

Sub-Riemannian structures on Lie groups (*Lecture 9*)

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«Introduction to geometric control theory»

Lecture course in Dept. of Mathematics and Mechanics

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8. *Нет ни быка, ни человека:*

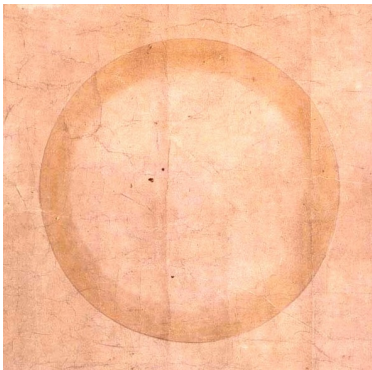
Нет ничего — ни хлыста, ни привязи; ни человека, ни быка:

Разве можно объять взором просторы небес?

В пылающей печи не найти никаких снежинок:

Когда положение дел таково, проявляется дух древнего мастера.

Пу-мин, “Десять рисунков о пастухе и быке”



Reminder: Plan of the previous lecture

1. Euler's elastic problem.

Outline of this lecture

1. Sub-Riemannian structures, minimizers, spheres
2. Sub-Riemannian problem on the group of Euclidean plane motions $SE(2)$
3. Sub-Riemannian problem on the group of hyperbolic plane motions $SH(2)$

Basic definitions

- a smooth manifold M ,
- distribution $\Delta = \{\Delta_q \subset T_q M \mid q \in M\}$, $\dim \Delta_q \equiv \text{const}$,
- scalar product in Δ :

$$g = \{g_q - \text{scalar product in } \Delta_q \mid q \in M\}$$

- SR manifold (M, Δ, g) , SR structure (Δ, g) on M
- horizontal (admissible) curve $q \in \text{Lip}([0, t_1], M)$:

$$\dot{q}(t) \in \Delta_{q(t)} \text{ for a.e. } t \in [0, t_1],$$

- length $l(q(\cdot)) = \int_0^{t_1} (g(\dot{q}(t), \dot{q}(t)))^{1/2} dt$,
- SR distance $d(q_0, q_1) = \inf \{l(q(\cdot)) \mid q(\cdot) \text{ horizontal curve, } q(0) = q_0, q(t_1) = q_1\}$,

- SR minimizer $q(t)$, $t \in [0, t_1]$: horizontal curve s.t. $l(q(\cdot)) = d(q(0), q(t_1))$,
- sphere $S_R(q_0) = \{q \in M \mid d(q, q_0) = R\}$,
ball $B_R(q_0) = \{q \in M \mid d(q, q_0) \leq R\}$,
- geodesic: horizontal curve whose short arcs are minimizers,
- cut time along geodesic $q(t)$:

$$t_{\text{cut}}(q(\cdot)) = \sup\{t > 0 \mid q(s), s \in [0, t], \text{ minimizer } \},$$

- cut point $q(t_1)$, $t_1 = t_{\text{cut}}(q(\cdot))$,
- cut locus

$$\text{Cut}_{q_0} = \{q_1 \in M \mid q_1 \text{ cut point for some geodesic } q(\cdot), q(0) = q_0\}$$

- first conjugate time along geodesic $q(t)$:

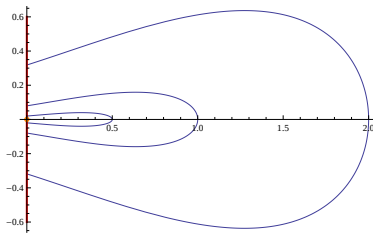
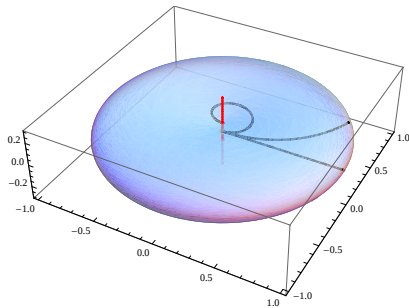
$$t_{\text{conj}}^1(q(\cdot)) = \sup\{t > 0 \mid q(s), s \in [0, t], \text{ locally optimal } \},$$

- $q(\cdot)$ locally optimal if $\exists$ a neighborhood of $O \supset \{q(t)\}$ including $q(\cdot)$ is the minimizer on $(O, \Delta|_O, g|_O)$,
- the first conjugate point along the geodesic $q(t)$: $q(t_1)$, $t_1 = t_{\text{conj}}^1(q(\cdot))$,
- the first caustic:

$$\begin{aligned} \text{Conj}_{q_0} = \{q_1 \in M \mid \\ q_1 \text{ the first conjugate point for some geodesic } q(\cdot), \\ q(0) = q_0\}. \end{aligned}$$

Example: Heisenberg Group

- $M = \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \mid (a, b, c) \in \mathbb{R}^3 \right\}$
- $X_1 = \frac{\partial}{\partial x} - \frac{y}{2} \frac{\partial}{\partial z}, \quad X_2 = \frac{\partial}{\partial y} + \frac{x}{2} \frac{\partial}{\partial z}, \quad x = a, \quad y = b, \quad z = c - ab/2$
- $\Delta_q = \text{span}(X_1(q), X_2(q)), \quad g(X_i, X_j) = \delta_{ij}$



Optimal control problem

- SR manifold (M, Δ, g)
- Orthonormal frame:

$$\Delta_q = \text{span}(X_1(q), \dots, X_k(q)), \quad g(X_i, X_j) = \delta_{ij}, \quad i, j = 1, \dots, k,$$

- Minimizer $q(t)$ — solution of the problem

$$\dot{q} = \sum_{i=1}^k u_i X_i(q), \quad q \in M, \quad u_i \in \mathbb{R},$$

$$q(0) = q_0, \quad q(t_1) = q_1,$$

$$I = \int_0^{t_1} \left(\sum_{i=1}^k u_i^2(t) \right)^{1/2} dt \rightarrow \min$$

$$\Leftrightarrow J = \frac{1}{2} \int_0^{t_1} \sum_{i=1}^k u_i^2(t) dt \rightarrow \min.$$

Existence of Solutions

Theorem 1 (Rashevskii-Chow)

Let M be connected and for all $q \in M$

$$\text{span}(X_i(q), [X_i, X_j](q), [[X_i, X_j], X_l](q), \dots) = T_q M. \quad (1)$$

Then for $\forall q_0, q_1 \in M \exists$ a horizontal curve $q(t)$, $t \in [0, t_1]$, so $q(0) = q_0$, $q(t_1) = q_1$.

Further, the condition of full rank (1) is assumed to be satisfied.

Theorem 2 (Filippov)

A minimizer connecting points $q_0, q_1 \in M$ exists if one of the following conditions is satisfied:

- *q_1 is sufficiently close to q_0 ,*
- *the balls $B_R(q_0)$ are compact,*
- *(Δ, g) is left-invariant on the Lie group M .*

Pontryagin's Maximum Principle

- $h_i(\lambda) = \langle \lambda, X_i(q) \rangle, \lambda \in T^*M.$

Theorem 3 (Pontryagin)

If $q(t), t \in [0, t_1]$, is a length minimizer corresponding to control $u(t)$, then $\exists \lambda \in \text{Lip}([0, t_1], T^*M), \lambda(t) \in T_{q(t)}^*M$, s.t.:

(N) either $\dot{\lambda}(t) = \vec{H}(\lambda(t)), H(\lambda) = \frac{1}{2} \sum_{i=1}^k h_i^2(\lambda), u_i(t) = h_i(\lambda(t)),$

(A) or $h_1(\lambda(t)) = \dots = h_k(\lambda(t)) \equiv 0, \dot{\lambda}(t) = \sum_{i=1}^k u_i(t) \vec{h}_i(\lambda(t)).$

(N) $\Rightarrow \lambda(t)$ is a normal extremal, $q(t)$ is a normal extremal trajectory,

(A) $\Rightarrow \lambda(t)$ is an abnormal extremal, $q(t)$ is an abnormal extremal trajectory.

Optimality of normal geodesics

- $q(t)$ — normal extremal trajectory $\Rightarrow$
 $q(t)$ — geodesic (strong Legendre condition)
- $\lambda(t)$ — normal extremal $\Rightarrow$
 $\lambda(t) = e^{t\vec{H}}(\lambda_0), \quad H(\lambda(t)) \equiv \text{const}$
- $\lambda_0 \in C = \{H(\lambda) \equiv 1/2\} \cap T_{q_0}^*M$
- Exponential map $\text{Exp} : C \times \mathbb{R}_+ \rightarrow M$, $\text{Exp}(\lambda, t) = q(t) = \pi \circ e^{t\vec{H}}(\lambda)$.
- q_1 is a Maxwell point on $q(t)$:
 $\exists \tilde{q}(t) \neq q(t), \tilde{q}(0) = q(0), \tilde{q}(t_1) = q(t_1) = q_1$.

Theorem 4

Let $q(t) = \text{Exp}(\lambda, t)$ be a normal geodesic that does not contain abnormal arcs. If t_1 is the cut time, then $q(t_1)$ is the first Maxwell point or the first conjugate point.

Smoothness of Spheres

Theorem 5

If $\Delta_{q_0} \neq T_{q_0}M$, then any sphere $S_R(q_0)$ is not a smooth manifold (if $S_R(q_0) \neq \emptyset$).

Theorem 6

Let $q_1 \in S_R(q_0)$. Suppose that:

- (1) q_1 is connected to q_0 by a unique normal minimizer $q(t)$,*
- (2) q_1 is not a conjugate point along $q(t)$.*

Then $S_R(q_0)$ is a smooth manifold in the neighborhood of q_1 .

Corollary 1

Reasons for cut points and singularities of spheres:

- (1) abnormal shortest paths,*
- (2) Maxwell points,*
- (3) conjugate points.*

Group of Euclidean plane motions

$$\text{SE}(2) = \left\{ \begin{pmatrix} \cos \theta & -\sin \theta & x \\ \sin \theta & \cos \theta & y \\ 0 & 0 & 1 \end{pmatrix} \mid (x, y) \in \mathbb{R}^2, \theta \in S^1 \right\}$$

$$X_1(q) = \cos \theta \frac{\partial}{\partial x} + \sin \theta \frac{\partial}{\partial y}, \quad X_2(q) = \frac{\partial}{\partial \theta}.$$

$$M = \text{SE}(2), \quad \Delta = \text{span}(X_1, X_2), \quad g(X_i, X_j) = \delta_{ij}.$$

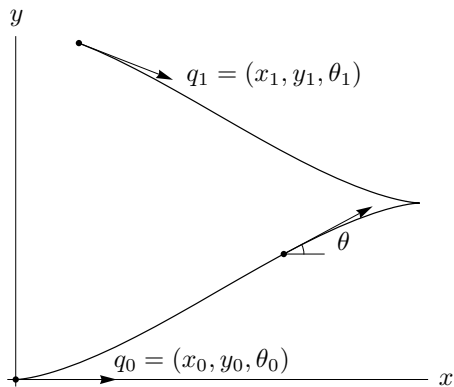
$$\dot{q} = u_1 X_1 + u_2 X_2, \quad q = (x, y, \theta) \in \text{SE}(2), \quad (u_1, u_2) \in \mathbb{R}^2, \\ q(0) = q_0 = \text{Id} = (0, 0, 0), \quad q(t_1) = q_1,$$

$$I = \int_0^{t_1} \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{\theta}^2} dt = \int_0^{t_1} \sqrt{u_1^2 + u_2^2} dt \rightarrow \min.$$

Contact sub-Riemannian structure on $SE(2)$

- $X_3 = [X_1, X_2] = \sin \theta \frac{\partial}{\partial x} - \cos \theta \frac{\partial}{\partial y}$
- $\text{span}(X_1(q), X_2(q), X_3(q)) = T_q M \Rightarrow$ complete controllability
- Growth vector $(2, 3) \Rightarrow$ contact distribution
- The only left-invariant contact sub-Riemannian structure on $SE(2)$, up to dilations and local isometries

Problem on optimal motion of a mobile robot on a plane



$$I = \int_0^{t_1} \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{\theta}^2} dt \rightarrow \min$$

Pontryagin's maximum principle

- Abnormal extremal trajectories are constant.

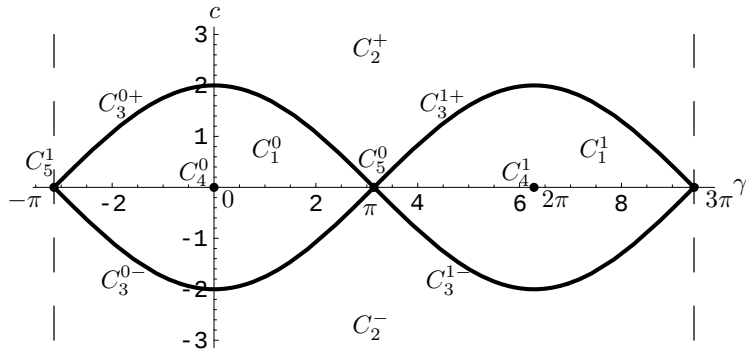
- Normal extremals:

$$\begin{aligned}\dot{\gamma} &= c, & \dot{c} &= -\sin \gamma, & (\gamma, c) &\in C \cong (2S^1_\gamma) \times \mathbb{R}_c, \\ \dot{x} &= \sin \frac{\gamma}{2} \cos \theta, & \dot{y} &= \sin \frac{\gamma}{2} \sin \theta, & \dot{\theta} &= -\cos \frac{\gamma}{2}.\end{aligned}$$

- The energy integral $E = \frac{c^2}{2} - \cos \gamma \in [-1, +\infty)$
- $\gamma(t)$, $c(t)$, $q(t)$: parameterization by Jacobi functions sn, cn, dn, E.

Partition of the pendulum phase cylinder $C = \cup_{i=1}^5 C_i$

- $C_1 = \{\lambda \in C \mid E \in (-1, 1)\} \Rightarrow$ pendulum oscillations,
- $C_2 = \{\lambda \in C \mid E \in (1, +\infty)\} \Rightarrow$ pendulum rotations,
- $C_3 = \{\lambda \in C \mid E = 1, c \neq 0\} \Rightarrow$ critical motion,
- $C_4 = \{\lambda \in C \mid E = -1\} \Rightarrow$ stable equilibrium,
- $C_5 = \{\lambda \in C \mid E = 1, c = 0\} \Rightarrow$ unstable equilibrium.

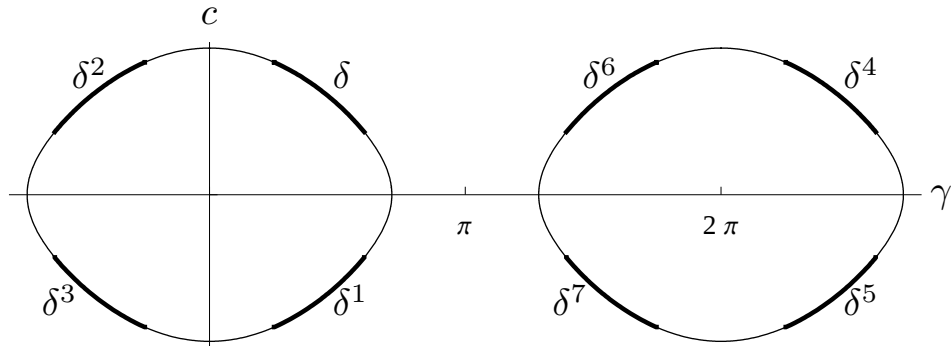


Reflections of ε^i in the phase cylinder of a pendulum $\ddot{\gamma} = -\sin \gamma$

- $\varepsilon^i : C \rightarrow C$, $\varepsilon_*^i \vec{H}_v = \pm \vec{H}_v$, $\vec{H}_v = c \frac{\partial}{\partial \gamma} - \sin \gamma \frac{\partial}{\partial c} \in \text{Vec } C$,
- Symmetry group of a parallelepiped

$$G = \{\text{Id}, \varepsilon^1, \dots, \varepsilon^7\} = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2.$$

- Action of reflections $\varepsilon^i : \delta \mapsto \delta^i$ on the pendulum trajectory:



First Maxwell time corresponding to symmetries

Symmetries of the exponential map:

$$\text{Exp} \circ \varepsilon^i(\lambda, t) = \varepsilon^i \circ \text{Exp}(\lambda, t), \quad (\lambda, t) \in C \times \mathbb{R}_+, \quad \varepsilon^i \in G.$$

$$t_{\text{Max}}(\lambda) = \min\{t > 0 \mid \exists \varepsilon^i \in G : \varepsilon^i(\lambda, t) \neq (\lambda, t), \quad \text{Exp} \circ \varepsilon^i(\lambda, t) = \text{Exp}(\lambda, t)\}$$

Theorem 7

- $E = -1 \Rightarrow t_{\text{Max}}(\lambda) = \pi,$
- $E \in (-1, 1) \Rightarrow t_{\text{Max}}(\lambda) = 2K(k), \quad k = \sqrt{(E+1)/2},$
- $E = 1 \Rightarrow t_{\text{Max}}(\lambda) = +\infty,$
- $E > 1 \Rightarrow t_{\text{Max}}(\lambda) = 2kp_1(k), \quad k = \sqrt{2/(E+1)},$

$$p_1(k) = \min\{p > 0 \mid \text{cn}(p, k)(E(p) - p) - \text{dn}(p, k) \text{sn}(p, k) = 0\}.$$

Estimates of the first conjugate time

Theorem 8

- $E \in [-1, 1] \Rightarrow t_{\text{conj}}^1(\lambda) = +\infty,$
- $E > 1 \Rightarrow t_{\text{conj}}^1(\lambda) \in [t_{\text{Max}}(\lambda), 4kK],$
- $\forall \lambda \in C \quad t_{\text{conj}}^1(\lambda) \geq t_{\text{Max}}(\lambda).$

Proof method:

Homotopy invariance of the Maslov index (number of conjugate points)

Global structure of the exponential map

- $\text{Exp} : C \times \mathbb{R}_+ = N \rightarrow M$: non-opt. geodes. for $t > t_{\text{Max}}(\lambda)$,
- $\hat{N} = \{(\lambda, t) \in C \times \mathbb{R}_+ \mid t \leq t_{\text{Max}}(\lambda)\}$, $\hat{M} = M \setminus \{q_0\}$,
 $\text{Exp} : \hat{N} \rightarrow \hat{M}$ surjective, not injective (Maxwell points),
- $\tilde{M} = \{q \in M \mid \varepsilon^i(q) \neq q\} =$
 $= \{q \in M \mid \sin \theta \neq 0, R_i(q) \neq 0\} = \cup_{i=1}^8 M_i$,
 $\tilde{N} = \text{Exp}^{-1}(\tilde{M}) =$
 $= \{(\lambda, t) \in N \mid t < t_{\text{Max}}(\lambda), \sin(\gamma_{t/2}/2) \neq 0\} = \cup_{i=1}^8 D_i$,
 $\text{Exp} : \tilde{N} \rightarrow \tilde{M}$: neither Maxwell point nor conjugate points.

Theorem 9

$\text{Exp} : D_i \rightarrow M_i$ — diffeomorphism, $i = 1, \dots, 8$.

$\text{Exp} : \tilde{N} \rightarrow \tilde{M}$ — diffeomorphism.

Optimal synthesis in a problem

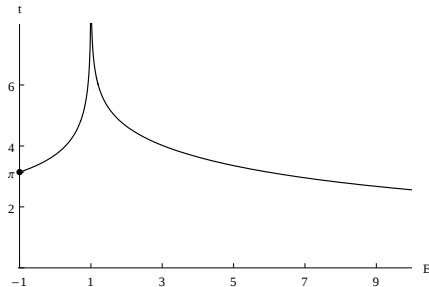
$q(0) = q_1, q(t_1) = q_0 = (0, 0, 0)$

- $q_1 \in \hat{M} = M \setminus \{q_0\}$
- $\text{Exp} : \hat{N} \rightarrow \hat{M}$ surjective
- $\text{Exp}^{-1}(q) = \begin{cases} \{(\lambda, t)\}, & \text{if } q \in \hat{M} \setminus \text{Max}, \\ \{(\lambda', t) \neq (\lambda'', t)\}, & \text{if } q \in \text{Max} \end{cases}$
- $\text{Exp}^{-1}(q_1) = (\lambda, t), \quad \lambda = (\gamma, c) \in (2S^1) \times \mathbb{R}, t > 0$
- $\ddot{\gamma}_s = -\sin \gamma_s, (\gamma_0, \dot{\gamma}_0) = (\gamma, c), s \in [0, t]$
- $u_1(q_1) = -\sin(\gamma_t/2), u_2(q_1) = \cos(\gamma_t/2)$
- the optimal synthesis $q_1 \mapsto (u_1, u_2)$ is two-valued on Max , single-valued on $\hat{M} \setminus \text{Max}$.

Cut time

Theorem 10

- $t_{\text{cut}}(\lambda) = t_{\text{Max}}(\lambda), \quad \lambda \in C,$
- $t_{\text{cut}} \circ \varepsilon^i = t_{\text{cut}}, \quad \varepsilon^i \in G,$
- $\vec{H}_V t_{\text{cut}} = 0,$
- $t_{\text{cut}} : C \rightarrow (0, +\infty]$ is continuous, $t_{\text{cut}}|_{E \neq \pm 1}$ is smooth.



Group of hyperbolic motions of the plane: Statement of the problem

$$\mathrm{SH}(2) = \left\{ \begin{pmatrix} \operatorname{ch} z & \operatorname{sh} z & x \\ \operatorname{sh} z & \operatorname{ch} z & y \\ 0 & 0 & 1 \end{pmatrix} \mid (x, y, z) \in \mathbb{R}^3 \right\}$$

Left-invariant frame:

$$X_1(q) = \operatorname{ch} z \frac{\partial}{\partial x} + \operatorname{sh} z \frac{\partial}{\partial y}, \quad X_2(q) = \frac{\partial}{\partial z}.$$

$$M = \mathrm{SH}(2), \quad \Delta = \operatorname{span}(X_1, X_2), \quad g(X_i, X_j) = \delta_{ij}.$$

$$\dot{q} = u_1 X_1 + u_2 X_2, \quad q = (x, y, \theta) \in \mathrm{SH}(2), \quad (u_1, u_2) \in \mathbb{R}^2,$$

$$q(0) = q_0 = \operatorname{Id} = (0, 0, 0), \quad q(t_1) = q_1,$$

$$I = \int_0^{t_1} \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{\theta}^2} dt = \int_0^{t_1} \sqrt{u_1^2 + u_2^2} dt \rightarrow \min.$$

Contact sub-Riemannian structure on $\mathrm{SH}(2)$

- $X_3 = [X_1, X_2] = -\operatorname{sh} z \frac{\partial}{\partial x} - \operatorname{ch} z \frac{\partial}{\partial y}$
- $\operatorname{span}(X_1(q), X_2(q), X_3(q)) = T_q M \Rightarrow$ complete controllability
- Growth vector $(2, 3) \Rightarrow$ contact distribution
- The only left-invariant contact sub-Riemannian structure on $\mathrm{SH}(2)$, up to dilations and local isometries

Pontryagin's maximum principle

- Abnormal extremal trajectories are constant.
- Normal extremals:

$$\begin{aligned}\dot{\gamma} &= c, & \dot{c} &= -\sin \gamma, & (\gamma, c) &\in C \cong (2S^1_\gamma) \times \mathbb{R}_c, \\ \dot{x} &= \cos \frac{\gamma}{2} \operatorname{ch} z, & \dot{y} &= \cos \frac{\gamma}{2} \operatorname{sh} z, & \dot{z} &= \sin \frac{\gamma}{2}.\end{aligned}$$

- $\gamma(t)$, $c(t)$, $q(t)$: parameterization by Jacobi functions sn , cn , dn , E
- Symmetry group of Exp:

$$G = \{\operatorname{Id}, \varepsilon^1, \dots, \varepsilon^7\}.$$

First Maxwell Time and Conjugate Times

Theorem 11

- $E = -1 \Rightarrow t_{\text{Max}}(\lambda) = 2\pi,$
- $E \in (-1, 1) \Rightarrow t_{\text{Max}}(\lambda) = 4K(k), k = \sqrt{(E+1)/2},$
- $E = 1 \Rightarrow t_{\text{Max}}(\lambda) = +\infty,$
- $E > 1 \Rightarrow t_{\text{Max}}(\lambda) = 4kK(k), k = \sqrt{2/(E+1)}.$

Theorem 12

- $t_{\text{Max}}^n(\lambda) \leq t_{\text{conj}}^n(\lambda) \leq t_{\text{Max}}^{n+1}(\lambda)$ for any $\lambda \in C, n \in \mathbb{N}.$
- A generalization of Rolle's theorem is valid: *between successive Maxwell points there is one conjugate point.*

Global structure of the exponential map

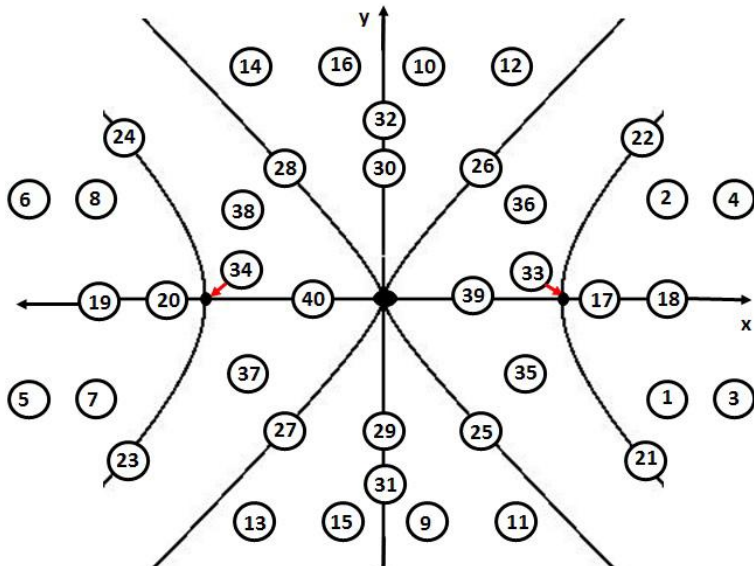
Diffeomorphic stratifications in the preimage and image of Exp :

- $\hat{N} = \{(\lambda, t) \in \mathcal{C} \times \mathbb{R}_+ \mid t \leq t_{\text{Max}}(\lambda)\} =$
 $= \cup_{i=1}^2 D_i \cup (\cup_{i=1}^{40} N'_i),$
- $\hat{M} = M \setminus \{q_0\} =$
 $= \cup_{i=1}^2 M_i \cup (\cup_{i=1}^{40} M'_i).$

Theorem 13

- $\text{Exp} : D_i \rightarrow M_i$ — *diffeomorphism*, $i = 1, 2$.
- $\text{Exp} : N'_i \rightarrow M'_i$ — *diffeomorphism*, $i = 1, \dots, 40$.

Stratification of plane $M' = \{z = 0\}$

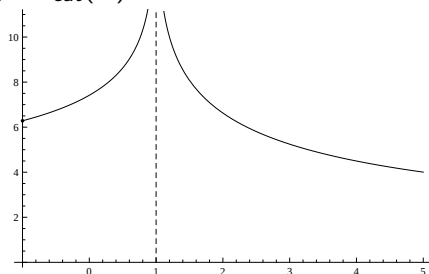


Cut time

Theorem 14

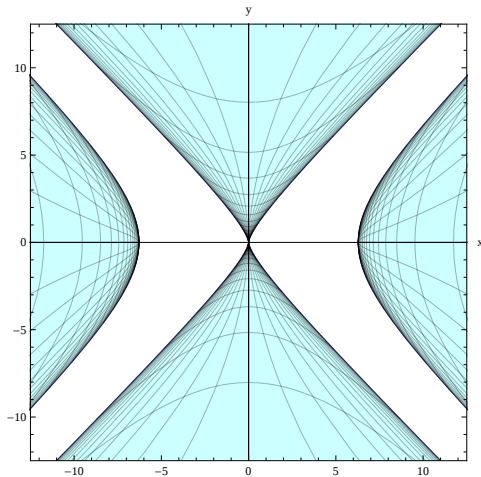
- $t_{\text{cut}}(\lambda) = t_{\text{Max}}(\lambda)$, $\lambda \in C$,
- $t_{\text{cut}} \circ \varepsilon^i = t_{\text{cut}}$, $\varepsilon^i \in G$,
- $\vec{H}_V t_{\text{cut}} = 0$,
- $t_{\text{cut}} : C \rightarrow (0, +\infty]$ is continuous, $t_{\text{cut}}|_{E \neq \pm 1}$ is smooth

Graph of the function $t_{\text{cut}} = t_{\text{cut}}(E)$:



Cut locus

- $Cut = \text{Max} \cup (Cut \cap \text{Conj}) = Cut_{\text{loc}} \cup Cut_{\text{glob}},$
- $\text{cl}(Cut_{\text{loc}}) \ni q_0, d(q_0, Cut_{\text{glob}}) = 2\pi,$
- $Cut \subset \{z = 0\}.$



Sub-Riemannian spheres

- $R > 0 \Rightarrow S_R \cong S^2$,
- Singularities of spheres:

$$S_R \cap \text{Cut} = (S_R \cap \text{Max}) \cup (S_R \cap \text{Cut} \cap \text{Conj}).$$

Exercises

1. Study the sub-Riemannian problems on the groups $SE(2)$ and $SH(2)$ independently.