Lorentzian and sub-Lorentzian geometry
(Lecture 10)

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«Introduction to geometric control theory»

Lecture course in Dept. of Mathematics and Mechanics

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9. Возвращение к первоначалу, возврат к источнику:

Вернуться к первоначалу, вернуться к источнику — значит сделать неверный шаг!

Лучше всего — оставаться дома, быть слепым и глухим и ни о чем не заботиться;

Сидя в хижине, он не замечает никаких внешних вещей;

Взгляни на потоки воды — кто-нибудь знает их? А ярко-красные цветы — для кого они?

Пу-мин, "Десять рисунков о пастухе и быке"

Reminder: Plan of the previous lecture

1. Euler's elastic problem

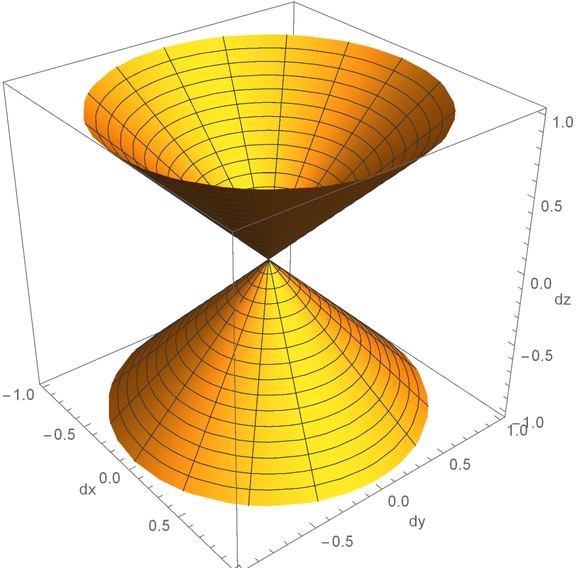
Plan of this lecture

1. Lorentzian problem in the Lobachevsky plane
2. Sub-Lorentzian problem on the Heisenberg group

Lorentzian geometry

- Smooth manifold M,
- Lorentzian metric (nondegenerate quadratic form of index 1):
  $g = \{g_q$ — Lorentzian metric in $T_qM \mid q \in M\}$
- a tangent vector $v \in T_qM$ is called:
  - timelike if $g(v) < 0$
  - spacelike if $g(v) > 0$ or $v = 0$,
  - lightlike if $g(v) = 0$ and $v \neq 0$,
  - nonspacelike if $g(v) \leq 0$.
- Lipschitzian curve in M is called timelike if it has timelike velocity vector a.e.
- Spacelike, lightlike and nonspacelike curves are defined similarly.
- A time orientation X_0 is an arbitrary timelike vector field in M.
- A nonspacelike vector $v \in T_qM$ is future directed if $g(v, X_0(q)) < 0$, and past directed if $g(v, X_0(q)) > 0$.
- In Lorentzian geometry, information propagates only along future directed nonspacelike curves.
- Relativity: speed of particles is not greater than speed of light, time non-reversible.

Lightlike cone for $g = -dz^2 + dx^2 + dy^2$, Minkowski space



Lorentzian geometry

- The Lorentzian length of a nonspacelike curve $q \in \mathrm{Lip}([0, t_1], M)$ is

$$l(q) = \int_0^{t_1} |g(\dot{q}(t))|^{1/2} dt.$$

- For points $q_0, q_1 \in M$ denote by $\Omega_{q_0 q_1}$ the set of all future directed nonspacelike curves in M that connect q_0 to q_1.
- The Lorentzian distance (time separation function) from q_0 to q_1 is defined as

$$d(q_0, q_1) = \begin{cases} \sup\{l(q) \mid q \in \Omega_{q_0 q_1}\} & \text{if } \Omega_{q_0 q_1} \neq \emptyset, \\ 0 & \text{if } \Omega_{q_0 q_1} = \emptyset. \end{cases} \tag{1}$$

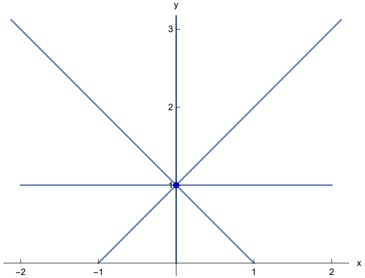
- A future directed nonspacelike curve q is called a Lorentzian length maximizer if it realizes the supremum in (7) between its endpoints $q(0) = q_0$, $q(t_1) = q_1$.
- Relativity Theory: $l(q)$ is the proper time of a particle q moving in a space-time M, $d(q_0, q_1)$ is the proper time for particles freely falling under the action of gravity.

Example of Lorentzian geometry: 2D Minkowski (flat) space-time

- $M = \mathbb{R}^2_{xy}$
- $g = -dx^2 + dy^2 d((x_0, y_0), (x_1, y_1)) = \sqrt{(x_1 - x_0)^2 - (y_1 - y_0)^2}$
- $X_0 = \frac{\partial}{\partial x}$

Lobachevsky plane

- Proper affine functions on the line $a \mapsto y \cdot a + x$, $a \in \mathbb{R}$, $y > 0$, $x \in \mathbb{R}$.
- Group of proper affine functions on the line $G = \mathrm{Aff}_+(\mathbb{R}) = \{(x, y) \in \mathbb{R}^2 \mid y > 0\}$
- Left-invariant frame $X_1 = y\frac{\partial}{\partial x}$, $X_2 = y\frac{\partial}{\partial y}$.
- Riemannian geometry on $\mathrm{Aff}_+(\mathbb{R})$ with the orthonormal frame X_1, X_2 is the Lobachevsky-Gauss-Bolyai non-Euclidean (hyperbolic) geometry, in Poincaré's model in the upper halfplane.
- Lie algebra $\mathfrak{g} = \mathrm{span}(X_1, X_2)$, $[X_2, X_1] = X_1 \quad \Rightarrow \quad \mathfrak{g}$ solvable non-Abelian.
- One-parameter subgroups in G: $u_1(y - 1) = u_2 x$, $(u_1, u_2) \neq (0, 0)$, see fig.:



Left-invariant Lorentzian problem on the Lobachevsky plane

- A Lorentzian form g on the Lie algebra $\mathfrak{g}$.
- A time orientation vector field $X_0 \in \mathfrak{g}$, $g(X_0) < 0$.
- The linear form $g_0(X) = g(X, X_0)$, $X \in \mathfrak{g}$.
- Lorentzian length maximizer $q \in \mathrm{Lip}([0, t_1], G)$:

$$
\begin{aligned}
& g(\dot{q}(t)) \leq 0, \qquad g_0(\dot{q}(t)) < 0, \\
& q(0) = q_0 = \mathrm{Id} = (0, 1), \qquad q(t_1) = q_1, \\
& l = \int_0^{t_1} |g(\dot{q}(t))|^{1/2} dt \to \max .
\end{aligned}
$$

Optimal control problem

$$\dot{q} = u_1 X_1(q) + u_2 X_2(q), \qquad q = (x, y) \in G, \tag{2}$$

$$u = (u_1, u_2) \in U = \{u \in \mathbb{R}^2 \mid g(u) \leq 0,\ g_0(u) < 0\}, \tag{3}$$

$$q(0) = q_0 = \mathrm{Id} = (0, 1), \quad q(t_1) = q_1, \tag{4}$$

$$I = \int_0^{t_1} \sqrt{|g(u)|}\, dt \to \max. \tag{5}$$

Parameterization of the family of problems

- $g(u) = -(au_1 + bu_2)^2 + (cu_1 + du_2)^2$,
- $g_0(u) = au_1 + bu_2$,
- $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_+(2, \mathbb{R})$: $\det A > 0$.

Attainable set

- $\lambda_1(q) := (c-a)x + (d-b)(y-1)$,
- $\lambda_2(q) := (c+a)x + (d+b)(y-1)$,
- $\lambda_3(q) := \lambda_1(q) + 2|A|/(a+c)$, $q = (x, y) \in G$

Theorem 1

The attainable set of the system (2), (3) *from* q_0 *for arbitrary non-negative time (causal future of the point* q_0*) is*

$$\mathscr{A}_{q_0} = J^+(q_0) = \{q \in G \mid \lambda_1(q) \leq 0 \leq \lambda_2(q)\}. \tag{6}$$

Proof.

Admissible trajectories with constant controls (one-parameter subgroups) fill domain (6).
At the boundary of domain (6) admissible velocities look inside or along the boundary of this domain. $\square$

Existence of length maximizers for globally hyperbolic Lorentzian structures

- Consider a Lorentzian structure (g, X_0) on a manifold M.
- Denote by $J^+(q_0) = \mathscr{A}_{q_0}$ the causal future of q_0, and by $J^-(q_1)$ the attainable set from $q_1 \in M$ for arbitrary nonpositive times (the causal past of q_1).

Theorem 2 (Avez 1963, Seifert 1967)

Let (g, X_0) be globally hyperbolic, i.e.:

(1) *Any $q \in M$ has a neighborhood $O \subset M$ such that no future directed nonspacelike curve that leaves O ever returns to O,*

(2) *For any $q_0, q_1 \in M$ the intersection $J^+(q_0) \cap J^-(q_1)$ is compact.*

If $q_1 \in J^+(q_0)$, then there exists a Lorentzian length maximizer that connects q_0 to q_1.

Lemma 3

All left-invariant Lorentzian structures on $\mathrm{Aff}_+(\mathbb{R})$ *satisfy the above hypothesis* (1).

Proof.

$\dot{x} = u_1 y$ or $\dot{y} = u_2 y$ preserves sign and is separated from zero for $(x, y) \in O$,
$u_1^2 + u_2^2 \geq C > 0$, $g(u) \leq 0$, $g_0(u) < 0$. $\square$

Consider a left-invariant Lorentzian structure (g, X_0) on $G = \mathrm{Aff}_+(\mathbb{R})$. It is called:

- Timelike if $g(X_1) < 0$,
- Spacelike if $g(X_1) > 0$,
- Lightlike if $g(X_1) = 0$,
- Nontimelike if $g(X_1) \geq 0$.

Theorem 4

A Lorentzian structure (g, X_0) is globally hyperbolic iff it is nontimelike.

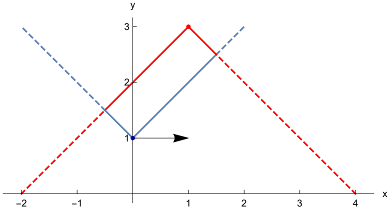


Figure: Spacelike, globally hyperbolic

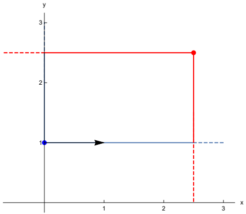


Figure: Lightlike, globally hyperbolic

Timelike Lorentzian structures on $G = \mathrm{Aff}_+(\mathbb{R})$

- Consider a timelike left-invariant Lorentzian structure (g, X_0) on $G = \mathrm{Aff}_+(\mathbb{R})$.
- (g, X_0) is not globally hyperbolic on G, see the left figure below.
- (g, X_0) is globally hyperbolic on $M = G \cap \{\lambda_3 > 0\}$, see the right figure below.

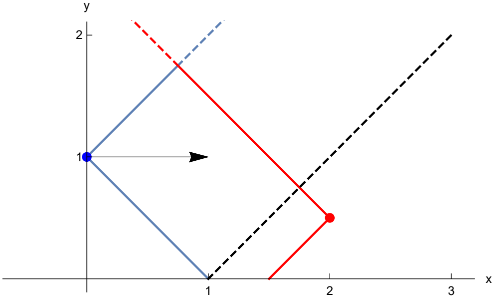


Figure: G is not globally hyperbolic

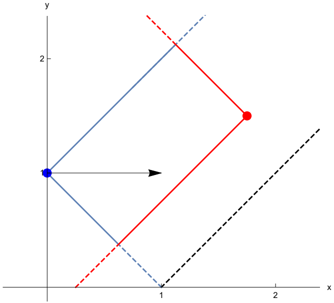


Figure: M is globally hyperbolic

Existence of length maximizers on $G = \mathrm{Aff}_+(\mathbb{R})$

Theorem 5

(1) *Let G be nontimelike. Then a Lorentzian length maximizer exists for any $q_1 \in \mathscr{A}_{q_0}$.*

(2) *Let G be timelike, and let $q_1 \in \mathscr{A}_{q_0}$. A Lorentzian length maximizer exists iff $q_1 \in M = G \cap \{\lambda_3 > 0\}$.*

Proof.

(1) If G is nontimelike, then it is globally hyperbolic.
(2.1) M is globally hyperbolic, thus a Lorentzian length maximizer exists for any $q_1 \in \mathscr{A}_{q_0} \cap M$.
(2.2) If $q_1 \in \mathscr{A}_{q_0} \setminus \mathrm{cl}(M)$, then there exist arbitrarily long trajectories from q_0 to q_1.
(2.3) If $q_1 \in \mathscr{A}_{q_0} \cap \partial M$, then there are no extremal trajectories from q_0 to q_1.

□

Extremal trajectories and Lorentzian length maximizers

Theorem 6

(1) *Any timelike extremal trajectory is an arc of a hyperbola with an asymptote parallel to a light-like one-parameter subgroup, or a rectilinear segment.*

(2) *Lightlike extremal trajectories are arbitrary Lipschitz (maybe nonmonotone) reparameterizations of lightlike one-parameter subgroups.*

(3) *Any extremal trajectory is a Lorentzian length maximizer.*

(4) *If* $q_1 \in \operatorname{int} M$ *in the timelike case, and* $q_1 \in \operatorname{int} \mathscr{A}_{q_0}$ *in the nontimelike case, then there is a unique arclength parameterized Lorentzian length maximizer connecting* q_0 *and* q_1.

Proof.

Pontryagin maximum principle, Hadamard's global diffeomorphism theorem. $\square$

Lorentzian distance

Denote $d(q_0, q)$ by $d(q)$, $q \in G$.

Theorem 7

(1) *In the timelike case* $d \in C^\omega(\operatorname{int} M)$, $d \in C(\operatorname{cl} M)$. *Moreover,* $0 \le d|_M < \pi$, $d|_{\partial M} = \pi$, $d|_{\mathscr{A}_{q_0} \setminus \operatorname{cl} M} = +\infty$.

(2) *In the nontimelike case* $d \in C^\omega(\operatorname{int} \mathscr{A}_{q_0})$, $d \in C(\mathscr{A}_{q_0})$. *Moreover,* $0 \le d|_{\mathscr{A}_{q_0}} < +\infty$.

(3) *Near smoothness points of* ∂M *in the timelike case and near smoothness points of* $\partial \mathscr{A}_{q_0}$ *in the nontimelike case, the Lorentzian distance* d *is a Hölder function with exponent* $\frac{1}{2}$ *of the Euclidean distance to the corresponding bound.*

(4) *The distance* d *is expressed in elementary functions: inverse hyperbolic functions in the timelike case, inverse trigonometric functions in the spaceelike case, square roots and rational functions in the lightlike case.*

Theorem 8

(1) *Lorentzian spheres*

$$S(R) = \{q \in G \mid d(q) = R\}$$

are noncompact in both directions arcs of hyperbolas for the timelike case and $R \in (0, \pi)$, *and for the nontimelike case and* $R \in (0, +\infty)$.

(2) *In the timelike case, also* $S(\pi) = \partial M$, $S(+\infty) = \mathscr{A}_{q_0} \setminus \operatorname{cl} M$.

(3) *In any problem the Lorentzian sphere* $S(0) \cap \mathscr{A}_{q_0} = \partial \mathscr{A}_{q_0}$ *is a broken line of two rectilinear rays — light-like one-parameter subgroups in* G.

Example: problem P_1

$U = \{u = (u_1, u_2) \in \mathbb{R}^2 \mid u_2^2 - u_1^2 \leq 0,\ u_1 \geq 0\}$, $g = u_2^2 - u_1^2$, $g_0 = -u_1$

Theorem 9

Let $q_1 = (x_1, y_1) \in M \setminus \{q_0\}$ *for the problem* P_1.

(1) *If* $x_1 = |y_1 - 1|$, *then* $x(t) = \pm(e^{\pm t} - 1)$, $y(t) = e^{\pm t}$, $\pm = \operatorname{sgn}(y_1 - 1)$, $t_1 = \pm \ln y_1$, $d(q_1) = 0$.

(2) *If* $x_1 > |y_1 - 1|$, *then*

$$x(t) = \cos c \tan \tau - \sin c, \quad y(t) = \frac{\cos c}{\cos \tau}, \quad \tau = c + t, \quad t_1 = \arcsin \alpha - c = d(q_1),$$

$$c = \arcsin \frac{y_1^2 - x_1^2 - 1}{2x_1}, \qquad \alpha = \frac{x_1^2 + y_1^2 - 1}{2x_1 y_1},$$

is the arc of the hyperbola $y^2 - (x - \sin c)^2 = \cos^2 c$.

Example: problem P_2

$U = \{u = (u_1, u_2) \in \mathbb{R}^2 \mid u_1^2 - u_2^2 \le 0,\ u_2 \ge 0\}$, $g = u_1^2 - u_2^2$, $g_0 = -u_2$

Theorem 10

Let $q_1 = (x_1, y_1) \in \mathscr{A}_{q_0} \setminus \{q_0\}$ *for the problem* P_2.

(1) *If* $y_1 - 1 = |x_1|$, *then* $x(t) = \pm(e^t - 1)$, $y(t) = e^t$, $\pm = \operatorname{sgn} x_1$, $t_1 = \ln y_1$, $d(q_1) = 0$.

(2) *If* $x_1 = 0$, *then* $x(t) \equiv 0$, $y(t) = e^t$, $t_1 = \ln y_1 = d(q_1)$.

(3) *If* $0 < |x_1| < y_1 - 1$, *then*

$$x(t) = \pm(\sinh c \coth \tau - \cosh c), \quad y(t) = \frac{\sinh c}{\sinh \tau}, \qquad \pm = \operatorname{sgn} x_1, \qquad \tau = c - t,$$

$$c = \operatorname{arsinh}\left(y_1\sqrt{\alpha^2 - 1}\right), \qquad t_1 = c - \operatorname{arcosh}\alpha = d(q_1), \qquad \alpha = \frac{x_1^2 + y_1^2 - 1}{2|x_1|y_1},$$

is the arc of the hyperbola $(\pm x + \cosh c)^2 - y^2 = \sinh^2 c$.

Example: problem P_3

$U = \{u = (u_1, u_2) \in \mathbb{R}^2 \mid u_1 \geq 0,\ u_2 \geq 0\}$, $g = -u_1 u_2$, $g_0 = -u_1$

Theorem 11

Let $q_1 = (x_1, y_1) \in \mathscr{A}_{q_0} \setminus \{q_0\}$ for the problem P_3.

(1) *If $x_1 = 0$, then $x(t) \equiv 0$, $y(t) = e^t$, $t_1 = \ln y_1$, $d(q_1) = 0$.*

(2) *If $y_1 = 1$, then $x(t) = t$, $y(t) \equiv 1$, $t_1 = x_1$, $d(q_1) = 0$.*

(3) *If $x_1 > 0$ and $y_1 > 1$, then $x(t) = c(c - \tau)$, $y(t) = \frac{c}{\tau}$,*

$$\tau = c - t, \qquad c = \sqrt{\frac{x_1 y_1}{y_1 - 1}}, \qquad t_1 = c - \sqrt{\frac{x_1}{y_1(y_1 - 1)}} = d(q_1),$$

is the arc of the hyperbola $x = c^2 \left(1 - \frac{1}{y}\right)$.

Sub-Lorentzian geometry

- Smooth manifold M,
- vector distribution $\Delta = \{\Delta_q \subset T_qM \mid q \in M\}$, $\dim \Delta_q \equiv \text{const}$,
- Lorentzian metric (nondegenerate quadratic form of index 1) in Δ:

$$g = \{g_q - \text{Lorentzian metric in } \Delta_q \mid q \in M\}$$

- sub-Lorentzian (SL) structure (Δ, g) on M
- horizontal vector: $v \in \Delta_q$,
- horizontal vector v is called:
    - timelike if $g(v) < 0$
    - spacelike if $g(v) > 0$ or $v = 0$,
    - lightlike if $g(v) = 0$ and $v \neq 0$,
    - nonspacelike if $g(v) \leq 0$
- Lipschitzian curve in M is called timelike if it has timelike velocity vector a.e.,
- spacelike, lightlike and nonspacelike curves are defined similarly.

Sub-Lorentzian geometry

- A time orientation X is an arbitrary timelike vector field in M.
- A nonspacelike vector $v \in \Delta_q$ is future directed if $g(v, X(q)) < 0$, and past directed if $g(v, X(q)) > 0$.
- A future directed timelike curve $q(t)$, $t \in [0, t_1]$, is called arclength parametrized if $g(\dot{q}(t), \dot{q}(t)) \equiv -1$.
- Any future directed timelike curve can be parametrized by arclength, similarly to the arclength parametrization of a horizontal curve in sub-Riemannian geometry.
- The length of a nonspacelike curve $\gamma \in \mathrm{Lip}([0, t_1], M)$ is

$$l(\gamma) = \int_0^{t_1} |g(\dot{\gamma}, \dot{\gamma})|^{1/2} dt.$$

- For points $q_1, q_2 \in M$ denote by $\Omega_{q_1 q_2}$ the set of all future directed nonspacelike curves in M that connect q_1 to q_2.
- In the case $\Omega_{q_1 q_2} \neq \emptyset$ denote the sub-Lorentzian distance from the point q_1 to the point q_2 as

$$d(q_1, q_2) = \sup\{l(\gamma) \mid \gamma \in \Omega_{q_1 q_2}\}. \tag{7}$$

Sub-Lorentzian geometry

- A future directed nonspacelike curve γ is called a SL length maximizer if it realizes the supremum in (7) between its endpoints $\gamma(0) = q_1$, $\gamma(t_1) = q_2$.
- The causal future of a point $q_0 \in M$ is the set $J^+(q_0)$ of points $q_1 \in M$ for which there exists a future directed nonspacelike curve γ that connects q_0 and q_1.
- The chronological future $I^+(q_0)$ of a point $q_0 \in M$ is defined similarly via future directed timelike curves γ.
- Let $q_0 \in M$, $q_1 \in J^+(q_0)$. The search for SL length maximizers that connect q_0 with q_1 reduces to the search for future directed nonspacelike curves γ that solve the problem

$$l(\gamma) \to \max, \qquad \gamma(0) = q_0, \quad \gamma(t_1) = q_1. \tag{8}$$

Sub-Lorentzian geometry

- Vector fields $X_1, \dots, X_k \in \mathrm{Vec}(M)$ form an orthonormal frame for (Δ, g) if

$$\Delta_q = \mathrm{span}(X_1(q), \dots, X_k(q)), \qquad q \in M,$$
$$g_q(X_1, X_1) = -1, \qquad g_q(X_i, X_i) = 1, \quad i = 2, \dots, k,$$
$$g_q(X_i, X_j) = 0, \quad i \neq j.$$

- Assume that time orientation is defined by a timelike vector field $X \in \mathrm{Vec}(M)$ for which $g(X, X_1) < 0$ (e.g., $X = X_1$). Then the SL problem for the SL structure with the orthonormal frame $X_1, \dots, X_k$ is stated as follows:

$$\dot{q} = \sum_{i=1}^{k} u_i X_i(q), \qquad q \in M,$$
$$u \in U = \left\{ (u_1, \dots, u_k) \in \mathbb{R}^k \mid u_1 \geq \sqrt{u_2^2 + \dots + u_k^2} \right\},$$
$$q(0) = q_0, \quad q(t_1) = q_1, \qquad l(q(\cdot)) = \int_0^{t_1} \sqrt{u_1^2 - u_2^2 - \dots - u_k^2}\, dt \to \max.$$

Sub-Lorentzian geometry

- The SL length is preserved under monotone Lipschitzian time reparametrizations $t(s)$, $s \in [0, s_1]$. Thus if $q(t)$, $t \in [0, t_1]$, is a sub-Lorentzian length maximizer, then so is any its reparametrization $q(t(s))$, $s \in [0, s_1]$.
- We choose the following parametrization of trajectories: the arclength parametrization $(u_1^2 - u_2^2 - \cdots - u_k^2 \equiv 1)$ for timelike trajectories, and the parametrization with $u_1(t) \equiv 1$ for future directed lightlike trajectories.

Statement of the SL problem on the Heisenberg group

- The Heisenberg group is the space $M \simeq \mathbb{R}^3_{x,y,z}$ with the product rule
$$(x_1, y_1, z_1) \cdot (x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2 + (x_1 y_2 - x_2 y_1)/2).$$
- It is a three-dimensional nilpotent Lie group with a left-invariant frame
$$X_1 = \frac{\partial}{\partial x} - \frac{y}{2}\frac{\partial}{\partial z}, \qquad X_2 = \frac{\partial}{\partial y} + \frac{x}{2}\frac{\partial}{\partial z}, \qquad X_3 = \frac{\partial}{\partial z}, \tag{9}$$
with the only nonzero Lie bracket $[X_1, X_2] = X_3$.
- Consider the left-invariant SL problem on the Heisenberg group M defined by the orthonormal frame (X_1, X_2), with the time orientation X_1:
$$\dot q = u_1 X_1 + u_2 X_2, \qquad q \in M, \tag{10}$$
$$u \in U = \{(u_1, u_2) \in \mathbb{R}^2 \mid u_1 \geq |u_2|\}, \tag{11}$$
$$q(0) = q_0 = \mathrm{Id} = (0, 0, 0), \quad q(t_1) = q_1, \tag{12}$$
$$l(q(\cdot)) = \int_0^{t_1} \sqrt{u_1^2 - u_2^2}\, dt \to \max. \tag{13}$$

Reduced SL problem on the Heisenberg group

- Reduced sub-Lorentzian problem

$$\dot{q} = u_1 X_1 + u_2 X_2, \qquad q \in M, \tag{14}$$

$$u \in \operatorname{int} U = \{(u_1, u_2) \in \mathbb{R}^2 \mid u_1 > |u_2|\}, \tag{15}$$

$$q(0) = q_0 = \mathrm{Id} = (0, 0, 0), \quad q(t_1) = q_1, \tag{16}$$

$$l(q(\cdot)) = \int_0^{t_1} \sqrt{u_1^2 - u_2^2}\, dt \to \max. \tag{17}$$

- In the full problem (10)–(13) admissible trajectories $q(\cdot)$ are future directed nonspacelike ones, while in the reduced problem (14)–(17) admissible trajectories $q(\cdot)$ are only future directed timelike ones.
- Passing to arclength-parametrized future directed timelike trajectories:

$$\dot{q} = u_1 X_1 + u_2 X_2, \qquad q \in M, \qquad u_1^2 - u_2^2 = 1, \qquad u_1 > 0, \tag{18}$$

$$q(0) = q_0 = \mathrm{Id} = (0, 0, 0), \quad q(t_1) = q_1, \tag{19}$$

$$t_1 \to \max. \tag{20}$$

Previously obtained results by M. Grochowski

(1) Sub-Lorentzian extremal trajectories were parametrized by hyperbolic and linear functions: were obtained formulas equivalent to our formulas (23), (24).

(2) It was proved that there exists a domain in M containing $q_0 = \mathrm{Id}$ in its boundary at which the sub-Lorentzian distance $d(q_0, q)$ is smooth.

(3) The attainable sets of the sub-Lorentzian structure from the point $q_0 = \mathrm{Id}$ were computed: the chronological future of the point q_0

$$I^+(q_0) = \{(x, y, z) \in M \mid -x^2 + y^2 + 4|z| < 0, \; x > 0\},$$

and the causal future of the point q_0

$$J^+(q_0) = \{(x, y, z) \in M \mid -x^2 + y^2 + 4|z| \leq 0, \; x \geq 0\}. \tag{21}$$

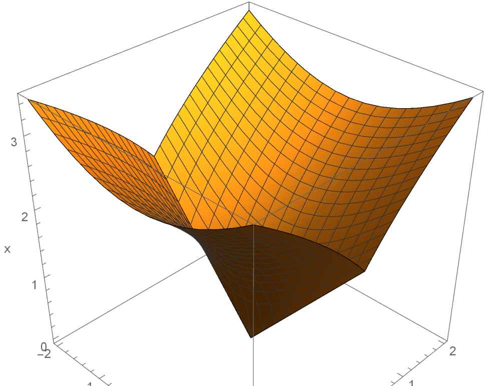
In the standard language of control theory, $I^+(q_0)$ is the attainable set of the reduced system (14), (15) from the point q_0 for arbitrary positive time. Thus the attainable set of the reduced system (14), (15) from the point q_0 for arbitrary nonnegative time is

$$\mathscr{A} = I^+(q_0) \cup \{q_0\}.$$

Previously obtained results by M. Grochowski

(3) The attainable set of the full system (10), (11) from the point q_0 for arbitrary nonnegative time is $\mathrm{cl}(\mathscr{A}) = J^+(q_0)$.

(4) The attainable set $\mathscr{A}$ was also computed by H. Abels and E.B. Vinberg, they called its boundary as the Heisenberg beak. See the set $\partial\mathscr{A}$ below, and its views from the y- and z-axes in the next slide.



Views of the Heisenberg beak

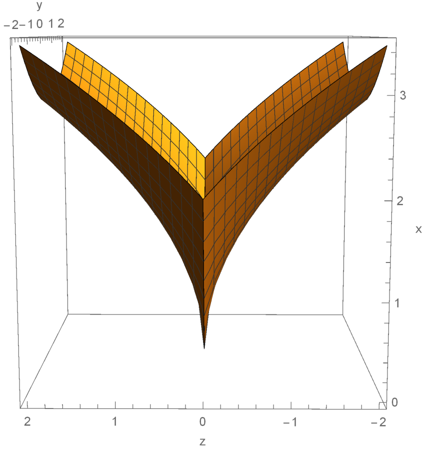


Figure: View of $\partial\mathscr{A}$ along y-axis

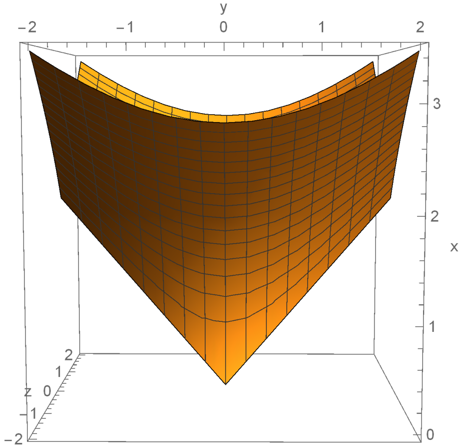


Figure: View of $\partial\mathscr{A}$ along z-axis

Pontryagin maximum principle

- Denote points of the cotangent bundle T^*M as λ. Introduce linear on fibers of T^*M Hamiltonians $h_i(\lambda) = \langle \lambda, X_i \rangle$, $i = 1, 2, 3$.
- Define the Hamiltonian of the Pontryagin maximum principle (PMP) for the sub-Lorentzian problem (10)–(13)

$$h_u^\nu(\lambda) = u_1 h_1(\lambda) + u_2 h_2(\lambda) - \nu\sqrt{u_1^2 - u_2^2}, \qquad \lambda \in T^*M, \quad u \in U, \quad \nu \in \mathbb{R}.$$

- It follows from PMP that if $u(t)$, $t \in [0, t_1]$, is an optimal control in problem (10)–(13), and $q(t)$, $t \in [0, t_1]$, is the corresponding optimal trajectory, then there exists a curve $\lambda_\cdot \in \mathrm{Lip}([0, t_1], T^*M)$, $\pi(\lambda_t) = q(t)$, and a number $\nu \in \{0, -1\}$ for which there hold the conditions for a.e. $t \in [0, t_1]$:
    1. the Hamiltonian system $\dot\lambda_t = \vec{h}^\nu_{u(t)}(\lambda_t)$,
    2. the maximality condition $h^\nu_{u(t)}(\lambda_t) = \max_{v \in U} h_v^\nu(\lambda_t) \equiv 0$,
    3. the nontriviality condition $(\nu, \lambda_t) \neq (0, 0)$.

Abnormal case

Theorem 12

In the abnormal case $\nu = 0$ there exist $\tau_1, \tau_2 \geq 0$ such that:

(1) $h_3(\lambda_t) \equiv \text{const} > 0$:

$$\begin{aligned} t \in (0, \tau_1) \quad &\Rightarrow \quad h_1(\lambda_t) = h_2(\lambda_t) < 0, \quad && u_1(t) = -u_2(t), \\ t \in (\tau_1, \tau_1 + \tau_2) \quad &\Rightarrow \quad h_1(\lambda_t) = -h_2(\lambda_t) < 0, \quad && u_1(t) = u_2(t). \end{aligned}$$

(2) $h_3(\lambda_t) \equiv \text{const} < 0$:

$$\begin{aligned} t \in (0, \tau_1) \quad &\Rightarrow \quad h_1(\lambda_t) = -h_2(\lambda_t) < 0, \quad && u_1(t) = u_2(t), \\ t \in (\tau_1, \tau_1 + \tau_2) \quad &\Rightarrow \quad h_1(\lambda_t) = h_2(\lambda_t) < 0, \quad && u_1(t) = -u_2(t). \end{aligned}$$

(3) $h_3(\lambda_t) \equiv 0$:

$$\begin{aligned} &(h_1, h_2)(\lambda_t) \equiv \text{const} \neq (0, 0), \qquad h_1(\lambda_t) \equiv -|h_2(\lambda_t)|, \\ &u(t) \equiv \text{const}, \qquad u_1(t) \equiv \pm u_2(t), \qquad \pm = -\operatorname{sgn}(h_1 h_2(\lambda_t)). \end{aligned}$$

Normal case

- In the normal case ($\nu = -1$) extremals exist only for $h_1 \leq -|h_2|$.
- In the case $h_1 = -|h_2|$ normal controls and extremal trajectories coincide with the abnormal ones.
- And in the domain $\{\lambda \in T^*M \mid h_1 < -|h_2|\}$ extremals are reparametrizations of trajectories of the Hamiltonian vector field $\vec{H}$ with the Hamiltonian $H = \frac{1}{2}(h_2^2 - h_1^2)$.
- In the arclength parametrization, the extremal controls are

$$(u_1, u_2)(t) = (-h_1(\lambda_t), h_2(\lambda_t)), \tag{22}$$

and the extremals satisfy the Hamiltonian ODE $\dot{\lambda} = \vec{H}(\lambda)$ and belong to the level surface $\{H(\lambda) = \frac{1}{2}\}$, in coordinates:

$$\begin{aligned} \dot{h}_1 &= -h_2 h_3, \qquad \dot{h}_2 = -h_1 h_3, \qquad \dot{h}_3 = 0, \\ \dot{q} &= \cosh\psi\, X_1 + \sinh\psi\, X_2, \\ h_1 &= -\cosh\psi, \qquad h_2 = \sinh\psi, \qquad \psi \in \mathbb{R}. \end{aligned}$$

Parametrization of normal trajectories

- If $h_3 = 0$, then

$$x = t\cosh\psi, \quad y = t\sinh\psi, \quad z = 0. \tag{23}$$

- If $c := h_3 \neq 0$, then

$$x = \frac{\sinh(\psi + ct) - \sinh\psi}{c}, \quad y = \frac{\cosh(\psi + ct) - \cosh\psi}{c}, \quad z = \frac{\sinh(ct) - ct}{2c^2}. \tag{24}$$

Theorem 13

Normal controls and trajectories either coincide with abnormal ones (in the case $h_1(\lambda_t) = -|h_2(\lambda_t)|$), or can be arclength parametrized to get controls (22) *and future directed timelike trajectories* (23) *if $c = 0$, or* (24) *if $c \neq 0$.*
In particular, each normal extremal can be parameterized so that $H(\lambda_t) \equiv \mathrm{const} \in \left\{0, \frac{1}{2}\right\}$.

Exponential mapping

- Normal trajectories are either nonstrictly normal (i.e., simultaneously normal and abnormal) in the case $H = 0$, or strictly normal (i.e., normal but not abnormal) in the case $H = \frac{1}{2}$.
- Strictly normal arclength-parametrized trajectories are described by the exponential mapping

$$\mathrm{Exp} \,:\, N \to \widetilde{\mathscr{A}}, \qquad (\lambda, t) \mapsto q(t) = \pi \circ e^{t\vec{H}}(\lambda), \tag{25}$$

$$N = C \times \mathbb{R}_+, \qquad \mathbb{R}_+ = (0, +\infty), \qquad C = T^*_{\mathrm{Id}} M \cap H^{-1}\left(\frac{1}{2}\right) \simeq \mathbb{R}^2_{\psi, c},$$

$$\widetilde{\mathscr{A}} = \operatorname{int} \mathscr{A} = I^+(q_0)$$

given explicitly by formulas (23), (24).

Projections of strictly normal trajectories

- Projections of strictly normal (future directed timelike) trajectories to the plane (x, y) are:
  - either rays $y = kx$, $x \geq 0$, $k \in (-1, 1)$ (for $c = 0$),
  - or arcs of hyperbolas with asymptotes parallel to rays $x = \pm y > 0$ (for $c \neq 0$).

Projections of nonstrictly normal trajectories

- Projections of nonstrictly normal trajectories to the plane (x, y) are broken lines with one or two edges parallel to the rays $x = \pm y > 0$.
- Projections of all extremal trajectories (as well as of all admissible trajectories) to the plane (x, y) are contained in the angle $\{(x, y) \in \mathbb{R}^2 \mid x \geq |y|\}$, which is the projection of the attainable set $J^+(q_0)$ to this plane.

Inversion of the exponential mapping

Theorem 14

The exponential mapping $\operatorname{Exp} : N \to \widetilde{\mathscr{A}}$ *is a real-analytic diffeomorphism. The inverse mapping* $\operatorname{Exp}^{-1} : \widetilde{\mathscr{A}} \to N$, $(x, y, z) \mapsto (\psi, c, t)$, *is given by the following formulas:*

$$z = 0 \quad \Rightarrow \quad \psi = \operatorname{artanh} \frac{y}{x}, \quad c = 0, \quad t = \sqrt{x^2 - y^2}, \tag{26}$$

$$z \neq 0 \quad \Rightarrow \quad \psi = \operatorname{artanh} \frac{y}{x} - p, \quad c = (\operatorname{sgn} z)\sqrt{\frac{\sinh 2p - 2p}{2z}}, \quad t = \frac{2p}{c}, \tag{27}$$

where $p = \beta\left(\frac{z}{x^2 - y^2}\right)$, *and* $\beta : \left(-\frac{1}{4}, \frac{1}{4}\right) \to \mathbb{R}$ *is the inverse function to the diffeomorphism*

$$\alpha : \mathbb{R} \to \left(-\frac{1}{4}, \frac{1}{4}\right), \qquad \alpha(p) = \frac{\sinh 2p - 2p}{8 \sinh^2 p}.$$

Lagrangian manifolds

- Let M be a smooth manifold, then the cotangent bundle T^*M bears the Liouville 1-form $s = pdq \in \Lambda^1(T^*M)$ and the symplectic 2-form $\sigma = ds = dp \wedge dq \in \Lambda^2(T^*M)$.
- A submanifold $\mathscr{L} \subset T^*M$ is called a Lagrangian manifold if $\dim \mathscr{L} = \dim M$ and $\sigma|_{\mathscr{L}} = 0$.
- Consider an optimal control problem

$$\dot{q} = f(q, u), \qquad q \in M, \quad u \in U,$$
$$q(t_0) = q_0, \qquad q(t_1) = q_1,$$
$$J[q(\cdot)] = \int_{t_0}^{t_1} \varphi(q, u)\, dt \to \min, \qquad t_0 \text{ is fixed}, \quad t_1 \text{ is free}.$$

- Let $g_u(\lambda) = \langle \lambda, f(q, u) \rangle - \varphi(q, u)$, $\lambda \in T^*M$, $q = \pi(\lambda)$, $u \in U$, be the normal Hamiltonian of PMP.
- Suppose that the maximized normal Hamiltonian $G(\lambda) = \max_{u \in U} g_u(\lambda)$ is smooth in an open domain $O \subset T^*M$, and let the v. field $\vec{G} \in \mathrm{Vec}(O)$ be complete.

Sufficient optimality condition

Theorem 15

- *Let* $\mathscr{L} \subset G^{-1}(0) \cap O$ *be a Lagrangian submanifold such that the form* $s|_{\mathscr{L}}$ *is exact.*
- *Let the projection* $\pi : \mathscr{L} \to \pi(\mathscr{L})$ *be a diffeomorphism on a domain in* M*.*
- *Consider an extremal* $\widetilde{\lambda}_t = e^{t\vec{G}}(\lambda_0)$*,* $t \in [t_0, t_1]$*, contained in* $\mathscr{L}$*, and the corresponding extremal trajectory* $\widetilde{q}(t) = \pi(\widetilde{\lambda}_t)$*.*
- *Consider also any other trajectory* $q(t) \in \pi(\mathscr{L})$*,* $t \in [t_0, \tau]$*, such that* $q(t_0) = \widetilde{q}(t_0)$*,* $q(\tau) = \widetilde{q}(t_1)$*.*
- *Then* $J[\widetilde{q}(\cdot)] < J[q(\cdot)]$*.*

Optimality in the reduced SL problem

- For the reduced SL problem the maximized Hamiltonian $G = 1 - \sqrt{h_1^2 - h_2^2}$ is smooth on the domain $O = \{\lambda \in T^*M \mid h_1 < -|h_2|\}$, and the Hamiltonian vector field $\vec{G} \in \mathrm{Vec}(O)$ is complete
- In the domain O the Hamiltonian vector fields $\vec{G}$ and $\vec{H}$ have the same trajectories up to a monotone time reparametrization; moreover, on the level surface $\left\{H = \frac{1}{2}\right\} = \{G = 0\}$ they just coincide between themselves.
- Define the set

$$\mathscr{L} = \left\{e^{t\vec{G}}(\lambda_0) \mid \lambda_0 \in C,\ t > 0\right\}. \qquad (28)$$

Lemma 16

$\mathscr{L} \subset T^*M$ is a Lagrangian manifold such that $s|_{\mathscr{L}}$ is exact.

Theorem 17

For any point $q_1 \in \operatorname{int} \mathscr{A} = I^+(q_0)$ the strictly normal trajectory $q(t) = \mathrm{Exp}(\lambda, t)$, $t \in [0, t_1]$, is the unique optimal trajectory of the reduced SL problem connecting q_0 with q_1, where $(\lambda, t_1) = \mathrm{Exp}^{-1}(q_1) \in N$.

The cost function for the equivalent reduced SL problem

Denote

$$\begin{aligned}\widetilde{d}(q_1) &= \sup\{l(q(\cdot)) \mid \text{ traj. } q(\cdot) \text{ of (14)–(17), } q(0) = q_0,\ q(t_1) = q_1\} \\ &= \sup\{t_1 > 0 \mid \exists \text{ traj. } q(\cdot) \text{ of (18)–(20) s.t. } q(0) = q_0,\ q(t_1) = q_1\},\end{aligned}$$

where $q_1 \in \operatorname{int} \mathscr{A} = I^+(q_0)$.

Theorem 18

Let $q = (x, y, z) \in I^+(q_0)$. Then

$$\widetilde{d}(q) = \sqrt{x^2 - y^2} \cdot \frac{p}{\sinh p}, \qquad p = \beta\left(\frac{z}{x^2 - y^2}\right). \tag{29}$$

The function $\widetilde{d} : I^+(q_0) \to \mathbb{R}_+$ is real-analytic.

Optimality in the full SL problem

Theorem 19

Let $q_1 \in \operatorname{int} A = I^+(q_0)$. Then the SL length maximizers for the full problem are reparametrizations of the corresponding SL length maximizers for the reduced problem described above.

In particular, $d|_{I^+(q_0)} = \widetilde{d}$.

Theorem 20

Let $q_1 = (x_1, y_1, z_1) \in \partial A = J^+(q_0) \setminus I^+(q_0)$, $q_1 \neq q_0$. Then an optimal trajectory in the full SL problem is a future directed lightlike piecewise smooth trajectory with one or two subarcs generated by the vector fields $X_1 \pm X_2$.

Length maximizers in the full SL problem

Corollary 21

For any $q_1 \in J^+(q_0)$, $q_1 \neq q_0$, there is a unique, up to reparametrization, SL length maximizer in the full problem that connects q_0 and q_1:

- *if $q_1 \in \operatorname{int} \mathscr{A} = I^+(q_0)$, then $q(\cdot)$ is a future directed timelike strictly normal trajectory.*
- *if $q_1 \in \partial \mathscr{A} = J^+(q) \setminus I^+(q_0)$, then $q(\cdot)$ is a future directed lightlike nonstrictly normal trajectory.*

Corollary 22

Any SL length maximizer of the full problem problem of positive length is timelike and strictly normal.

- The broken trajectories described above are optimal in the SL problem, while in SR problems trajectories with angle points cannot be optimal.
- Moreover, these broken trajectories are normal and nonsmooth, which is also impossible in SR geometry.

Sub-Lorentzian distance

Denote $d(q) := d(q_0, q)$, $q \in J^+(q_0)$.

Theorem 23

Let $q = (x, y, z) \in J^+(q_0)$. *Then*

$$d(q) = \sqrt{x^2 - y^2} \cdot \frac{p}{\sinh p}, \qquad p = \beta\left(\frac{z}{x^2 - y^2}\right). \tag{30}$$

In particular:

(1) $z = 0 \quad \Leftrightarrow \quad d(q) = \sqrt{x^2 - y^2}$,

(2) $q \in J^+(q_0) \setminus I^+(q_0) \quad \Leftrightarrow \quad d(q) = 0$.

Regularity of the sub-Lorentzian distance

Theorem 24

(1) *The function $d(\cdot)$ is continuous on $J^+(q_0)$ and real-analytic on $I^+(q_0)$.*

(2) *The function $d(\cdot)$ is not Lipschitz near points $q = (x, y, z)$ with $x = |y| > 0$, $z = 0$.*

Remark 1

The sub-Lorentzian distance $d : J^+(q_0) \to [0, +\infty)$ is not uniformly continuous since the same holds for its restriction $d|_{z=0} = \sqrt{x^2 - y^2}$ on the angle $\{x \geq |y|\}$.

Bounds of the sub-Lorentzian distance

Theorem 25

(1) *The ratio* $\dfrac{\sqrt{x^2 - y^2 - 4|z|}}{d(q)}$ *takes any values in the segment* $[0,1]$ *for* $q = (x, y, z) \in J^+(q_0)$.

(2) *For any* $q = (x, y, z) \in J^+(q_0)$ *there holds the bound* $d(q) \le \sqrt{x^2 - y^2}$, *moreover, the ratio* $\dfrac{d(q)}{\sqrt{x^2 - y^2}}$ *takes any values in the segment* $[0,1]$.

Symmetries

Theorem 26

(1) *The hyperbolic rotations* $X_0 = y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$ *and reflections* $\varepsilon^1 \; : \; (x, y, z) \mapsto (x, -y, z)$, $\varepsilon^2 \; : \; (x, y, z) \mapsto (x, y, -z)$ *preserve* $d(\cdot)$.

(2) *The dilations* $Y = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + 2z\frac{\partial}{\partial z}$ *stretch* $d(\cdot)$:

$$d(e^{sY}(q)) = e^s d(q), \qquad s \in \mathbb{R}, \quad q \in J^+(q_0).$$

The unit sub-Lorentzian sphere

$$S = \{\mathrm{Exp}(\lambda, 1) \mid \lambda \in C\}$$

Theorem 27

(1) *The unit SL sphere S is a regular real-analytic manifold diffeomorphic to $\mathbb{R}^2$.*

(2) *Let $q = \mathrm{Exp}(\psi, c, 1) \in S$, $(\psi, c) \in C$, then the tangent space*

$$T_q S = \left\{ v = \sum_{i=1}^{3} v_i X_i(q) \mid -v_1 \cosh(\psi + c) + v_2 \sinh(\psi + c) + v_3 c = 0 \right\}. \quad (31)$$

(3) *S is the graph of the function $x = \sqrt{y^2 + f(z)}$, where $f(z) = e \circ k(z)$,* $e(w) = \frac{\sinh^2 w}{w^2}$, $k(z) = b(z)/2$, $b = a^{-1}$, $a(c) = \frac{\sinh c - c}{2c^2}$.

(4) *The function $f(z)$ is real-analytic, even, strictly convex, unboundedly and strictly increasing for $z \geq 0$. This function has a Taylor decomposition* $f(z) = 1 + 12z^2 + O(z^4)$ *as $z \to 0$ and an asymptote $4|z|$ as $z \to \infty$.*

The unit sub-Lorentzian sphere

(5) The function $f(z)$ satisfies the bounds

$$4|z| < f(z) < 4|z| + 1, \qquad z \neq 0. \tag{32}$$

(6) A section of the sphere S by a plane $\{z = \text{const}\}$ is a branch of the hyperbola $x^2 - y^2 = f(z)$, $x > 0$. A section of the sphere S by a plane $\{x = \text{const} > 1\}$ is a strictly convex curve $y^2 + f(z) = x^2$ diffeomorphic to S^1.

(7) The sub-Lorentzian distance from the point q_0 to a point $q = (x, y, z) \in \widetilde{\mathscr{A}}$ may be expressed as $d(q) = R$, where $x^2 - y^2 = R^2 f(z/R^2)$.

(8) The sub-Lorentzian ball $B = \{q \in M \mid d(q) \leq 1\}$ has infinite volume in the coordinates x, y, z.

Sub-Lorentzian sphere of zero radius

$$S(0) = \{q \in M \mid d(q) = 0\}.$$

Theorem 28

(1) $S(0) = J^+(q_0) \setminus I^+(q_0) = \partial J^+(q_0) = \partial I^+(q_0) = \partial \mathscr{A}$.

(2) *$S(0)$ is the graph of a continuous function $x = \Phi(y, z) := \sqrt{y^2 + 4|z|}$, thus a 2-dimensional topological manifold.*

(3) *The function $\Phi(y, z)$ is even in y and z, real-analytic for $z \neq 0$, Lipschitz near $z = 0$, $y \neq 0$, and Hölder with constant $\frac{1}{2}$, non-Lipschitz near $(y, z) = (0, 0)$.*

(4) *$S(0)$ is filled by broken lightlike trajectories with one or two edges, and is parametrized by them as follows:*

$$S(0) = \left\{ e^{\tau_2(X_1 - X_2)} e^{\tau_1(X_1 + X_2)} = (\tau_1 + \tau_2, \tau_1 - \tau_2, -\tau_1\tau_2) \mid \tau_i \geq 0 \right\}$$
$$\cup \left\{ e^{\tau_2(X_1 + X_2)} e^{\tau_1(X_1 - X_2)} = (\tau_1 + \tau_2, \tau_2 - \tau_1, \tau_1\tau_2) \mid \tau_i \geq 0 \right\}.$$

Sub-Lorentzian sphere of zero radius

(5) The flows of the vector fields Y, X_0 preserve $S(0)$. Moreover, the symmetries Y, X_0 provide a regular parametrization of

$$S(0) \cap \{\operatorname{sgn} z = \pm 1\} = \left\{ e^{sY} \circ e^{rX_0}(q_\pm) \mid r, s > 0 \right\}, \tag{33}$$

where $q_\pm = (x_\pm, y_\pm, z_\pm)$ is any point in $S(0) \cap \{\operatorname{sgn} z = \pm 1\}$.

(6) $S(0) = \left\{16z^2 = (x^2 - y^2)^2,\ x^2 - y^2 \geq 0,\ x \geq 0\right\}$ is a semi-algebraic set.

(7) The zero-radius sphere is a Whitney stratified set with the stratification

$$\begin{aligned} S(0) = &\left(S(0) \cap \{z > 0\}\right) \cup \left(S(0) \cap \{z < 0\}\right) \\ &\cup \left(S(0) \cap \{z = 0,\ y > 0\}\right) \cup \left(S(0) \cap \{z = 0,\ y < 0\}\right) \cup \{q_0\}. \end{aligned}$$

(8) Intersection of the sphere $S(0)$ with a plane $\{z = \text{const} \neq 0\}$ is a branch of a hyperbola $\{x^2 - y^2 = 4|z|,\ x > 0, z = \text{const}\}$, intersection with a plane $\{z = 0\}$ is an angle $\{x = |y|, z = 0\}$, intersection with a plane $\{y = kx\}$, $k \in (-1, 1)$, is a union of two half-parabolas $\{4z = \pm(1 - k^2)x^2,\ x \geq 0,\ y = kx\}$, and intersection with a plane $\{y = \pm x\}$ is a ray $\{y = \pm x,\ z = 0\}$.

Conclusion

The results obtained for the SL problem on the Heisenberg group differ drastically from the known results for the SR problem on the same group:

1. The SL problem is not completely controllable.
2. Filippov's existence theorem for optimal controls cannot be immediately applied to the SL problem.
3. In the SL problem all extremal trajectories are infinitely optimal, thus the cut locus and the conjugate locus for them are empty.
4. The SL length maximizers coming to the zero-radius sphere are nonsmooth (concatenations of two smooth arcs forming a corner, nonstrictly normal extremal trajectories).
5. SL spheres and SL distance are real-analytic if $d > 0$.

It would be interesting to understand which of these properties persist for more general SL problems (e.g., for left-invariant problems on Carnot groups).

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