Applications of geometric control theory
(Lecture 11)

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«Introduction to geometric control theory»

Lecture course in Dept. of Mathematics and Mechanics

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10. Вступает в город и готов одарять благами:

С голой грудью и босиком он появляется на рынке;
Вымазанный грязью и весь в золе, как широко он улыбается!
Нет нужды в таинственной силе богов,
Ибо стоит ему прикоснуться, — смотри! — сухие деревья пышно расцветают.

Pu-ming, "The Ten Oxherding Pictures"

Reminder: Plan of the previous lecture

1. Lorentzian problem in the Lobachevsky plane
2. Sub-Lorentzian problem on the Heisenberg group

Plan of this lecture

1. Applications to metric geometry (metric lines in sub-Riemannian Carnot groups),
2. Applications to PDEs (kernel of sub-Riemannian heat equation),
3. Applications to models of vision and image processing,
4. Applications to robotics,
5. Some open problems,
6. Further reading.

Metric lines

- Let $\Gamma : \mathbb{R} \to M$ be a geodesic in a sub-Riemannian manifold M.
- Γ is called a *metric line* if it is an isometry, i.e., if any of the following conditions hold:
  - for any $a, b \in \mathbb{R}$ the restriction $\Gamma|_{[a,b]}$ is a sub-Riemannian minimizer,
  - for any $a, b \in I$ the sub-Riemannian distance $d(\Gamma(a), \Gamma(b)) = |a - b|$.
- Important question in the theory of metric groups: characterization of metric lines in sub-Riemannian Carnot groups

Carnot groups

- A *stratification* of a Lie algebra $\mathfrak{g}$ is a direct-sum decomposition into vector subspaces for some $s \in \mathbb{N}$

$$\mathfrak{g} = V_1 \oplus \cdots \oplus V_s, \quad [V_1, V_k] = V_{k+1}, \qquad k = 1, \ldots, s, \tag{1}$$

$$V_s \neq \{0\}, \qquad V_{s+1} = \{0\}. \tag{2}$$

- A *Carnot algebra* $\mathfrak{g}$ is a Lie algebra that has a stratification.
- The integer s in (1)–(2) is called the *step* of the Carnot algebra $\mathfrak{g}$. The subspaces V_k, $1 \leq k \leq s$ are called *layers* of degree k. The dimension of the first layer V_1 is called the *rank* of the Carnot algebra $\mathfrak{g}$. The first layer V_1 generates $\mathfrak{g}$ as a Lie algebra. Any Carnot algebra is nilpotent.
- A *Carnot group* is a connected and simply connected Lie group whose Lie algebra is a Carnot algebra.
- If the first layer V_1 is endowed with a scalar product, then the left translations of this layer and the scalar product define a left-invariant sub-Riemannian structure on the corresponding Carnot group G. In this case we call G a *sub-Riemannian Carnot group*.

Metric lines in Carnot groups

Theorem 1

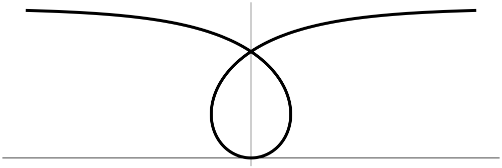
If a geodesic $g : \mathbb{R} \to G$ in a sub-Riemannian Carnot group G projects to the first layer V_1 to a straight line, then g is a metric line.

Theorem 2

If G is a sub-Riemannian Carnot group of step $s \leq 2$, then any metric line in G projects to the first layer V_1 to a straight line.

Theorem 3

If G is a sub-Riemannian free Carnot group of step $s > 2$, then there exists a metric line in G that projects to the first layer V_1 to a curve distinct from straight line:



Sub-Riemannian heat equation

- Let M be an oriented sub-Riemannian manifold with an orthonormal frame $X_1, \ldots, X_m \in \mathrm{Vec}(M)$, $\dim M = n \geq m$.
- Let ω be a non-vanishing n-form on M (*volume form*) and let div_ω be the corresponding divergence of vector field.
- The *horizontal gradient* of a function $\varphi \in C^\infty(M)$ is the vector field $\mathrm{grad}_H(\varphi) = \sum_{i=1}^m X_i(\varphi) X_i \in \mathrm{Vec}(M)$.
- The *sub-Riemannian Laplacian* is the operator $\Delta_H \varphi = \mathrm{div}_\omega(\mathrm{grad}_H(\varphi))$.
- The *sub-Riemannian heat equation* is

$$\frac{\partial}{\partial t}\varphi(q,t) = \Delta_H \varphi(q,t).$$

- This equation describes a diffusion process with a density $\varphi(q,t)$ depending on the position $q \in M$ and time t subject to conditions:
  - the diffusion is performed in the direction of the distribution $\mathrm{span}(X_1, \ldots, X_m)$ in which φ is decreasing fastest,
  - the diffusing quantity (heat, mass, ...) satisfies a conservation law.

Existence of solutions to the heat equation

Theorem 4

Assume that the distribution $\mathrm{span}(X_1, \dots, X_m)$ *is full-rank, i.e.,* $\mathrm{Lie}(X_1, \dots, X_m)(q) = T_q M$ *for any* $q \in M$*. Then the sub-Riemannian heat equation is* hypoelliptic*, i.e., any its solution* $\varphi : \Omega \subset M \times \mathbb{R} \to \mathbb{R}$ *belongs to* $C^\infty(\Omega)$*.*

Theorem 5

Assume that the distribution $\mathrm{span}(X_1, \dots, X_m)$ *is full-rank, and let any points* $q_0, q_1 \in M$ *be connected by a sub-Riemannian length minimizer. Then the Cauchy problem*

$$\frac{\partial}{\partial t}\varphi(q,t) = \Delta_H \varphi(q,t), \qquad (q,t) \in M \times [0, +\infty)$$
$$\varphi(q,0) = \varphi_0(q) \in L^2(M)$$

has a unique solution in $L^2(M \times [0, +\infty))$*. Moreover, such a solution belongs to* $C^\infty(M \times (0, +\infty))$*.*

The heat equation on the Heisenberg group

- Consider the sub-Riemannian structure on $\mathbb{R}^3$ with an orthonormal frame $X_1 = \frac{\partial}{\partial x} - \frac{y}{2}\frac{\partial}{\partial z}$, $X_2 = \frac{\partial}{\partial y} + \frac{x}{2}\frac{\partial}{\partial z}$.
- Then the sub-Riemannian heat equation is

$$\frac{\partial}{\partial t}\varphi(x,y,z,t) = \Delta_H(\varphi) = \left((X_1)^2 + (X_2)^2\right)\varphi(x,y,z,t).$$

- The *heat kernel* is a function $K_t(q,\bar{q})$ such that the solution to the Cauchy problem

$$\frac{\partial}{\partial t}\varphi(q,t) = \Delta_H\varphi(q,t), \qquad (q,t) \in \mathbb{R}^3 \times [0,+\infty)$$
$$\varphi(q,0) = \varphi_0(q) \in L^2(\mathbb{R}^3, dh)$$

can be expressed as

$$\varphi(q,t) = \int_{\mathbb{R}^3} K_t(q,\bar{q})\varphi_0(\bar{q})dh(\bar{q}), \qquad t > 0,$$

where dh is the Haar measure on the Heisenberg group.

The Gaveau-Hulanicki fundamental solution for the Heisenberg group

Theorem 6

The heat kernel for the sub-Riemannian structure on the Heisenberg group is given by the formula

$$K_t(q,\bar{q}) = P_t(q^{-1}\cdot\bar{q}), \qquad q,\bar{q}\in\mathbb{R}^3,$$

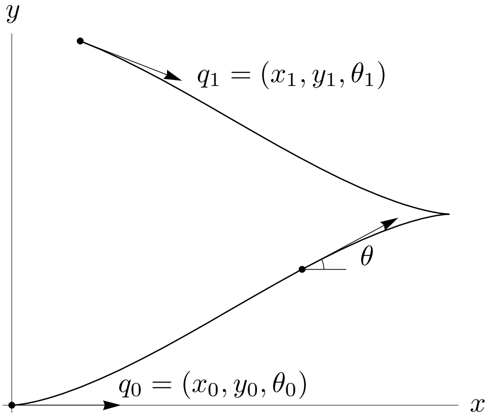
where

$$P_t(x,y,z) = \frac{1}{(2\pi t)^2}\int_{\mathbb{R}} \frac{2\tau}{\sinh(2\tau)} \exp\left(-\frac{\tau(x^2+y^2)}{2t\tanh(2\tau)}\right)\cos\left(\frac{2z\tau}{t}\right)\, d\tau, \qquad t>0.$$

Notice that $P_t(q)$ represents the evolution at time t of an initial condition that at time zero is concentrated at the origin (the Dirac delta δ_0):

$$P_t(q) = K_t(q,0) = \int_{\mathbb{R}^3} K_t(q,\bar{q})\delta_0(\bar{q})dh(\bar{q}).$$

Problem statement:
Optimal motion of a mobile robot in the plane



$$q(0) = q_0, \qquad q(t_1) = q_1, \qquad I = \int_0^{t_1} \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{\theta}^2}\, dt \to \min$$

Optimal control problem

$$\dot{x} = u\cos\theta, \qquad \dot{y} = u\sin\theta, \qquad \dot{\theta} = v,$$

$$(x, y) \in \mathbb{R}^2, \qquad \theta \in S^1 = \mathbb{R}/(2\pi\,\mathbb{Z}),$$

$$q = (x, y, \theta) \in M = \mathbb{R}^2 \times S^1,$$

$$(u, v) \in \mathbb{R}^2,$$

$$q(0) = q_0, \qquad q(t_1) = q_1,$$

$$l = \int_0^{t_1} \sqrt{u^2 + v^2}\,dt \to \min.$$

Cut time and cut points

$$t_{\text{cut}}(\lambda) = \text{t}(\lambda) = \begin{cases} t^1_{\varepsilon^5} = 2K(k) = T/2, & \lambda \in C_1, \\ t^1_{\varepsilon^2} = 2kp^1_1(k) \in (T, 2T), & \lambda \in C_2, \\ +\infty, & \lambda \in C_3 \cup C_5, \\ t^1_{\varepsilon^5} = \pi = T/2, & \lambda \in C_4 \end{cases}$$

$$p = p^1_1(k) \quad : \quad \text{cn}(p,k)(\text{E}(p,k) - p) - \text{dn}(p,k)\,\text{sn}(p,k) = 0$$

Optimal solutions

Generic boundary conditions:

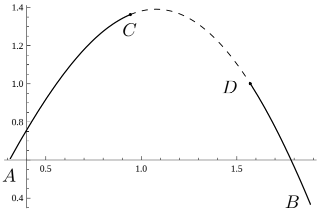
systems of equations in Jacobi's functions $\Rightarrow$

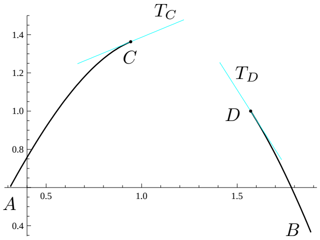
$\Rightarrow$ software (MATHEMATICA).

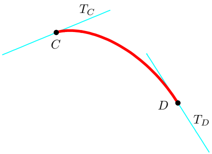
Sub-Riemannian spheres

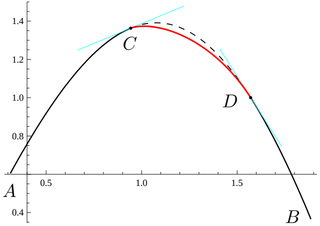
- $d(q_0, q_1) = \inf\{l(q(\cdot)) \mid q(0) = q_0,\ q(t_1) = q_1\}$,
- $S_R = \{q \in M \mid d(q_0, q) = R\}$,
- $R = 0 \quad \Rightarrow \quad S_R = \{q_0\}$,
- $R \in (0, \pi) \quad \Rightarrow \quad S_R \cong S^2$,
- $R = \pi \quad \Rightarrow \quad S_R \cong S^2/\{N = S\}$,
- $R > \pi \quad \Rightarrow \quad S_R \cong T^2$.

Application: Anthropomorphic restoration of curves









Neurogeometry and sub-Riemannian problem on $\mathbb{R}^2 \times \mathbb{R}P^1$

- J.Petitot, The neurogeometry of pinwheels as a sub-Riemannian contact structure, *J. Physiology - Paris* 97 (2003), 265–309.
- J.Petitot, *Neurogeometrie de la vision — Modeles mathematiques et physiques des architectures fonctionnelles*, 2008, Editions de l'Ecole Polytechnique.

$$\dot{x} = u\cos\theta, \qquad \dot{y} = u\sin\theta, \qquad \dot{\theta} = v,$$

$$q = (x, y, \theta), \qquad (x, y) \in \mathbb{R}^2, \quad \theta \in \mathbb{R}P^1 = \mathbb{R}/(\pi\,\mathbb{Z}),$$

$$(u, v) \in \mathbb{R}^2,$$

$$q(0) = q_0, \qquad q(t_1) = q_1,$$

$$l = \int_0^{t_1} \sqrt{u^2 + v^2}\, dt \to \min.$$

Some open problems

1. Smoothness of sub-Riemannian length minimizers
2. Sard's property in sub-Riemannian geometry
3. Singularities of sub-Riemannian spheres near abnormal minimizers
4. Structure of exponential mapping and sub-Riemannian spheres for generic Martinet, Engel, and Cartan structures
5. Optimal synthesis for 3D left-invariant sub-Riemannian structures
6. Metric lines in sub-Riemannian Carnot groups of step > 2

Further reading

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2. V. Jurdjevic, *Geometric Control Theory*, Cambridge University Press, 1997.
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4. A.A. Agrachev, Yu.L. Sachkov, *Control theory from the geometric viewpoint*, Berlin Heidelberg New York Tokyo. Springer-Verlag. 2004
5. A.A. Agrachev, *Some open problems*, Geometric Control Theory and Sub-Riemannian Geometry, pp. 1–13. Springer INdAM Series, vol. 5. Springer, 2014.
6. A. A. Agrachev, Topics in sub-Riemannian geometry, *Russian Math. Surveys*, 71:6 (2016), 989–1019

Further reading

7. A. Agrachev, D. Barilari, U. Boscain, *A Comprehensive Introduction to sub-Riemannian Geometry from Hamiltonian viewpoint*, Cambridge Studies in Advanced Mathematics, Cambridge Univ. Press, 2019
8. Yu. L. Sachkov, *Introduction to geometric control theory* (in Russian), Moscow, URSS, 2021. English translation: *Introduction to geometric control*, Springer, 2022.
9. Yu. L. Sachkov, Left-invariant optimal control problems on Lie groups: classifications and problems integrable in elementary functions, *Russian mathematical surveys*, 77:1(463) (2022), 109–176
10. Yu. L. Sachkov, Left-invariant optimal control problems on Lie groups: problems integrable by elliptic functions, *Russian math. surveys*, 78:1(469) (2023), 67–166

Thank you for your attention!