to \mathbb{R}^{1+p}, \qquad \widehat{\Phi} \begin{pmatrix} y \\ q \end{pmatrix} = \begin{pmatrix} y \\ \Phi(q) \end{pmatrix}.$$

- Since the control $\widetilde{u}(t)$ is optimal in our problem (21)–(24), then

$$\widehat{\Phi}(\widehat{q}_1) \in \partial \widehat{\Phi}(\widehat{\mathcal{A}}_{\widehat{q}_0}(t_1)). \tag{32}$$

  So we replace the necessary optimality condition (31) by (32) and return to the scheme of proof of PMP for problems with two-point boundary condition.

- Take any $k \in \mathbb{N}$ and any needle-like variation of the optimal control:

$$u_s(t), \quad s \in \mathbb{R}^k_+, \qquad u_0(t) = \widetilde{u}(t), \quad t \in [0, t_1].$$

- Define the mappings

$$G : \mathbb{R}^k \to \mathbb{R} \times M, \qquad G(s) = \widehat{q}_{u_s}(t_1) = \widehat{q}_0 \circ \overrightarrow{\exp} \int_0^{t_1} \widehat{f}_{u_s(t)}\, dt, \tag{33}$$

$$F : \mathbb{R}^k \to \mathbb{R}^{1+p}, \qquad F(s) = \widehat{\Phi}(G(s)) = \widehat{q}_0 \circ \overrightarrow{\exp} \int_0^{t_1} \widehat{f}_{u_s(t)}\, dt \circ \widehat{\Phi}. \tag{34}$$

- Then it follows from inclusion (32) that

$$\widehat{\Phi}(\widehat{q}_1) = F(0) \in \partial F(\mathbb{R}^k_+). \tag{35}$$

- By the first auxiliary lemma for the geometric statement of PMP,

$$F_0'(\mathbb{R}^k_+) = \operatorname{cone}\left\{ \left.\frac{\partial F}{\partial s_i}\right|_0 \mid i = 1, \dots, k \right\} \neq \mathbb{R}^{1+p},$$

thus there exists a plane of support, i.e.,

$$\exists\ \widehat{\xi} \in \left(\mathbb{R}^{1+p}\right)^*, \quad \widehat{\xi} \neq 0,$$

such that

$$\left\langle \widehat{\xi}, \left.\frac{\partial F}{\partial s_i}\right|_0 \right\rangle \leq 0, \qquad i = 1, \dots, k. \tag{36}$$

- We compute the derivative by the chain rule:

$$\left.\frac{\partial F}{\partial s_i}\right|_0 = \widehat{\Phi}_* \left.\frac{\partial G}{\partial s_i}\right|_0, \tag{37}$$

and rewrite inequalities (36) as follows:

$$\left\langle \widehat{\Phi}^* \widehat{\xi}, \left.\frac{\partial G}{\partial s_i}\right|_0 \right\rangle = \left\langle \widehat{\xi}, \widehat{\Phi}_* \left.\frac{\partial G}{\partial s_i}\right|_0 \right\rangle \le 0, \qquad i = 1, \ldots, k. \tag{38}$$

- Then we denote the covector

$$\widehat{\lambda}_{t_1} = \widehat{\Phi}^* \widehat{\xi} = \begin{pmatrix} \nu \\ \lambda_{t_1} \end{pmatrix} \in T_{\widehat{q}_1}(\mathbb{R} \times M) \tag{39}$$

and obtain conclusions (25)–(28) in the same way as in PMP for optimal control problems with two-point boundary condition.

- The only distinction now is that the covector $\widehat{\lambda}_{t_1}$ is not arbitrary: equality (39) implies the second of the transversality conditions (29).
- Indeed, we have

$$\lambda_{t_1} = \Phi^* \xi, \qquad \xi \in \left(\mathbb{R}^p\right)^*,$$

thus

$$\langle \lambda_{t_1}, T_{q_{\widetilde{u}}(t_1)} N_1 \rangle = \langle \Phi^* \xi, T_{q_{\widetilde{u}}(t_1)} N_1 \rangle = \langle \xi, \underbrace{\Phi_* T_{q_{\widetilde{u}}(t_1)} N_1}_{=0} \rangle = 0.$$

- The first transversality condition (29) is now trivially satisfied, so the proof of this theorem in the case $N_0 = \{q_0\}$ is complete.

- (2) Let now the initial manifold N_0 be an arbitrary immersed submanifold of M. We can modify the scheme presented above to cover this case as well. Since now the initial point $q(0)$ is not fixed, we add variations of $q(0)$.
- Replace mappings (33), (34) by the following ones:

$$G : N_0 \times \mathbb{R}^k \to \mathbb{R} \times M, \qquad G(q,s) = \widehat{q} \circ \overrightarrow{\exp} \int_0^{t_1} \widehat{f}_{u_s(t)}\, dt,$$

$$F : N_0 \times \mathbb{R}^k \to \mathbb{R}^{1+p}, \qquad F(q,s) = \widehat{\Phi}(G(q,s)) = \widehat{q} \circ \overrightarrow{\exp} \int_0^{t_1} \widehat{f}_{u_s(t)}\, dt \circ \widehat{\Phi},$$

  where $\widehat{q} = (0,q) \in \mathbb{R} \times M$.
- Then the necessary optimality condition (35) is replaced by the inclusion

$$F(\widetilde{q}(0), 0) \in \partial F(N_0 \times \mathbb{R}^k_+). \tag{40}$$

- Apply the first auxiliary lemma before geometric statement of PMP to restriction of the mapping F to the space

$$\mathbb{R}^m \cong O_{\widetilde{q}(0)} \times \mathbb{R}^k, \qquad m = l + k, \quad l = \dim N_0,$$

where $O_{\widetilde{q}(0)} \subset N_0$ is a small neighborhood of $\widetilde{q}(0)$.
- By the remark after that lemma, inclusion (40) implies that

$$F'_{(\widetilde{q}(0),0)}(\mathbb{R}^l \oplus \mathbb{R}^k_+) \neq \mathbb{R}^{1+p},$$

i.e., there exists a covector

$$\widehat{\xi} \in \left(\mathbb{R}^{1+p}\right)^*, \qquad \widehat{\xi} \neq 0, \quad \widehat{\xi} = \begin{pmatrix} \nu \\ \xi \end{pmatrix},$$

such that

$$\left\langle \widehat{\xi}, \frac{\partial F}{\partial q} v \right\rangle \leq 0, \qquad v \in T_{\widetilde{q}(0)} N_0,$$
$$\left\langle \widehat{\xi}, \frac{\partial F}{\partial s_i} \right\rangle \leq 0, \qquad i = 1, \ldots, k. \tag{41}$$

- In the first inequality v belongs to a linear space, thus it turns into equality:

$$\left\langle \widehat{\xi}, \frac{\partial F}{\partial q} v \right\rangle = 0, \qquad v \in T_{\widetilde{q}(0)} N_0. \tag{42}$$

- Compute by Leibniz rule the partial derivative:

$$\begin{aligned}
\left.\frac{\partial F}{\partial q}\right|_{(\widetilde{q}(0),0)} &: \ T_{\widetilde{q}(0)} N_0 \to \mathbb{R}^{1+p}, \\
\left.\frac{\partial F}{\partial q}\right|_{(\widetilde{q}(0),0)} v &= \begin{pmatrix} 0 \\ v \end{pmatrix} \circ \overrightarrow{\exp} \int_0^{t_1} \widehat{f}_{\widetilde{u}(t)}\, dt \circ \widehat{\Phi} = \begin{pmatrix} 0 \\ v \circ P^{t_1} \circ \Phi \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ \Phi_* P^{t_1}_* v \end{pmatrix}, \qquad v \in T_{\widetilde{q}(0)} N_0,
\end{aligned}$$

where $P^{t_1} = \overrightarrow{\exp} \int_0^{t_1} f_{\widetilde{u}(t)}\, dt$.

- Then conditions (42), (41) read as follows:

$$\left\langle \xi, \Phi_* P_*^{t_1} v \right\rangle = 0, \qquad v \in T_{\widetilde{q}(0)} N_0, \tag{43}$$

$$\left\langle \widehat{\Phi}^* \widehat{\xi}, \left. \frac{\partial G}{\partial s_i} \right|_{(\widetilde{q}(0),0)} \right\rangle \leq 0, \qquad i = 1, \dots, k.$$

- As before, introduce the covector $\widehat{\lambda}_{t_1} = (\nu, \lambda_{t_1})$ by equality (39), then conclusions (25)–(28) of this theorem and the second transversality condition (29) follows.

- The first transversality condition is also satisfied: equality (43) can be rewritten as

$$\left\langle \lambda_{t_1}, P_*^{t_1} v \right\rangle = 0, \qquad v \in T_{\widetilde{q}(0)} N_0.$$

- But $\lambda_0 = P_{t_1}^* \lambda_{t_1}$, thus

$$\langle \lambda_0, v \rangle = \left\langle P_{t_1}^* \lambda_{t_1}, v \right\rangle = 0, \qquad v \in T_{\widetilde{q}(0)} N_0.$$

- The theorem is completely proved.

Plan of this lecture

1. Proof of the geometric statement of PMP with fixed terminal time
2. Geometric statement of PMP for free time
3. PMP for optimal control problems
4. PMP with transversality conditions