Differential Forms and Symplectic Geometry

(Lecture 5)

Yuri Sachkov

yusachkov@gmail.com

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Lomonosov Moscow State University

Plan of previous lecture

1. Autonomous vector fields
2. Action of diffeomorphisms on vector fields
3. Commutation of flows
4. Variations formula
5. Derivative of flow with respect to parameter
6. Differential 1-forms

Plan of this lecture

1. Differential 1-forms
2. Differential k-forms
3. Exterior differential
4. Lie derivative of differential forms
5. Liouville form and symplectic form

Differential 1-forms

Linear forms

- E a real vector space of finite dimension n.
- A *linear form* on E is a linear function $\xi : E \to \mathbb{R}$.
- The set of linear forms on E has a natural structure of a vector space called the *dual space* to E and denoted by E^*.
- If vectors $e_1, \dots, e_n$ form a basis of E, then the corresponding *dual basis* of E^* is formed by the covectors $e_1^*, \dots, e_n^*$ such that

$$\langle e_i^*, e_j \rangle = \delta_{ij}, \qquad i,\ j = 1, \dots n.$$

- So the dual space has the same dimension as the initial one:

$$\dim E^* = n = \dim E.$$

Cotangent bundle

- M a smooth manifold and T_qM its tangent space at a point $q \in M$.
- The space of linear forms on T_qM, i.e., the dual space $(T_qM)^*$ to T_qM, is called the *cotangent space* to M at q and is denoted as T_q^*M.
- The disjoint union of all cotangent spaces is called the *cotangent bundle* of M:

$$T^*M \stackrel{\text{def}}{=} \bigsqcup_{q \in M} T_q^*M.$$

- The set T^*M has a natural structure of a smooth manifold of dimension $2n$, where $n = \dim M$.
- Local coordinates on T^*M are constructed from local coordinates on M.
- Let $O \subset M$ be a coordinate neighborhood and let

$$\Phi : O \to \mathbb{R}^n, \qquad \Phi(q) = (x_1(q), \ldots, x_n(q)),$$

be a local coordinate system.

- Differentials of the coordinate functions

$$dx_i|_q \in T_q^*M, \qquad i = 1, \dots, n, \quad q \in O,$$

form a basis in the cotangent space T_q^*M.

- The dual basis in the tangent space T_qM is formed by the vectors

$$\left.\frac{\partial}{\partial x_i}\right|_q \in T_qM, \qquad i = 1, \dots, n, \quad q \in O,$$

$$\left\langle dx_i, \frac{\partial}{\partial x_j} \right\rangle \equiv \delta_{ij}, \qquad i,\ j = 1, \dots, n.$$

- Any linear form $\xi \in T_q^*M$ can be decomposed via the basis forms:

$$\xi = \sum_{i=1}^{n} \xi_i \, dx_i.$$

- So any covector $\xi \in T^*M$ is characterized by n coordinates $(x_1, \dots, x_n)$ of the point $q \in M$ where ξ is attached, and by n coordinates $(\xi_1, \dots, \xi_n)$ of the linear form ξ in the basis $dx_1, \dots, dx_n$.

- Mappings of the form
$$\xi \mapsto (\xi_1, \ldots, \xi_n; x_1, \ldots, x_n)$$
define local coordinates on the cotangent bundle. Consequently, T^*M is a $2n$-dimensional manifold.
- Coordinates of the form (ξ, x) are called *canonical coordinates* on T^*M.

- If $F : M \to N$ is a smooth mapping between smooth manifolds, then the differential
$$F_* : T_q M \to T_{F(q)} N$$
has the adjoint (dual) mapping
$$F^* \stackrel{\text{def}}{=} (F_*)^* : T^*_{F(q)} N \to T^*_q M$$
defined as follows:
$$F^*\xi = \xi \circ F_*, \qquad \xi \in T^*_{F(q)} N,$$
$$\langle F^*\xi, v\rangle = \langle \xi, F_* v\rangle, \qquad v \in T_q M.$$
- A vector $v \in T_q M$ is pushed forward by the differential F_* to the vector $F_* v \in T_{F(q)} N$, while a covector $\xi \in T^*_{F(q)} N$ is pulled back to the covector $F^*\xi \in T^*_q M$.
- So a smooth mapping $F : M \to N$ between manifolds induces a smooth mapping $F^* : T^*N \to T^*M$ between their cotangent bundles.

Differential 1-forms

- A *differential 1-form* on M is a smooth mapping $q \mapsto \omega_q \in T_q^* M, \;\; q \in M$, i.e, a family $\omega = \{\omega_q\}$ of linear forms on the tangent spaces $T_q M$ smoothly depending on the point $q \in M$.
- The set of all differential 1-forms on M has a natural structure of an infinite-dimensional vector space denoted as $\Lambda^1 M$.
- Like linear forms on a vector space are dual objects to vectors of the space, differential forms on a manifold are dual objects to smooth curves in the manifold.
- The pairing operation is the *integral* of a differential 1-form $\omega \in \Lambda^1 M$ along a smooth oriented curve $\gamma : [t_0, t_1] \to M$, defined as follows:

$$\int_\gamma \omega \stackrel{\text{def}}{=} \int_{t_0}^{t_1} \langle \omega_{\gamma(t)}, \dot\gamma(t) \rangle \, dt.$$

- The integral of a 1-form along a curve does not change under orientation-preserving smooth reparametrizations of the curve and changes its sign under change of orientation.

Differential k-forms

- A differential k-form on M is an object to integrate over k-dim. surfaces in M.
- Infinitesimally, a k-dimensional surface is presented by its tangent space, i.e., a k-dimensional subspace in T_qM.
- We need a dual object to the set of k-dim. subspaces in the linear space.
- Fix a linear space E.
- A k-dimensional subspace is defined by its basis $v_1, \dots, v_k \in E$.
- The dual objects should be mappings

$$(v_1, \dots, v_k) \mapsto \omega(v_1, \dots, v_k) \in \mathbb{R}$$

  such that $\omega(v_1, \dots, v_k)$ depend only on the linear hull $\operatorname{span}\{v_1, \dots, v_k\}$ and the oriented volume of the k-dimensional parallelepiped generated by $v_1, \dots, v_k$.
- Moreover, the dependence on the volume should be linear.
- Recall that the ratio of volumes of the parallelepipeds generated by vectors $w_i = \sum_{j=1}^{k} \alpha_{ij} v_j$, $i = 1, \dots, k$, and the vectors $v_1, \dots, v_k$, equals $\det(\alpha_{ij})_{i,j=1}^{k}$, and that determinant of a $k \times k$ matrix is a multilinear skew-symmetric form of the columns of the matrix.

Exterior k-forms

- Let E be a finite-dimensional real vector space, $\dim E = n$, and let $k \in \mathbb{N}$.
- An *exterior k-form* on E is a mapping

$$\omega : \underbrace{E \times \cdots \times E}_{k \text{ times}} \to \mathbb{R},$$

which is multilinear:

$$\begin{aligned} &\omega(v_1, \ldots, \alpha_1 v_i^1 + \alpha_2 v_i^2, \ldots, v_k) \\ &\qquad = \alpha_1 \omega(v_1, \ldots, v_i^1, \ldots, v_k) + \alpha_2 \omega(v_1, \ldots, v_i^2, \ldots, v_k), \qquad \alpha_1,\ \alpha_2 \in \mathbb{R}, \end{aligned}$$

and skew-symmetric:

$$\omega(v_1, \ldots, v_i, \ldots, v_j, \ldots, v_k) = -\omega(v_1, \ldots, v_j, \ldots, v_i, \ldots, v_k), \quad i,\ j = 1, \ldots, k.$$

- The set of all exterior k-forms on E is denoted by $\Lambda^k E$.
- By the skew-symmetry, any exterior form of order $k > n$ is zero, thus $\Lambda^k E = \{0\}$ for $k > n$.

- Exterior forms can be multiplied by real numbers, and exterior forms of the same order k can be added one with another, so each $\Lambda^k E$ is a vector space.
- We construct a basis of $\Lambda^k E$ after we consider another operation between exterior forms — the exterior product.
- The exterior product of two forms $\omega_1 \in \Lambda^{k_1} E$, $\omega_2 \in \Lambda^{k_2} E$ is an exterior form $\omega_1 \wedge \omega_2$ of order $k_1 + k_2$.
- Given linear 1-forms $\omega_1, \omega_2 \in \Lambda^1 E$, we have a natural (tensor) product for them:

$$\omega_1 \otimes \omega_2 \ : \ (v_1, v_2) \mapsto \omega_1(v_1)\omega_2(v_2), \qquad v_1, v_2 \in E.$$

- The result is a bilinear but not a skew-symmetric form.
- The *exterior product* is the anti-symmetrization of the tensor one:

$$\omega_1 \wedge \omega_2 \ : \ (v_1, v_2) \mapsto \omega_1(v_1)\omega_2(v_2) - \omega_1(v_2)\omega_2(v_1), \qquad v_1, v_2 \in E.$$

- Similarly, the tensor and exterior products of forms $\omega_1 \in \Lambda^{k_1} E$ and $\omega_2 \in \Lambda^{k_2} E$ are the following forms of order $k_1 + k_2$:

$$\omega_1 \otimes \omega_2 \ : \ (v_1, \ldots, v_{k_1+k_2}) \mapsto \omega_1(v_1, \ldots, v_{k_1})\omega_2(v_{k_1+1}, \ldots, v_{k_1+k_2}),$$

$$\omega_1 \wedge \omega_2 \ : \ (v_1, \ldots, v_{k_1+k_2}) \mapsto$$

$$\frac{1}{k_1!\, k_2!} \sum_{\sigma} (-1)^{\nu(\sigma)} \omega_1(v_{\sigma(1)}, \ldots, v_{\sigma(k_1)})\omega_2(v_{\sigma(k_1+1)}, \ldots, v_{\sigma(k_1+k_2)}), \qquad (1)$$

where the sum is taken over all permutations σ of order $k_1 + k_2$ and $\nu(\sigma)$ is parity of a permutation σ.

- The factor $\frac{1}{k_1!\, k_2!}$ normalizes the sum in (1) since it contains $k_1!\, k_2!$ identically equal terms: e.g., if permutations σ do not mix the first k_1 and the last k_2 arguments, then all terms of the form

$$(-1)^{\nu(\sigma)} \omega_1(v_{\sigma(1)}, \ldots, v_{\sigma(k_1)})\omega_2(v_{\sigma(k_1+1)}, \ldots, v_{\sigma(k_1+k_2)})$$

are equal to

$$\omega_1(v_1, \ldots, v_{k_1})\omega_2(v_{k_1+1}, \ldots, v_{k_1+k_2}).$$

- This guarantees the associative property of the exterior product:

$$\omega_1 \wedge (\omega_2 \wedge \omega_3) = (\omega_1 \wedge \omega_2) \wedge \omega_3, \qquad \omega_i \in \Lambda^{k_i} E,$$

- Further, the exterior product is skew-commutative:

$$\omega_2 \wedge \omega_1 = (-1)^{k_1 k_2} \omega_1 \wedge \omega_2, \qquad \omega_i \in \Lambda^{k_i} E.$$

- Let $e_1, \dots, e_n$ be a basis of the space E and $e_1^*, \dots, e_n^*$ the corresponding dual basis of E^*.
- If $1 \le k \le n$, then the following $C_n^k = \frac{n!}{k!(n-k)!}$ elements form a basis of the space $\Lambda^k E$:

$$e_{i_1}^* \wedge \dots \wedge e_{i_k}^*, \qquad 1 \le i_1 < i_2 < \dots < i_k \le n.$$

- The equalities

$$(e_{i_1}^* \wedge \ldots \wedge e_{i_k}^*)(e_{i_1}, \ldots, e_{i_k}) = 1,$$
$$(e_{i_1}^* \wedge \ldots \wedge e_{i_k}^*)(e_{j_1}, \ldots, e_{j_k}) = 0, \quad \text{if } (i_1, \ldots, i_k) \neq (j_1, \ldots, j_k)$$

for $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ imply that any k-form $\omega \in \Lambda^k E$ has a unique decomposition of the form

$$\omega = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} \omega_{i_1 \ldots i_k} e_{i_1}^* \wedge \ldots \wedge e_{i_k}^*$$

with

$$\omega_{i_1 \ldots i_k} = \omega(e_{i_1}, \ldots, e_{i_k}).$$

Exercise 1

Show that for any 1-forms $\omega_1, \ldots \omega_p \in \Lambda^1 E$ and any vectors $v_1, \ldots, v_p \in E$ there holds the equality

$$(\omega_1 \wedge \ldots \wedge \omega_p)(v_1, \ldots, v_p) = \det \left(\langle \omega_i, v_j \rangle \right)_{i,j=1}^p . \tag{2}$$

- Notice that the space of n-forms of an n-dimensional space E is one-dimensional.
- Any nonzero n-form on E is called a *volume form*.
- For example, the value of the standard volume form $e_1^* \wedge \ldots \wedge e_n^*$ on an n-tuple of vectors $(v_1, \ldots, v_n)$ is

$$(e_1^* \wedge \ldots \wedge e_n^*)(v_1, \ldots, v_n) = \det\left(\langle e_i^*, v_j \rangle\right)_{i,j=1}^n,$$

the oriented volume of the parallelepiped generated by the vectors $v_1, \ldots, v_n$.

- A *differential k-form* on M is a mapping

$$\omega \ : \ q \mapsto \omega_q \in \Lambda^k T_q M, \qquad q \in M,$$

  smooth w.r.t. $q \in M$.
- The set of all differential k-forms on M is denoted by $\Lambda^k M$.
- It is natural to consider smooth functions on M as 0-forms, so $\Lambda^0 M = C^\infty(M)$.
- In local coordinates $(x_1, \ldots, x_n)$ on a domain $O \subset M$, any differential k-form $\omega \in \Lambda^k M$ can be uniquely decomposed as follows:

$$\omega_x = \sum_{i_1 < \cdots < i_k} a_{i_1 \ldots i_k}(x) dx_{i_1} \wedge \ldots \wedge dx_{i_k}, \qquad x \in O, \quad a_{i_1 \ldots i_k} \in C^\infty(O). \quad (3)$$

- Any smooth mapping $F : M \to N$ induces a mapping of differential forms $\widehat{F} : \Lambda^k N \to \Lambda^k M$ in the following way: given a differential k-form $\omega \in \Lambda^k N$, the k-form $\widehat{F}\omega \in \Lambda^k M$ is defined as

$$(\widehat{F}\omega)_q(v_1, \dots, v_k) = \omega_{F(q)}(F_* v_1, \dots, F_* v_k), \qquad q \in M, \ v_i \in T_q M.$$

- For 0-forms, pull-back is a substitution of variables:

$$\widehat{F}a(q) = a \circ F(q), \qquad a \in C^\infty(M), \quad q \in M.$$

- The pull-back $\widehat{F}$ is linear w.r.t. forms and preserves the exterior product:

$$\widehat{F}(\omega_1 \wedge \omega_2) = \widehat{F}\omega_1 \wedge \widehat{F}\omega_2.$$

Exercise 2

Prove the composition law for pull-back of differential forms:

$$\widehat{F_2 \circ F_1} = \widehat{F}_1 \circ \widehat{F}_2, \tag{4}$$

where $F_1 : M_1 \to M_2$ and $F_2 : M_2 \to M_3$ are smooth mappings.

- Now we can define the integral of a k-form over an oriented k-dimensional surface.
- Let $\Pi \subset \mathbb{R}^k$ be a k-dimensional open oriented domain and $\Phi : \Pi \to \Phi(\Pi) \subset M$ a diffeomorphism.
- Then the *integral* of a k-form $\omega \in \Lambda^k M$ over the k-dimensional oriented surface $\Phi(\Pi)$ is defined as follows:

$$\int_{\Phi(\Pi)} \omega \overset{\mathrm{def}}{=} \int_{\Pi} \widehat{\Phi}\omega,$$

  it remains only to define the integral over Π in the right-hand side.
- Since $\widehat{\Phi}\omega \in \Lambda^k \mathbb{R}^k$ is a k-form on $\mathbb{R}^k$, it is expressed via the standard volume form $dx_1 \wedge \ldots \wedge dx_k \in \Lambda^k \mathbb{R}^k$:

$$(\widehat{\Phi}\omega)_x = a(x)\, dx_1 \wedge \cdots \wedge dx_k, \qquad x \in \Pi.$$

- We set

$$\int_{\Pi} \widehat{\Phi}\omega \overset{\mathrm{def}}{=} \int_{\Pi} a(x)\, dx_1 \ldots dx_k,$$

  a usual multiple integral.

- The integral $\int_{\Phi(\Pi)} \omega$ is defined correctly with respect to orientation-preserving reparametrizations of the surface $\Phi(\Pi)$.
- Although, if a parametrization changes orientation, then the integral changes sign.
- The notion of integral is extended to arbitrary submanifolds as follows.
- Let $N \subset M$ be a k-dimensional submanifold and let $\omega \in \Lambda^k M$.
- Consider a covering of N by coordinate neighborhoods $O_i \subset M$:

$$N = \bigcup_i (N \cap O_i).$$

- Take a partition of unity subordinated to this covering:

$$\alpha_i \in C^\infty(M), \quad \operatorname{supp} \alpha_i \subset O_i, \quad 0 \leq \alpha_i \leq 1,$$
$$\sum_i \alpha_i \equiv 1.$$

- Then

$$\int_N \omega \stackrel{\text{def}}{=} \sum_i \int_{N \cap O_i} \alpha_i \omega.$$

- The integral thus defined does not depend upon the choice of partition of unity.

Exterior differential

- Exterior differential of a function (i.e., a 0-form) is a 1-form: if $a \in C^\infty(M) = \Lambda^0 M$, then its differential $d_q a \in T_q^* M$ is the functional (directional derivative)
$$\langle d_q a, v \rangle = va, \qquad v \in T_q M, \tag{5}$$
so $da \in \Lambda^1 M$.
- By the Newton-Leibniz formula, if $\gamma \subset M$ is a smooth oriented curve starting at a point $q_0 \in M$ and terminating at $q_1 \in M$, then
$$\int_\gamma da = a(q_1) - a(q_0).$$
- The right-hand side can be considered as the integral of the function a over the oriented boundary of the curve: $\partial\gamma = q_1 - q_0$, thus
$$\int_\gamma da = \int_{\partial\gamma} a. \tag{6}$$

- In the exposition above, Newton-Leibniz formula (6) comes as a consequence of definition (5) of differential of a function. But one can go the reverse way: if we postulate Newton-Leibniz formula (6) for any smooth curve $\gamma \subset M$ and pass to the limit $q_1 \to q_0$, we necessarily obtain definition (5) of differential of a function.
- Such approach can be realized for higher order differential forms as well.
- Let $\omega \in \Lambda^k M$. We define the *exterior differential*

$$d\omega \in \Lambda^{k+1} M$$

as the differential $(k+1)$-form for which Stokes formula holds:

$$\int_N d\omega = \int_{\partial N} \omega \tag{7}$$

for $(k+1)$-dimensional submanifolds with boundary $N \subset M$ (for simplicity, one can take here N equal to a diffeomorphic image of a $(k+1)$-dimensional polytope).
- The boundary ∂N is oriented by a frame of tangent vectors $e_1, \dots e_k \in T_q(\partial N)$ in such a way that the frame $e_{\text{norm}}, e_1, \dots, e_k \in T_q N$ define a positive orientation of N, where e_{norm} is the outward normal vector to N at q.

- The existence of a form $d\omega$ that satisfies Stokes formula (7) comes from the fact that the mapping $N \mapsto \int_{\partial N} \omega$ is additive w.r.t. domain: if $N = N_1 \cup N_2$, $N_1 \cap N_2 = \partial N_1 \cap \partial N_2$, then

$$\int_{\partial N} \omega = \int_{\partial N_1} \omega + \int_{\partial N_2} \omega$$

  (notice that orientation of the boundaries is coordinated: ∂N_1 and ∂N_2 have mutually opposite orientations at points of their intersection).
- Thus the integral $\int_{\partial N} \omega$ is a kind of measure w.r.t. N, and one can recover $(d\omega)_q$ passing to limit in (7) as the submanifold N contracts to a point q.

Exterior differential

- First of all, it is obvious from the Stokes formula that $d : \Lambda^k M \to \Lambda^{k+1} M$ is a linear operator.
- Further, if $F : M \to N$ is a diffeomorphism, then

$$d\widehat{F}\omega = \widehat{F}d\omega, \qquad \omega \in \Lambda^k N. \tag{8}$$

- Indeed, if $W \subset M$, then

$$\int_{F(W)} \omega = \int_W \widehat{F}\omega, \qquad \omega \in \Lambda^k N,$$

thus

$$\begin{aligned} \int_W d\widehat{F}\omega &= \int_{\partial W} \widehat{F}\omega = \int_{F(\partial W)} \omega = \int_{\partial F(W)} \omega = \int_{F(W)} d\omega \\ &= \int_W \widehat{F}d\omega, \end{aligned}$$

and equality (8) follows.

- Another basic property of exterior differential is given by the equality

$$d \circ d = 0,$$

which follows since $\partial(\partial N) = \emptyset$ for any submanifold with boundary $N \subset M$.

- Exterior differential is an antiderivation:

$$d(\omega_1 \wedge \omega_2) = (d\omega_1) \wedge \omega_2 + (-1)^{k_1} \omega_1 \wedge d\omega_2, \qquad \omega_i \in \Lambda^{k_i} M,$$

this equality is dual to the formula of boundary $\partial(N_1 \times N_2)$.

- In local coordinates exterior differential is computed as follows: if

$$\omega = \sum_{i_1 < \cdots < i_k} a_{i_1 \dots i_k} dx_{i_1} \wedge \dots \wedge dx_{i_k}, \qquad a_{i_1 \dots i_k} \in C^\infty,$$

then

$$d\omega = \sum_{i_1 < \cdots < i_k} (da_{i_1 \dots i_k}) \wedge dx_{i_1} \wedge \dots \wedge dx_{i_k},$$

this formula is forced by above properties of differential forms.

Lie derivative of differential forms

- The "infinitesimal version" of the pull-back $\widehat{P}$ of a differential form by a flow P is given by the following operation.
- *Lie derivative* of a differential form $\omega \in \Lambda^k M$ along a vector field $f \in \operatorname{Vec} M$ is the differential form $L_f \omega \in \Lambda^k M$ defined as follows:

$$L_f \omega \stackrel{\text{def}}{=} \left. \frac{d}{d\,\varepsilon} \right|_{\varepsilon=0} \widehat{e^{\varepsilon f}} \omega. \tag{9}$$

- Since

$$\widehat{e^{tf}}(\omega_1 \wedge \omega_2) = \widehat{e^{tf}} \omega_1 \wedge \widehat{e^{tf}} \omega_2,$$

Lie derivative L_f is a derivation of the algebra of differential forms:

$$L_f(\omega_1 \wedge \omega_2) = (L_f \omega_1) \wedge \omega_2 + \omega_1 \wedge L_f \omega_2.$$

- Further, we have

$$\widehat{e^{tf}} \circ d = d \circ \widehat{e^{tf}},$$

thus

$$L_f \circ d = d \circ L_f.$$

- For 0-forms, Lie derivative is just the directional derivative:

$$L_f a = fa, \qquad a \in C^\infty(M),$$

since $\widehat{e^{tf}} a = a \circ e^{tf}$ is a substitution of variables.

- Now we obtain a useful formula for the action of Lie derivative on differential forms of an arbitrary order.
- Consider, along with exterior differential

$$d : \Lambda^k M \to \Lambda^{k+1} M$$

the *interior product* of a differential form ω with a vector field $f \in \operatorname{Vec} M$:

$$i_f : \Lambda^k M \to \Lambda^{k-1} M,$$
$$(i_f \omega)(v_1, \dots, v_{k-1}) \stackrel{\text{def}}{=} \omega(f, v_1, \dots, v_{k-1}), \qquad \omega \in \Lambda^k M, \ v_i \in T_q M,$$

which acts as substitution of f for the first argument of ω. By definition, for 0-order forms

$$i_f a = 0, \qquad a \in \Lambda^0 M.$$

- Interior product is an antiderivation, as well as the exterior differential:

$$i_f(\omega_1 \wedge \omega_2) = (i_f\omega_1) \wedge \omega_2 + (-1)^{k_1}\omega_1 \wedge i_f\omega_2, \qquad \omega_i \in \Lambda^{k_i} M.$$

- Now we prove that Lie derivative of a differential form of an arbitrary order can be computed by the following formula:

$$L_f = d \circ i_f + i_f \circ d \tag{10}$$

called *Cartan's formula*, for short "$L = di + id$".

- Notice first of all that the right-hand side in (10) has the required order:

$$d \circ i_f + i_f \circ d \; : \; \Lambda^k M \to \Lambda^k M.$$

- Further, $d \circ i_f + i_f \circ d$ is a derivation as it is obtained from two antiderivations.

- Moreover, this derivation commutes with differential:

$$d \circ (d \circ i_f + i_f \circ d) = d \circ i_f \circ d,$$
$$(d \circ i_f + i_f \circ d) \circ d = d \circ i_f \circ d.$$

- Now we check the formula $L = di + id$ on 0-forms: if $a \in \Lambda^0 M$, then

$$(d \circ i_f)a = 0,$$
$$(i_f \circ d)a = \langle da, f \rangle = fa = L_f a.$$

  So the formula $L = di + id$ holds for 0-forms.
- The properties of the mappings L_f and $d \circ i_f + i_f \circ d$ established and the coordinate representation of differential forms reduce the general case of k-forms to the case of 0-forms.
- Cartan's formula $L = di + id$ is proved for k-forms.

- The differential definition (9) of Lie derivative can be integrated, i.e., there holds the following equality on $\Lambda^k M$:

$$\left(\overrightarrow{\exp} \int_0^t f_\tau \, d\tau\right)^{\widehat{\ \ \ }} = \overrightarrow{\exp} \int_0^t L_{f_\tau} \, d\tau, \tag{11}$$

in the following sense.

- Denote the flow $P_{t_0}^{t_1} = \overrightarrow{\exp} \displaystyle\int_{t_0}^{t_1} f_\tau \, d\tau$ of a nonautonomous vector field f_τ on M.
- The family of operators on differential forms $\widehat{P_0^t} : \Lambda^k M \to \Lambda^k M$ is a unique solution of the Cauchy problem

$$\frac{d}{d\,t}\widehat{P_0^t} = \widehat{P_0^t} \circ L_{f_t}, \qquad \widehat{P_0^t}\Big|_{t=0} = \mathrm{Id}, \tag{12}$$

compare with Cauchy problems for the flow P_0^t and for the family of operators $\operatorname{Ad} P_0^t$, and this solution is denoted as

$$\overrightarrow{\exp} \int_0^t L_{f_\tau} \, d\tau \stackrel{\text{def}}{=} \widehat{P_0^t} = \left(\overrightarrow{\exp} \int_0^t f_\tau \, d\tau\right)^{\widehat{\ \ \ }}.$$

- In order to verify the ODE in (12), we prove first the following equality for operators on forms:

$$\left.\frac{d}{d\,\varepsilon}\right|_{\varepsilon=0} \widehat{P_t^{t+\varepsilon}}\omega = L_{f_t}\omega, \qquad \omega \in \Lambda^k M. \tag{13}$$

- This equality is straightforward for 0-order forms:

$$\left.\frac{d}{d\,\varepsilon}\right|_{\varepsilon=0} \widehat{P_t^{t+\varepsilon}}a = \left.\frac{d}{d\,\varepsilon}\right|_{\varepsilon=0} a \circ P_t^{t+\varepsilon} = f_t a = L_{f_t} a, \qquad a \in C^\infty(M).$$

- Further, the both operators $\left.\frac{d}{d\,\varepsilon}\right|_{\varepsilon=0} \widehat{P_t^{t+\varepsilon}}$ and L_{f_t} commute with d and satisfy the Leibniz rule w.r.t. product of a function with a differential form.
- Then equality (13) follows for forms of arbitrary order, as in the proof of Cartan's formula.

- Now we easily verify the ODE in (12):

$$\frac{d}{d\,t}\widehat{P_0^t} = \frac{d}{d\,\varepsilon}\Big|_{\varepsilon=0}\widehat{P_0^{t+\varepsilon}} = \frac{d}{d\,\varepsilon}\Big|_{\varepsilon=0}\widehat{\left(P_0^t \circ P_t^{t+\varepsilon}\right)}$$

by the composition rule for pull-back of differential forms

$$= \frac{d}{d\,\varepsilon}\Big|_{\varepsilon=0}\widehat{P_0^t}\circ\widehat{P_t^{t+\varepsilon}} = \widehat{P_0^t}\circ\frac{d}{d\,\varepsilon}\Big|_{\varepsilon=0}\widehat{P_t^{t+\varepsilon}}$$
$$= \widehat{P_0^t}\circ L_{f_t}.$$

Exercise 3

Prove uniqueness for Cauchy problem (12).

- For an autonomous vector field $f \in \operatorname{Vec} M$, equality (11) takes the form

$$\widehat{e^{tf}} = e^{tL_f}.$$

- Notice that the Lie derivatives of differential forms L_f and vector fields $(-\operatorname{ad} f)$ are in a certain sense dual one to another, see equality (14) below.
- That is, the function

$$\langle \omega, X \rangle \ : \ q \mapsto \langle \omega_q, X(q) \rangle, \qquad q \in M,$$

defines a pairing of $\Lambda^1 M$ and $\operatorname{Vec} M$ over $C^\infty(M)$. Then the equality

$$\langle \widehat{P}\omega, X \rangle = P \langle \omega, \operatorname{Ad} P^{-1} X \rangle, \qquad P \in \operatorname{Diff} M, \ X \in \operatorname{Vec} M, \ \omega \in \Lambda^1 M,$$

has an infinitesimal version of the form

$$\langle L_Y \omega, X \rangle = Y \langle \omega, X \rangle - \langle \omega, (\operatorname{ad} Y) X \rangle, \qquad X, \ Y \in \operatorname{Vec} M, \ \ \omega \in \Lambda^1 M. \qquad (14)$$

- Taking into account Cartan's formula $L = di + id$, we immediately obtain the following important equality:

$$d\omega(Y, X) = Y \langle \omega, X \rangle - X \langle \omega, Y \rangle - \langle \omega, [Y, X] \rangle, \quad X, \ Y \in \operatorname{Vec} M, \ \omega \in \Lambda^1 M. \ (15)$$

- We have already seen that the cotangent bundle $T^*M = \sqcup_{q \in M} T_q^* M$ of an n-dimensional manifold M is a $2n$-dimensional manifold. Any local coordinates $x = (x_1, \ldots, x_n)$ on M determine canonical local coordinates on T^*M of the form $(\xi, x) = (\xi_1, \ldots, \xi_n; x_1, \ldots, x_n)$ in which any covector $\lambda \in T_{q_0}^* M$ has the decomposition $\lambda = \sum_{i=1}^n \xi_i \, dx_i|_{q_0}$.
- The *"tautological" 1-form* (or *Liouville 1-form*) on the cotangent bundle

$$s \in \Lambda^1(T^*M)$$

is defined as follows.
- Let $\lambda \in T^*M$ be a point in the cotangent bundle and $w \in T_\lambda(T^*M)$ a tangent vector to T^*M at λ.
- Denote by π the canonical projection from T^*M to M:

$$\pi : T^*M \to M,$$
$$\pi : \lambda \mapsto q, \qquad \lambda \in T_q^* M.$$

- Differential of π is a linear mapping

$$\pi_* : T_\lambda(T^*M) \to T_qM, \qquad q = \pi(\lambda).$$

- The tautological 1-form s at the point λ acts on the tangent vector w in the following way:

$$\langle s_\lambda, w \rangle \stackrel{\text{def}}{=} \langle \lambda, \pi_* w \rangle.$$

- That is, we project the vector $w \in T_\lambda(T^*M)$ to the vector $\pi_* w \in T_qM$, and then act by the covector $\lambda \in T_q^*M$.
- So

$$s_\lambda \stackrel{\text{def}}{=} \lambda \circ \pi_*.$$

- The title “tautological” is explained by the coordinate representation of the form s.
- In canonical coordinates (ξ, x) on T^*M, we have:

$$\lambda = \sum_{i=1}^{n} \xi_i dx_i, \tag{16}$$

$$w = \sum_{i=1}^{n} \alpha_i \frac{\partial}{\partial \xi_i} + \beta_i \frac{\partial}{\partial x_i}.$$

- The projection written in canonical coordinates

$$\pi \ : \ (\xi, x) \mapsto x$$

is a linear mapping, its differential acts as follows:

$$\pi_* \left(\frac{\partial}{\partial \xi_i} \right) = 0, \qquad i = 1, \dots, n,$$

$$\pi_* \left(\frac{\partial}{\partial x_i} \right) = \frac{\partial}{\partial x_i}, \qquad i = 1, \dots, n.$$

- Thus
$$\pi_* w = \sum_{i=1}^{n} \beta_i \frac{\partial}{\partial x_i},$$
consequently,
$$\langle s_\lambda, w \rangle = \langle \lambda, \pi_* w \rangle = \sum_{i=1}^{n} \xi_i \beta_i.$$
- But $\beta_i = \langle dx_i, w \rangle$, so the form s has in coordinates (ξ, x) exactly the same expression
$$s_\lambda = \sum_{i=1}^{n} \xi_i dx_i \tag{17}$$
as the covector λ, see (16).
- Although, definition of the form s does not depend on any coordinates.

Remark 1

In mechanics, the tautological form s is denoted as $p\,dq$.

- Consider the exterior differential of the 1-form s:

$$\sigma \stackrel{\text{def}}{=} ds.$$

- The differential 2-form $\sigma \in \Lambda^2(T^*M)$ is called the *canonical symplectic structure* on T^*M.
- In canonical coordinates, we obtain from (17):

$$\sigma = \sum_{i=1}^{n} d\xi_i \wedge dx_i. \tag{18}$$

- This expression shows that the form σ is nondegenerate, i.e., the bilinear skew-symmetric form

$$\sigma_\lambda : T_\lambda(T^*M) \times T_\lambda(T^*M) \to \mathbb{R}$$

has no kernel:

$$\sigma(w, \cdot) = 0 \quad \Rightarrow \quad w = 0, \qquad\qquad w \in T_\lambda(T^*M).$$

- In the following basis in the tangent space $T_\lambda(T^*M)$

$$\frac{\partial}{\partial x_1}, \frac{\partial}{\partial \xi_1}, \ \dots, \ \frac{\partial}{\partial x_n}, \frac{\partial}{\partial \xi_n},$$

the form σ_λ has the block matrix
$$\begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & 1 \\ & & & -1 & 0 \end{pmatrix}.$$

- The form σ is closed: $d\sigma = 0$ since it is exact: $\sigma = ds$, and $d \circ d = 0$.

Remarks

(1) A closed nondegenerate exterior differential 2-form on a $2n$-dimensional manifold is called a *symplectic structure*. A manifold with a symplectic structure is called a *symplectic manifold*. The cotangent bundle T^*M with the canonical symplectic structure σ is the most important example of a symplectic manifold.

(2) In mechanics, the 2-form σ is known as the form $dp \wedge dq$.

Plan of this lecture

1. Differential 1-forms
2. Differential k-forms
3. Exterior differential
4. Lie derivative of differential forms
5. Liouville form and symplectic form