

Elements of Chronological Calculus-1

(Lecture 3)

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Plan of previous lecture

1. Time-Optimal Problem
2. Smooth manifolds
3. Tangent space and tangent vector
4. Ordinary differential equations on manifolds

Plan of this lecture

1. Points, Diffeomorphisms, and Vector Fields
2. Seminorms and $C^\infty(M)$ -Topology
3. Families of Functionals and Operators
4. ODEs with discontinuous right-hand side
5. Definition of the right chronological exponential
6. Formal series expansion
7. Estimates and convergence of the series
8. Left chronological exponential

Points, Diffeomorphisms, and Vector Fields

- We identify points, diffeomorphisms, and vector fields on the manifold M with functionals and operators on the algebra $C^\infty(M)$ of all smooth real-valued functions on M .
- Addition, multiplication, and product with constants are defined in the *algebra* $C^\infty(M)$, as usual, pointwise: if $a, b \in C^\infty(M)$, $q \in M$, $\alpha \in \mathbb{R}$, then

$$(a + b)(q) = a(q) + b(q),$$

$$(a \cdot b)(q) = a(q) \cdot b(q),$$

$$(\alpha \cdot a)(q) = \alpha \cdot a(q).$$

- Any *point* $q \in M$ defines a *linear functional*

$$\hat{q} : C^\infty(M) \rightarrow \mathbb{R}, \quad \hat{q}a = a(q), \quad a \in C^\infty(M).$$

- The functionals $\hat{q}$ are homomorphisms of the algebras $C^\infty(M)$ and $\mathbb{R}$:

$$\begin{aligned}\hat{q}(a + b) &= \hat{q}a + \hat{q}b, & a, b \in C^\infty(M), \\ \hat{q}(a \cdot b) &= (\hat{q}a) \cdot (\hat{q}b), & a, b \in C^\infty(M), \\ \hat{q}(\alpha \cdot a) &= \alpha \cdot \hat{q}a, & \alpha \in \mathbb{R}, a \in C^\infty(M).\end{aligned}$$

- So to any point $q \in M$, there corresponds a nontrivial *homomorphism of algebras* $\hat{q} : C^\infty(M) \rightarrow \mathbb{R}$. It turns out that there exists an inverse correspondence.

Proposition 1

Let $\varphi : C^\infty(M) \rightarrow \mathbb{R}$ be a nontrivial homomorphism of algebras. Then there exists a point $q \in M$ such that $\varphi = \hat{q}$.

Proof.

[AS] A.A. Agrachev, Yu.L. Sachkov, *Control theory from the geometric viewpoint*. Springer-Verlag, 2004. □

- Not only the manifold M can be reconstructed as a set from the algebra $C^\infty(M)$. One can recover topology on M from the weak topology in the space of functionals on $C^\infty(M)$:

$$\lim_{n \rightarrow \infty} q_n = q \quad \text{if and only if} \quad \lim_{n \rightarrow \infty} \hat{q}_n a = \hat{q} a \quad \forall a \in C^\infty(M).$$

- Moreover, the smooth structure on M is also recovered from $C^\infty(M)$, actually, “by definition”: a real function on the set $\{\hat{q} \mid q \in M\}$ is smooth if and only if it has a form $\hat{q} \mapsto \hat{q} a$ for some $a \in C^\infty(M)$.
- Any *diffeomorphism* $P : M \rightarrow M$ defines an *automorphism of the algebra* $C^\infty(M)$:

$$\begin{aligned} \hat{P} : C^\infty(M) &\rightarrow C^\infty(M), & \hat{P} &\in \text{Aut}(C^\infty(M)), \\ (\hat{P}a)(q) &= a(P(q)), & q &\in M, \quad a \in C^\infty(M), \end{aligned}$$

i.e., $\hat{P}$ acts as a change of variables in a function a .

- Conversely, any automorphism of $C^\infty(M)$ has such a form.

Proposition 2

Any automorphism $A : C^\infty(M) \rightarrow C^\infty(M)$ has a form of $\widehat{P}$ for some $P \in \text{Diff } M$.

Proof.

Let $A \in \text{Aut}(C^\infty(M))$. Take any point $q \in M$. Then the composition

$$\widehat{q} \circ A : C^\infty(M) \rightarrow \mathbb{R}$$

is a nonzero homomorphism of algebras, thus it has the form $\widehat{q}_1$ for some $q_1 \in M$. We denote $q_1 = P(q)$ and obtain

$$\widehat{q} \circ A = \widehat{q}_1 = \widehat{P(q)} = \widehat{q} \circ \widehat{P} \quad \forall q \in M,$$

i.e.,

$$A = \widehat{P},$$

and P is the required diffeomorphism.



- Now we characterize *tangent vectors* to M as *functionals* on $C^\infty(M)$.
- Tangent vectors to M are velocity vectors to curves in M , and points of M are identified with linear functionals on $C^\infty(M)$; thus we should obtain linear functionals on $C^\infty(M)$, but not homomorphisms into $\mathbb{R}$.
- To understand, which functionals on $C^\infty(M)$ correspond to tangent vectors to M , take a smooth curve $q(t)$ of points in M . Then the corresponding curve of functionals $\widehat{q}(t) = \widehat{q(t)}$ on $C^\infty(M)$ satisfies the multiplicative rule

$$\widehat{q}(t)(a \cdot b) = \widehat{q}(t)a \cdot \widehat{q}(t)b, \quad a, b \in C^\infty(M).$$

- We differentiate this equality at $t = 0$ and obtain that the velocity vector to the curve of functionals

$$\xi \stackrel{\text{def}}{=} \left. \frac{d\widehat{q}}{dt} \right|_{t=0}, \quad \xi : C^\infty(M) \rightarrow \mathbb{R},$$

satisfies the Leibniz rule:

$$\xi(ab) = \xi(a)b(q(0)) + a(q(0))\xi(b).$$

- Consequently, to each tangent vector $v \in T_q M$ we should put into correspondence a linear functional

$$\xi : C^\infty(M) \rightarrow \mathbb{R}$$

such that

$$\xi(ab) = (\xi a)b(q) + a(q)(\xi b), \quad a, b \in C^\infty(M). \quad (1)$$

- But there is a linear functional $\xi = \widehat{v}$ naturally related to any tangent vector $v \in T_q M$, the directional derivative along v :

$$\widehat{v}a = \left. \frac{d}{dt} \right|_{t=0} a(q(t)), \quad q(0) = q, \quad \dot{q}(0) = v,$$

and such functional satisfies Leibniz rule (1).

- Now we show that this rule characterizes exactly directional derivatives.

Proposition 3

Let $\xi : C^\infty(M) \rightarrow \mathbb{R}$ be a linear functional that satisfies Leibniz rule (1) for some point $q \in M$. Then $\xi = \hat{v}$ for some tangent vector $v \in T_q M$.

Proof.

- Notice first of all that any functional ξ that meets Leibniz rule (1) is local, i.e., it depends only on values of functions in an arbitrarily small neighborhood $O_q \subset M$ of the point q :

$$\tilde{a}|_{O_q} = a|_{O_q} \quad \Rightarrow \quad \xi \tilde{a} = \xi a, \quad a, \tilde{a} \in C^\infty(M).$$

- Indeed, take a cut function $b \in C^\infty(M)$ such that $b|_{M \setminus O_q} \equiv 1$ and $b(q) = 0$. Then $(\tilde{a} - a)b = \tilde{a} - a$, thus

$$\xi(\tilde{a} - a) = \xi((\tilde{a} - a)b) = \xi(\tilde{a} - a) b(q) + (\tilde{a} - a)(q) \xi b = 0.$$

- So the statement of the proposition is local, and we prove it in coordinates.
- Let $(x_1, \dots, x_n)$ be local coordinates on M centered at the point q . We have to prove that there exist $\alpha_1, \dots, \alpha_n \in \mathbb{R}$ such that $\xi = \sum_{i=1}^n \alpha_i \frac{\partial}{\partial x_i} \Big|_0$.

- First of all,

$$\xi(1) = \xi(1 \cdot 1) = (\xi 1) \cdot 1 + 1 \cdot (\xi 1) = 2\xi(1),$$

thus $\xi(1) = 0$. By linearity, $\xi(\text{const}) = 0$.

- In order to find the action of ξ on an arbitrary smooth function, we expand it by the Hadamard Lemma:

$$a(x) = a(0) + \sum_{i=1}^n \int_0^1 \frac{\partial a}{\partial x_i}(tx) x_i dt = a(0) + \sum_{i=1}^n b_i(x) x_i,$$

where $b_i(x) = \int_0^1 \frac{\partial a}{\partial x_i}(tx) dt$ are smooth functions.

- Now

$$\xi a = \sum_{i=1}^n \xi(b_i x_i) = \sum_{i=1}^n ((\xi b_i) x_i(0) + b_i(0)(\xi x_i)) = \sum_{i=1}^n \alpha_i \frac{\partial a}{\partial x_i}(0),$$

where we denote $\alpha_i = \xi x_i$ and make use of the equality $b_i(0) = \frac{\partial a}{\partial x_i}(0)$. □

- So *tangent vectors* $v \in T_q M$ can be identified with directional derivatives $\hat{v} : C^\infty(M) \rightarrow \mathbb{R}$, i.e., *linear functionals that meet Leibniz rule* (1).
- Now we characterize *vector fields* on M . A smooth vector field on M is a family of tangent vectors $v_q \in T_q M$, $q \in M$, such that for any $a \in C^\infty(M)$ the mapping $q \mapsto v_q a$, $q \in M$, is a smooth function on M .
- To a smooth vector field $V \in \text{Vec } M$ there corresponds a *linear operator*

$$\hat{V} : C^\infty(M) \rightarrow C^\infty(M)$$

that satisfies the Leibniz rule

$$\hat{V}(ab) = (\hat{V}a)b + a(\hat{V}b), \quad a, b \in C^\infty(M),$$

the directional derivative (Lie derivative) along V .

- A linear operator on an algebra meeting the Leibniz rule is called a *derivation* of the algebra, so the Lie derivative $\hat{V}$ is a derivation of the algebra $C^\infty(M)$.

- We show that the correspondence between smooth vector fields on M and derivations of the algebra $C^\infty(M)$ is invertible.

Proposition 4

Any derivation of the algebra $C^\infty(M)$ is the directional derivative along some smooth vector field on M .

Proof.

Let $D : C^\infty(M) \rightarrow C^\infty(M)$ be a derivation. Take any point $q \in M$. We show that the linear functional

$$d_q \stackrel{\text{def}}{=} \hat{q} \circ D : C^\infty(M) \rightarrow \mathbb{R}$$

is a directional derivative at the point q , i.e., satisfies Leibniz rule (1):

$$\begin{aligned} d_q(ab) &= \hat{q}(D(ab)) = \hat{q}((Da)b + a(Db)) = \hat{q}(Da)b(q) + a(q)\hat{q}(Db) = \\ &= (d_q a)b(q) + a(q)(d_q b), \quad a, b \in C^\infty(M). \end{aligned}$$



- So we can identify points $q \in M$, diffeomorphisms $P \in \text{Diff } M$, and vector fields $V \in \text{Vec } M$ with nontrivial homomorphisms $\hat{q} : C^\infty(M) \rightarrow \mathbb{R}$, automorphisms $\hat{P} : C^\infty(M) \rightarrow C^\infty(M)$, and derivations $\hat{V} : C^\infty(M) \rightarrow C^\infty(M)$ respectively.
- For example, we can write a point $P(q)$ in the operator notation as $\hat{q} \circ \hat{P}$.
- Moreover, in the sequel we omit hats and write $q \circ P$. This does not cause ambiguity: if q is to the right of P , then q is a point, P a diffeomorphism, and $P(q)$ is the value of the diffeomorphism P at the point q . And if q is to the left of P , then q is a homomorphism, P an automorphism, and $q \circ P$ a homomorphism of $C^\infty(M)$.
- Similarly, $V(q) \in T_q M$ is the value of the vector field V at the point q , and $q \circ V : C^\infty(M) \rightarrow \mathbb{R}$ is the directional derivative along the vector $V(q)$.

Seminorms and $C^\infty(M)$ -Topology

- We introduce seminorms and topology on the space $C^\infty(M)$.
- By Whitney's Theorem, a smooth manifold M can be properly embedded into a Euclidean space $\mathbb{R}^N$ for sufficiently large N . Denote by h_i , $i = 1, \dots, N$, the smooth vector field on M that is the orthogonal projection from $\mathbb{R}^N$ to M of the constant basis vector field $\frac{\partial}{\partial x_i} \in \text{Vec}(\mathbb{R}^N)$. So we have N vector fields $h_1, \dots, h_N \in \text{Vec } M$ that span the tangent space $T_q M$ at each point $q \in M$.
- We define the family of seminorms $\| \cdot \|_{s,K}$ on the space $C^\infty(M)$ in the following way:

$$\|a\|_{s,K} = \sup \{ |h_{i_l} \circ \dots \circ h_{i_1} a(q)| \mid q \in K, 1 \leq i_1, \dots, i_l \leq N, 0 \leq l \leq s \},$$
$$a \in C^\infty(M), \quad s \geq 0, \quad K \Subset M.$$

- This family of seminorms defines a topology on $C^\infty(M)$.

- A local base of this topology is given by the subsets

$$\left\{ a \in C^\infty(M) \mid \|a\|_{n,K_n} < \frac{1}{n} \right\}, \quad n \in \mathbb{N},$$

where K_n , $n \in \mathbb{N}$, is a chained system of compacta that cover M :

$$K_n \subset K_{n+1}, \quad \bigcup_{n=1}^{\infty} K_n = M.$$

- This topology on $C^\infty(M)$ does not depend on embedding of M into $\mathbb{R}^N$. It is called the *topology of uniform convergence of all derivatives on compacta*, or just *$C^\infty(M)$ -topology*.
- This topology turns $C^\infty(M)$ into a Fréchet space (a complete, metrizable, locally convex topological vector space).
- A sequence of functions $a_k \in C^\infty(M)$ converges to $a \in C^\infty(M)$ as $k \rightarrow \infty$ if and only if

$$\lim_{k \rightarrow \infty} \|a_k - a\|_{s,K} = 0 \quad \forall s \geq 0, K \Subset M.$$

- For vector fields $V \in \text{Vec } M$, we define the seminorms

$$\|V\|_{s,K} = \sup \{ \|Va\|_{s,K} \mid \|a\|_{s+1,K} = 1 \}, \quad s \geq 0, \quad K \Subset M. \quad (2)$$

- One can prove that any vector field $V \in \text{Vec } M$ has finite seminorms $\|V\|_{s,K}$, and that there holds an estimate of the action of a diffeomorphism $P \in \text{Diff } M$ on a function $a \in C^\infty(M)$:

$$\|Pa\|_{s,K} \leq C_{s,P} \|a\|_{s,P(K)}, \quad s \geq 0, \quad K \Subset M. \quad (3)$$

- Thus vector fields and diffeomorphisms are linear *continuous* operators on the topological vector space $C^\infty(M)$.

Families of Functionals and Operators

- In the sequel we will often consider *one-parameter families* of points, diffeomorphisms, and vector fields that satisfy various regularity properties (e.g. differentiability or absolute continuity) with respect to the parameter.
- Since we treat points as functionals, and diffeomorphisms and vector fields as operators on $C^\infty(M)$, we can introduce regularity properties for them in the weak sense, via the corresponding properties for one-parameter families of functions

$$t \mapsto a_t, \quad a_t \in C^\infty(M), \quad t \in \mathbb{R}.$$

- So we start from definitions for families of functions.
- *Continuity* and *differentiability* of a family of functions a_t w.r.t. parameter t are defined in a standard way since $C^\infty(M)$ is a topological vector space.

- A family of functions a_t is called *measurable* w.r.t. t if the real function $t \mapsto a_t(q)$ is measurable for any $q \in M$. A measurable family a_t is called *locally integrable* if

$$\int_{t_0}^{t_1} \|a_t\|_{s,K} dt < \infty \quad \forall s \geq 0, \quad K \in M, \quad t_0, t_1 \in \mathbb{R}.$$

- A family a_t is called *absolutely continuous* w.r.t. t if

$$a_t = a_{t_0} + \int_{t_0}^t b_\tau d\tau$$

for some locally integrable family of functions b_t .

- A family a_t is called *Lipschitzian* w.r.t. t if

$$\|a_t - a_\tau\|_{s,K} \leq C_{s,K}|t - \tau| \quad \forall s \geq 0, \quad K \in M, \quad t, \tau \in \mathbb{R},$$

and *locally bounded* w.r.t. t if

$$\|a_t\|_{s,K} \leq C_{s,K,I}, \quad \forall s \geq 0, \quad K \in M, \quad I \in \mathbb{R}, \quad t \in I,$$

where $C_{s,K}$ and $C_{s,K,I}$ are some constants depending on s , K , and I .

- Now we can define regularity properties of families of functionals and operators on $C^\infty(M)$.
- A family of linear functionals or linear operators on $C^\infty(M)$

$$t \mapsto A_t, \quad t \in \mathbb{R},$$

has some regularity property (i.e., is *continuous*, *differentiable*, *measurable*, *locally integrable*, *absolutely continuous*, *Lipschitzian*, *locally bounded* w.r.t. t) if the family

$$t \mapsto A_t a, \quad t \in \mathbb{R},$$

has the same property for any $a \in C^\infty(M)$.

- A locally bounded w.r.t. t family of vector fields

$$t \mapsto V_t, \quad V_t \in \text{Vec } M, \quad t \in \mathbb{R},$$

is called a *nonautonomous vector field*, or simply a *vector field*, on M .

- An absolutely continuous w.r.t. t family of diffeomorphisms

$$t \mapsto P^t, \quad P^t \in \text{Diff } M, \quad t \in \mathbb{R},$$

is called a *flow* on M .

- So, for a nonautonomous vector field V_t , the family of functions $t \mapsto V_t a$ is locally integrable for any $a \in C^\infty(M)$.
- Similarly, for a flow P^t , the family of functions $(P^t a)(q) = a(P^t(q))$ is absolutely continuous w.r.t. t for any $a \in C^\infty(M)$.
- Integrals of measurable locally integrable families, and derivatives of differentiable families are also defined in the weak sense:

$$\int_{t_0}^{t_1} A_t dt : a \mapsto \int_{t_0}^{t_1} (A_t a) dt, \quad a \in C^\infty(M),$$

$$\frac{d}{dt} A_t : a \mapsto \frac{d}{dt} (A_t a), \quad a \in C^\infty(M).$$

- One can show that if A_t and B_t are continuous families of operators on $C^\infty(M)$ which are differentiable at t_0 , then the family $A_t \circ B_t$ is continuous, moreover, differentiable at t_0 , and satisfies the Leibniz rule:

$$\left. \frac{d}{dt} \right|_{t_0} (A_t \circ B_t) = \left(\left. \frac{d}{dt} \right|_{t_0} A_t \right) \circ B_{t_0} + A_{t_0} \circ \left(\left. \frac{d}{dt} \right|_{t_0} B_t \right).$$

- If families A_t and B_t of operators are absolutely continuous, then the composition $A_t \circ B_t$ is absolutely continuous as well, the same is true for composition of functionals with operators.
- For an absolute continuous family of functions a_t , the family $A_t a_t$ is also absolutely continuous, and the Leibniz rule holds for it as well.

ODEs with discontinuous right-hand side

- We consider a *nonautonomous ordinary differential equation* of the form

$$\dot{q} = V_t(q), \quad q(0) = q_0, \quad (4)$$

where V_t is a nonautonomous vector field on M , and study the flow determined by this field.

- We denote by $\dot{q}$ the derivative $\frac{d q}{d t}$, so equation (4) reads in the expanded form as

$$\frac{d q(t)}{d t} = V_t(q(t)).$$

- To obtain local solutions to the Cauchy problem (4) on a manifold M , we reduce it to a Cauchy problem in a Euclidean space.
- Choose local coordinates $x = (x^1, \dots, x^n)$ in a neighborhood O_{q_0} of the point q_0 :

$$\begin{aligned} \Phi : O_{q_0} \subset M &\rightarrow O_{x_0} \subset \mathbb{R}^n, & \Phi : q &\mapsto x, \\ \Phi(q_0) &= x_0. \end{aligned}$$

- In these coordinates, the field V_t reads

$$(\Phi_* V_t)(x) = \tilde{V}_t(x) = \sum_{i=1}^n v_i(t, x) \frac{\partial}{\partial x^i}, \quad x \in O_{x_0}, \quad t \in \mathbb{R}, \quad (5)$$

and problem (4) takes the form

$$\dot{x} = \tilde{V}_t(x), \quad x(0) = x_0, \quad x \in O_{x_0} \subset \mathbb{R}^n. \quad (6)$$

- Since the nonautonomous vector field $V_t \in \text{Vec } M$ is locally bounded, the components $v_i(t, x)$, $i = 1, \dots, n$, of its coordinate representation (5) are:
 - (1) measurable and locally bounded w.r.t. t for any fixed $x \in O_{x_0}$,
 - (2) smooth w.r.t. x for any fixed $t \in \mathbb{R}$,
 - (3) differentiable in x with locally bounded partial derivatives:

$$\left| \frac{\partial v_i}{\partial x}(t, x) \right| \leq C_{I,K}, \quad t \in I \subseteq \mathbb{R}, \quad x \in K \subseteq O_{x_0}, \quad i = 1, \dots, n.$$

- By the classical Carathéodory Theorem, the Cauchy problem (6) has a unique solution, i.e., a vector-function $x(t, x_0)$, Lipschitzian w.r.t. t and smooth w.r.t. x_0 , and such that:
 - (1) ODE (6) is satisfied for almost all t ,
 - (2) initial condition holds: $x(0, x_0) = x_0$.
- Then the pull-back of this solution from $\mathbb{R}^n$ to M

$$q(t, q_0) = \Phi^{-1}(x(t, x_0)),$$

is a solution to problem (4) in M .

- The mapping $q(t, q_0)$ is Lipschitzian w.r.t. t and smooth w.r.t. q_0 , it satisfies almost everywhere the ODE and the initial condition in (4).
- For any $q_0 \in M$, the solution $q(t, q_0)$ to the Cauchy problem (4) can be continued to a maximal interval $t \in J_{q_0} \subset \mathbb{R}$ containing the origin and depending on q_0 .
- We will assume that the solutions $q(t, q_0)$ are defined for all $q_0 \in M$ and all $t \in \mathbb{R}$, i.e., $J_{q_0} = \mathbb{R}$ for any $q_0 \in M$. Then the nonautonomous field V_t is called *complete*.
- This holds, e.g., when all the fields V_t , $t \in \mathbb{R}$, vanish outside of a common compactum in M (in this case we say that the nonautonomous vector field V_t has a *compact support*).

Definition of the right chronological exponential

- The Cauchy problem $\dot{q} = V_t(q)$, $q(0) = q_0$, rewritten as a linear equation for Lipschitzian w.r.t. t families of functionals on $C^\infty(M)$:

$$\dot{q}(t) = q(t) \circ V_t, \quad q(0) = q_0, \quad (7)$$

is satisfied for the family of functionals

$$q(t, q_0) : C^\infty(M) \rightarrow \mathbb{R}, \quad q_0 \in M, \quad t \in \mathbb{R}$$

constructed in the previous subsection.

- We prove later that this Cauchy problem has no other solutions.
- Thus the flow defined as

$$P^t : q_0 \mapsto q(t, q_0) \quad (8)$$

is a unique solution of the operator Cauchy problem $\dot{P}^t = P^t \circ V_t$, $P^0 = \text{Id}$ (where Id is the identity operator), in the class of Lipschitzian flows on M .

- The flow P^t determined in (8) is called the *right chronological exponential* of the field V_t and is denoted as $P^t = \overrightarrow{\exp} \int_0^t V_\tau d\tau$.

Formal series expansion

- We rewrite differential equation in (7) as an integral one:

$$q(t) = q_0 + \int_0^t q(\tau) \circ V_\tau d\tau \quad (9)$$

then substitute this expression for $q(t)$ into the right-hand side

$$\begin{aligned} &= q_0 + \int_0^t \left(q_0 + \int_0^{\tau_1} q(\tau_2) \circ V_{\tau_2} d\tau_2 \right) \circ V_{\tau_1} d\tau_1 \\ &= q_0 \circ \left(\text{Id} + \int_0^t V_\tau dt \right) + \iint_{0 \leq \tau_2 \leq \tau_1 \leq t} q(\tau_2) \circ V_{\tau_2} \circ V_{\tau_1} d\tau_2 d\tau_1, \end{aligned}$$

repeat this procedure iteratively, and obtain the decomposition:

$$\begin{aligned}
q(t) = q_0 \circ & \left(\text{Id} + \int_0^t V_\tau d\tau + \iint_{\Delta_2(t)} V_{\tau_2} \circ V_{\tau_1} d\tau_2 d\tau_1 + \dots + \right. \\
& \left. \int_{\Delta_n(t)} \dots \int V_{\tau_n} \circ \dots \circ V_{\tau_1} d\tau_n \dots d\tau_1 \right) + \\
& \int_{\Delta_{n+1}(t)} \dots \int q(\tau_{n+1}) \circ V_{\tau_{n+1}} \circ \dots \circ V_{\tau_1} d\tau_{n+1} \dots d\tau_1. \quad (10)
\end{aligned}$$

- Here

$$\Delta_n(t) = \{(\tau_1, \dots, \tau_n) \in \mathbb{R}^n \mid 0 \leq \tau_n \leq \dots \leq \tau_1 \leq t\}$$

is the n -dimensional simplex.

- Purely formally passing in (10) to the limit $n \rightarrow \infty$, we obtain a formal series for the solution $q(t)$ to problem (7):

$$q_0 \circ \left(\text{Id} + \sum_{n=1}^{\infty} \int \cdots \int_{\Delta_n(t)} V_{\tau_n} \circ \cdots \circ V_{\tau_1} d\tau_n \dots d\tau_1 \right),$$

thus for the solution P^t to our Cauchy problem:

$$\text{Id} + \sum_{n=1}^{\infty} \int \cdots \int_{\Delta_n(t)} V_{\tau_n} \circ \cdots \circ V_{\tau_1} d\tau_n \dots d\tau_1. \quad (11)$$

Estimates and convergence of the series

- Unfortunately, series (11) never converges on $C^\infty(M)$ in the weak sense (if $V_t \not\equiv 0$): there always exists a smooth function on M , on which it diverges.
- Although, one can show that series (11) gives an asymptotic expansion for the chronological exponential $P^t = \overrightarrow{\exp} \int_0^t V_\tau d\tau$.
- There holds the following bound of the remainder term: denote the m -th partial sum of series (11) as $S_m(t) = \text{Id} + \sum_{n=1}^{m-1} \int \cdots \int_{\Delta_n(t)} V_{\tau_n} \circ \cdots \circ V_{\tau_1} d\tau_n \dots d\tau_1$, then

for any $a \in C^\infty(M)$, $s \geq 0$, $K \Subset M$

$$\begin{aligned}
 & \left\| \left(\overrightarrow{\exp} \int_0^t V_\tau d\tau - S_m(t) \right) a \right\|_{s,K} \\
 & \leq C e^{C \int_0^t \|V_\tau\|_{s,K'} d\tau} \frac{1}{m!} \left(\int_0^t \|V_\tau\|_{s+m-1,K'} d\tau \right)^m \|a\|_{s+m,K'} \quad (12) \\
 & = O(t^m), \quad t \rightarrow 0,
 \end{aligned}$$

where $K' \Subset M$ is some compactum containing K , see [AS].

- It follows from estimate (12) that

$$\left\| \left(\overrightarrow{\exp} \int_0^t \varepsilon V_\tau d\tau - S_m^\varepsilon(t) \right) a \right\|_{s,K} = O(\varepsilon^m), \quad \varepsilon \rightarrow 0,$$

where $S_m^\varepsilon(t)$ is the m -th partial sum of series (11) for the field εV_t .

- Thus we have an asymptotic series expansion:

$$\overrightarrow{\exp} \int_0^t V_\tau d\tau \approx \text{Id} + \sum_{n=1}^{\infty} \int_{\Delta_n(t)} \cdots \int V_{\tau_n} \circ \cdots \circ V_{\tau_1} d\tau_n \dots d\tau_1. \quad (13)$$

- In the sequel we will use terms of the zeroth, first, and second orders of the series obtained:

$$\overrightarrow{\exp} \int_0^t V_\tau d\tau \approx \text{Id} + \int_0^t V_\tau d\tau + \iint_{0 \leq \tau_2 \leq \tau_1 \leq t} V_{\tau_2} \circ V_{\tau_1} d\tau_2 d\tau_1 + \cdots .$$

- We prove now that the asymptotic series converges to the chronological exponential on any normed subspace $L \subset C^\infty(M)$ where V_t is well-defined and bounded:

$$V_t L \subset L, \quad \|V_t\| = \sup \{\|V_t a\| \mid a \in L, \|a\| \leq 1\} < \infty. \quad (14)$$

- We apply operator series (13) to any $a \in L$ and bound terms of the series obtained:

$$a + \sum_{n=1}^{\infty} \int \cdots \int_{\Delta_n(t)} V_{\tau_n} \circ \cdots \circ V_{\tau_1} a \, d\tau_n \dots d\tau_1. \quad (15)$$

$$\begin{aligned}
& \left\| \int_{\Delta_n(t)} \cdots \int V_{\tau_n} \circ \cdots \circ V_{\tau_1} a \, d\tau_n \cdots d\tau_1 \right\| \\
& \leq \int_{0 \leq \tau_n \leq \cdots \leq \tau_1 \leq t} \|V_{\tau_n}\| \cdots \|V_{\tau_1}\| \, d\tau_n \cdots d\tau_1 \cdot \|a\| \\
& = \int_{0 \leq \tau_{\sigma(n)} \leq \cdots \leq \tau_{\sigma(1)} \leq t} \|V_{\tau_n}\| \cdots \|V_{\tau_1}\| \, d\tau_n \cdots d\tau_1 \cdot \|a\| \\
& = \frac{1}{n!} \int_0^t \cdots \int_0^t \|V_{\tau_n}\| \cdots \|V_{\tau_1}\| \, d\tau_n \cdots d\tau_1 \cdot \|a\| \\
& = \frac{1}{n!} \left(\int_0^t \|V_{\tau}\| \, d\tau \right)^n \cdot \|a\|.
\end{aligned}$$

- So series (15) is majorized by the exponential series, thus the operator series (13) converges on L .
- Series (15) can be differentiated termwise, thus it satisfies the same ODE as the function $P^t a$:

$$\dot{a}_t = V_t a_t, \quad a_0 = a.$$

- Consequently,

$$P^t a = a + \sum_{n=1}^{\infty} \int_{\Delta_n(t)} \cdots \int V_{\tau_n} \circ \cdots \circ V_{\tau_1} a \, d\tau_n \cdots d\tau_1.$$

- So in the case (14) the asymptotic series converges to the chronological exponential and there holds the bound

$$\|P^t a\| \leq e^{\int_0^t \|V_\tau\| \, d\tau} \|a\|, \quad a \in L.$$

- Moreover, one can show that the bound and convergence hold not only for locally bounded, but also for integrable on $[0, t]$ vector fields: $\int_0^t \|V_\tau\| \, d\tau < \infty$.

- Notice that conditions (14) are satisfied for any finite-dimensional V_t -invariant subspace $L \subset C^\infty(M)$. In particular, this is the case when $M = \mathbb{R}^n$, L is the space of linear functions, and V_t is a linear vector field on $\mathbb{R}^n$.
- If M , V_t , and a are real analytic, then series (15) converges for sufficiently small t .

Left chronological exponential

- Consider the inverse operator $Q^t = (P^t)^{-1}$ to the right chronological exponential

$$P^t = \overrightarrow{\exp} \int_0^t V_\tau d\tau.$$

- We find an ODE for the flow Q^t by differentiation of the identity

$$P^t \circ Q^t = \text{Id}.$$

- Leibniz rule yields $\dot{P}^t \circ Q^t + P^t \circ \dot{Q}^t = 0$, thus, in view of the ODE for the flow P^t ,

$$P^t \circ V_t \circ Q^t + P^t \circ \dot{Q}^t = 0.$$

- We multiply this equality by Q^t from the left and obtain

$$V_t \circ Q^t + \dot{Q}^t = 0.$$

That is, the flow Q^t is a solution of the Cauchy problem

$$\frac{d}{dt} Q^t = -V_t \circ Q^t, \quad Q^0 = \text{Id}, \quad (16)$$

which is dual to the Cauchy problem for P^t : $\frac{d}{dt} P^t = P^t \circ V_t$, $P^0 = \text{Id}$.

- The flow Q^t is called the *left chronological exponential* and is denoted as

$$Q^t = \overleftarrow{\exp} \int_0^t (-V_\tau) d\tau.$$

- We find an asymptotic expansion for the left chronological exponential in the same way as for the right one, by successive substitutions into the right-hand side:

$$\begin{aligned} Q^t &= \text{Id} + \int_0^t (-V_\tau) \circ Q^\tau d\tau \\ &= \text{Id} + \int_0^t (-V_\tau) d\tau + \iint_{\Delta_2(t)} (-V_{\tau_1}) \circ (-V_{\tau_2}) \circ Q^{\tau_2} d\tau_2 d\tau_1 = \dots \\ &= \text{Id} + \sum_{n=1}^{m-1} \int_{\Delta_n(t)} \dots \int (-V_{\tau_1}) \circ \dots \circ (-V_{\tau_n}) d\tau_n \dots d\tau_1 \\ &\quad + \int_{\Delta_m(t)} \dots \int (-V_{\tau_1}) \circ \dots \circ (-V_{\tau_m}) \circ Q^{\tau_m} d\tau_m \dots d\tau_1. \end{aligned}$$

- For the left chronological exponential holds an estimate of the remainder term as (12) for the right one, and the series obtained is asymptotic:

$$\overleftarrow{\exp} \int_0^t (-V_\tau) d\tau \approx \text{Id} + \sum_{n=1}^{\infty} \int_{\Delta_n(t)} \cdots \int (-V_{\tau_1}) \circ \cdots \circ (-V_{\tau_n}) d\tau_n \dots d\tau_1.$$

- Notice that the reverse arrow in the left chronological exponential $\overleftarrow{\exp}$ corresponds to the reverse order of the operators $(-V_{\tau_1}) \circ \cdots \circ (-V_{\tau_n})$, $\tau_n \leq \dots \leq \tau_1$.
- The right and left chronological exponentials satisfy the corresponding differential equations:

$$\begin{aligned} \frac{d}{dt} \overrightarrow{\exp} \int_0^t V_\tau d\tau &= \overrightarrow{\exp} \int_0^t V_\tau d\tau \circ V_t, \\ \frac{d}{dt} \overleftarrow{\exp} \int_0^t (-V_\tau) d\tau &= -V_t \circ \overleftarrow{\exp} \int_0^t (-V_\tau) d\tau. \end{aligned}$$

The directions of arrows correlate with the direction of appearance of the operators V_t and $(-V_t)$ in the right-hand side of these ODEs.

- If the initial value is prescribed at a moment of time $t_0 \neq 0$, then the lower limit of integrals in the chronological exponentials is t_0 .
- There holds the following obvious rule for composition of flows:

$$\overrightarrow{\exp} \int_{t_0}^{t_1} V_\tau d\tau \circ \overrightarrow{\exp} \int_{t_1}^{t_2} V_\tau d\tau = \overrightarrow{\exp} \int_{t_0}^{t_2} V_\tau d\tau.$$

- There hold the identities

$$\overrightarrow{\exp} \int_{t_0}^{t_1} V_\tau d\tau = \left(\overrightarrow{\exp} \int_{t_1}^{t_0} V_\tau d\tau \right)^{-1} = \overleftarrow{\exp} \int_{t_1}^{t_0} (-V_\tau) d\tau. \quad (17)$$

- We saw that equation (7) for Lipschitzian families of functionals has a solution $q(t) = q_0 \circ \overrightarrow{\exp} \int_0^t V_\tau d\tau$. We can prove now that this equation has no other solutions.

Proposition 5

Let V_t be a complete nonautonomous vector field on M . Then Cauchy problem (7) has a unique solution in the class of Lipschitzian families of functionals on $C^\infty(M)$.

Proof.

Let a Lipschitzian family of functionals q_t be a solution to problem (7). Then

$$\frac{d}{dt} (q_t \circ (P^t)^{-1}) = \frac{d}{dt} (q_t \circ Q^t) = q_t \circ V_t \circ Q^t - q_t \circ V_t \circ Q^t = 0,$$

thus $q_t \circ Q^t \equiv \text{const.}$ But $Q^0 = \text{Id}$, consequently, $q_t \circ Q^t \equiv q_0$, hence

$$q_t = q_0 \circ P^t = q_0 \circ \overrightarrow{\exp} \int_0^t V_\tau d\tau$$

is a unique solution of Cauchy problem (7). □

Similarly, the both operator equations $\dot{P}^t = P^t \circ V_t$ and $\dot{Q}^t = -V_t \circ Q^t$ have no other solutions in addition to the chronological exponentials.

Plan of this lecture

1. Points, Diffeomorphisms, and Vector Fields
2. Seminorms and $C^\infty(M)$ -Topology
3. Families of Functionals and Operators
4. ODEs with discontinuous right-hand side
5. Definition of the right chronological exponential
6. Formal series expansion
7. Estimates and convergence of the series
8. Left chronological exponential