Orbit theorem
(Lecture 3)

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«Introduction to geometric control theory»

Lecture course in Dept. of Mathematics and Mechanics

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2. Seeing the Traces:

By the stream and under the trees, scattered are the traces of the lost;
The sweet-scented grasses are growing thick — did he find the way?
However remote over the hills and far away the beast may wander,
His nose reaches the heavens and none can conceal it.

Pu-ming, "The Ten Oxherding Pictures"

Reminder: Plan of the previous lecture

1. Lie groups, Lie algebras, and left-invariant optimal control problems
2. Controllability of linear systems
3. Local controllability of nonlinear systems
4. Orbit of a control system

Plan of this lecture

1. Preliminaries.
2. The Orbit theorem.
3. Corollaries of the Orbit theorem:
   - Orbit and Lie algebra of the system
   - Rashevskii–Chow theorem,
   - Lie algebra rank condition,
   - Frobenius theorem.

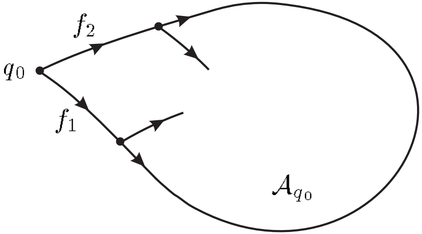
Orbit of a control system

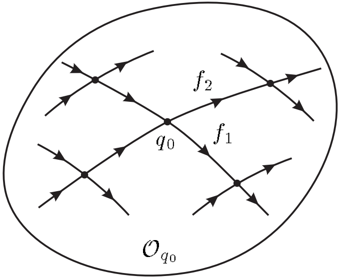
- A *control system* on a smooth manifold M is an arbitrary set of vector fields $\mathcal{F} \subset \mathrm{Vec}(M)$.
- The *attainable set* of the system $\mathcal{F}$ from a point $q_0 \in M$:

$$\mathcal{A}_{q_0} = \{e^{t_N f_N} \circ \cdots \circ e^{t_1 f_1}(q_0) \mid t_i \geq 0, \quad f_i \in \mathcal{F}, \quad N \in \mathbb{N}\}.$$

- The *orbit* of the system $\mathcal{F}$ through the point q_0:

$$\mathcal{O}_{q_0} = \{e^{t_N f_N} \circ \cdots \circ e^{t_1 f_1}(q_0) \mid t_i \in \mathbb{R}, \quad f_i \in \mathcal{F}, \quad N \in \mathbb{N}\}.$$



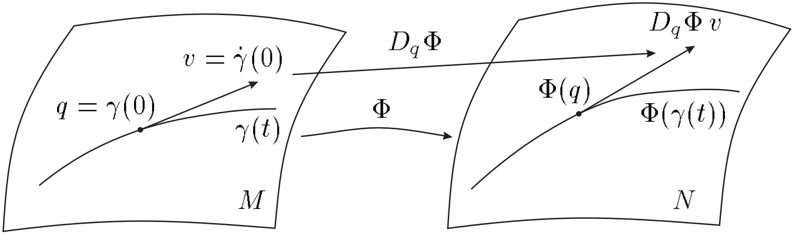


Action of diffeomorphisms on tangent vectors and vector fields

- Let $V \in \mathrm{Vec}(M)$, and let $\Phi \colon M \to N$ be a *diffeomorphism*, i.e., a smooth bijective mapping with a smooth inverse.
- The vector field $\Phi_* V \in \mathrm{Vec}(N)$ is defined as

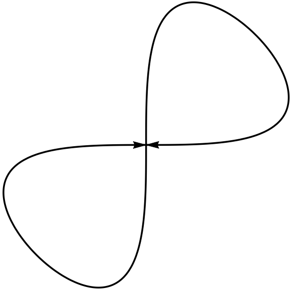
$$\Phi_* V|_{\Phi(q)} = \left.\frac{d}{dt}\right|_{t=0} \Phi \circ e^{tV}(q) = \Phi_{*q}(V(q)).$$

- Thus we have a mapping $\Phi_* : \mathrm{Vec}(M) \to \mathrm{Vec}(N)$, *push-forward of vector fields* from the manifold M to the manifold N under the action of the diffeomorphism Φ.



Immersed submanifolds

- A subset W of a smooth manifold M is called a k-dimensional *immersed submanifold* of M if there exists a k-dimensional manifold N and a smooth mapping $F\colon N \to M$ such that:
  - F is injective
  - $\operatorname{Ker} F_{*q} = 0$ for any $q \in N$
  - $W = F(N)$.
- Example: Figure of eight is a 1-dimensional immersed submanifold of the 2-dimensional plane.



Example: Irrational winding of the torus

- Torus $\mathbb{T}^2 = \mathbb{R}^2/(2\pi\,\mathbb{Z}^2) = \{(x, y) \in S^1 \times S^1\}$
- Vector field $V = p\frac{\partial}{\partial x} + q\frac{\partial}{\partial y} \in \mathrm{Vec}(\mathbb{T}^2)$, $p^2 + q^2 \neq 0$.
- The orbit $\mathcal{O}_0$ of V through the origin $0 \in \mathbb{T}^2$ may have two different types:
  (1) $p/q \in \mathbb{Q} \cup \{\infty\}$. Then $\mathrm{cl}\ \mathcal{O}_0 = \mathcal{O}_0$.
  (2) $p/q \in \mathbb{R}\backslash\mathbb{Q}$. Then $\mathrm{cl}\ \mathcal{O}_0 = \mathbb{T}^2$. In this case the orbit $\mathcal{O}_0$ is called the *irrational winding of the torus*.
- In the both cases the orbit $\mathcal{O}_0$ is an immersed submanifold of the torus, but in the second case it is not embedded.
- So even for one vector field the orbit may be an immersed submanifold, but not an embedded one
- An immersed submanifold $N = F(W) \subset M$ is called *embedded* if $F : W \to N$ is a homeomorphism in the topology induced by the inclusion $N \subset M$. In case (2) the topology of the orbit induced by the inclusion $\mathcal{O}_0 \subset \mathbb{R}^2$ is weaker than the topology of the orbit induced by the immersion $t \mapsto e^{tV}(0)$, $\quad \mathbb{R} \to \mathcal{O}_0$.

Theorem (*Orbit theorem*, Nagano–Sussmann)

Let $\mathcal{F} \subset \mathrm{Vec}(M)$, and let $q_0 \in M$.

(1) *The orbit $\mathcal{O}_{q_0}$ is a connected immersed submanifold of M.*

(2) *For any $q \in \mathcal{O}_{q_0}$*

$$T_q\mathcal{O}_{q_0} = \mathrm{span}(\mathcal{P}_*\mathcal{F})(q) = \mathrm{span}\{(P_*V)(q) \mid P \in \mathcal{P}, \quad V \in \mathcal{F}\},$$
$$\mathcal{P} = \{e^{t_N f_N} \circ \cdots \circ e^{t_1 f_1} \mid t_i \in \mathbb{R}, \quad f_i \in \mathcal{F}, \quad N \in \mathbb{N}\}.$$

Proof of the Orbit theorem: 1/7

Proof.

- Introduce a vector space important in the sequel

$$\Pi_q = \operatorname{span}(\mathcal{P}_*\mathcal{F})(q) \subset T_qM, \qquad q \in M,$$

  this is a candidate tangent space to the orbit $\mathcal{O}_{q_0}$.
- *1)* We prove that for all $q \in \mathcal{O}_{q_0}$ we have $\dim \Pi_q = \dim \Pi_{q_0}$.
- Choose any point $q \in \mathcal{O}_{q_0}$, then $q = Q(q_0)$, $Q \in \mathcal{P}$. Let us show that $Q_*^{-1}(\Pi_q) \subset \Pi_{q_0}$.
- Choose any element $(P_*f)(q) \in \Pi_q$, $P \in \mathcal{P}$, $f \in \mathcal{F}$. Then

$$\begin{aligned} Q_*^{-1}[(P_*f)(q)] &= (Q_*^{-1} \circ P_*f)(Q^{-1}(q)) \\ &= [(Q^{-1} \circ P)_*f](q_0) \in (\mathcal{P}_*\mathcal{F})(q_0) \subset \Pi_{q_0}. \end{aligned}$$

  Thus $Q_*^{-1}(\Pi_q) \subset \Pi_{q_0}$, whence $\dim \Pi_q \le \dim \Pi_{q_0}$. Interchanging in this arguments q and q_0, we get $\dim \Pi_{q_0} \le \dim \Pi_q$.
- Finally we have $\dim \Pi_q = \dim \Pi_{q_0}$, $q \in \mathcal{O}_{q_0}$.

Proof of the Orbit theorem: 2/7

- *2)* For any point $q \in M$ denote $m = \dim \Pi_q$, and choose such vector fields $V_1, \ldots, V_m \in \mathcal{P}_* \mathcal{F}$ that $\Pi_q = \operatorname{span}(V_1(q), \ldots, V_m(q))$.
- Further, define a mapping

$$G_q : (t_1, \ldots, t_m) \mapsto e^{t_m V_m} \circ \cdots \circ e^{t_1 V_1}(q), \qquad \mathbb{R}^m \to M.$$

- We have $\frac{\partial G_q}{\partial t_i}(0) = V_i(q)$, thus the vectors $\frac{\partial G_q}{\partial t_1}(0), \ldots, \frac{\partial G_q}{\partial t_m}(0)$ are linearly independent.
- Consequently, the restriction of G_q to a sufficiently small neighbourhood W_0 of the origin in $\mathbb{R}^m$ is a submersion.
- *3)* The image $G_q(W_0)$ is an (embedded) submanifold of M, may be, for a smaller neighbourhood W_0.

Proof of the Orbit theorem: 3/7

- *4)* We show that $G_q(W_0) \subset \mathcal{O}_q$.
- We have $G_q(W_0) = \{e^{t_m V_m} \circ \cdots \circ e^{t_1 V_1}(q) \mid t = (t_1, \ldots, t_m) \in W_0\}$.
- Since $V_1 = P_* f, P \in \mathcal{P}$, $f \in \mathcal{F}$, we get

$$e^{t_1 V_1}(q) = e^{t_1 P_* f}(q) = P \circ e^{t_1 f} \circ P^{-1}(q) \in \mathcal{O}_q.$$

  Exercise: prove that

$$e^{tP_* f}(q) = P \circ e^{tf} \circ P^{-1}(q), \qquad f \in \mathrm{Vec}(M), \quad P \in \mathrm{Diff}(M), \quad t \in \mathbb{R}. \tag{1}$$

- We conclude similarly that $e^{t_2 V_2} \circ e^{t_1 V_1}(q) \in \mathcal{O}_q$ etc. Finally we have $G_q(t) \in \mathcal{O}_q$, $t \in W_0$.

Proof of the Orbit theorem: 4/7

- *5)* We show that $G_{q*}(T_t\mathbb{R}^m) = \Pi_{G_q(t)}$, $t \in W_0$. We have $\dim G_{q*}(T_t\mathbb{R}^m) = m = \dim \Pi_{G_q(t)}$, thus it suffices to prove the inclusion $\frac{\partial G_q}{\partial t_i}(t) \in \Pi_{G_q(t)}, \qquad t \in W_0$.
- Let us compute this partial derivative:

$$\frac{\partial G_q}{\partial t_i} = \frac{\partial}{\partial t_i} e^{t_m V_m} \circ \cdots \circ e^{t_i V_i} \circ \cdots \circ e^{t_1 V_1}(q)$$

denote $R = e^{t_m V_m} \circ \cdots \circ e^{t_{i+1} V_{i+1}}$, $q' = e^{t_{i-1} V_{i-1}} \circ \cdots \circ e^{t_1 V_1}(q)$,

$$\begin{aligned} &= \frac{\partial}{\partial t_i} R \circ e^{t_i V_i}(q') = R_* V_i(e^{t_i V_i}(q')) \\ &= (R_* V_i)[R \circ e^{t_i V_i} \circ \cdots \circ e^{t_1 V_1}(q)] \\ &= (R_* V_i)(G_q(t)) \in (\mathcal{P}_* \mathcal{F})(G_q(t)) \subset \Pi_{G_q(t)}. \end{aligned}$$

- Thus $G_{q*}(T_t\mathbb{R}^m) = \Pi_{G_q(t)}$, i.e., the space $\Pi_{G_q(t)}$ is a tangent space to the smooth manifold $G_q(W_0)$ at the point $G_q(t)$.

Proof of the Orbit theorem: 5/7

- *6)* We prove that the sets $G_q(W_0)$ form a base of a ("strong") topology on M.
- *6a)* It is obvious that any point $q \in M$ is contained in the set $G_q(W_0)$.
- *6b)* Let us show that for any point $\widehat{q} \in G_q(W_0) \cap G_{\widetilde{q}}(\widetilde{W_0})$ there exists a set $G_{\widehat{q}}(\widehat{W_0}) \subset G_q(W_0) \cap G_{\widetilde{q}}(\widetilde{W_0})$.
- Take any point $\widehat{q} \in G_q(W_0) \cap G_{\widetilde{q}}(\widetilde{W_0})$ and consider $G_{\widehat{q}}(t) = e^{t_m \widehat{V}_m} \circ \cdots \circ e^{t_1 \widehat{V}_1}(\widehat{q})$.
- For any point $q' \in G_q(W_0)$ we have $\widehat{V}_1(q') \in (\mathcal{P}_* \mathcal{F})(q') \subset \Pi_{q'}$. But $G_q(W_0)$ is a submanifold with the tangent space $T_{q'} G_q(W_0) = \Pi_{q'}$. The vector field $\widehat{V}_1$ is tangent to this submanifold, thus $e^{t_1 \widehat{V}_1}(\widehat{q}) \in G_q(W_0)$ for small $|t_1|$. We conclude similarly that $e^{t_2 \widehat{V}_2} \circ e^{t_1 \widehat{V}_1}(\widehat{q}) \in G_q(W_0)$ for small $|t_1|$, $|t_2|$ etc. Finally we get

$$G_{\widehat{q}}(t) \in G_q(W_0) \text{ for small } |t|.$$

- Similarly $G_{\widehat{q}}(t) \in G_{\widetilde{q}}(\widetilde{W_0})$ for small $|t|$. Thus $G_{\widehat{q}}(\widehat{W_0}) \subset G_q(W_0) \cap G_{\widetilde{q}}(\widetilde{W_0})$ for some neighbourhood $\widehat{W_0}$, and property 6b) is proved.

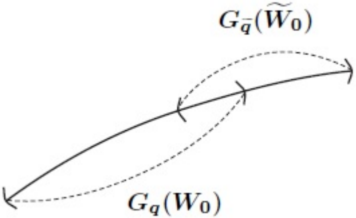


Figure: Intersection of neighborhoods in topology base

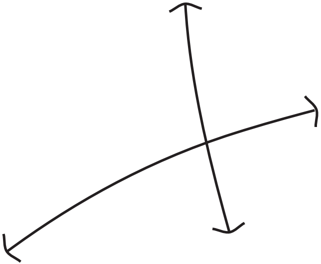


Figure: Intersection of neighborhoods not in topology base

Proof of the Orbit theorem: 6/7

- It follows from properties 6a) and 6b) that the sets $G_q(W_0)$ form a base of topology on the set M. Denote the corresponding topological space as $M^{\mathcal{F}}$.
- 7) We show that for any $q_0 \in M$ the orbit $\mathcal{O}_{q_0}$ is connected, open and closed in the space $M^{\mathcal{F}}$.
- The mappings $t_i \mapsto e^{t_i f_i}(q)$ are continuous in $M^{\mathcal{F}}$, thus $\mathcal{O}_{q_0}$ is connected.
- Any point $q \in \mathcal{O}_{q_0}$ is contained in the neighbourhood $G_q(W_0) \subset \mathcal{O}_q = \mathcal{O}_{q_0}$, thus the orbit is open in $M^{\mathcal{F}}$.
- Finally, any orbit is a complement in M to orbits with which it does not intersect. Thus any orbit is closed in $M^{\mathcal{F}}$.
- So any orbit $\mathcal{O}_{q_0}$ is a connected component of the topological space $M^{\mathcal{F}}$.

Proof of the Orbit theorem: 7/7

- *8)* Introduce a smooth structure on $\mathcal{O}_{q_0}$ as follows:
  - the sets $G_q(W_0)$ are called coordinate neighbourhoods
  - the mappings $G_q^{-1} : G_q(W_0) \to W_0$ are called coordinate mappings.
- It is easy to see that these coordinate neighbourhoods and mappings agree: for any intersecting neighbourhoods $G_q(W_0)$ and $G_{\widetilde{q}}(\widetilde{W_0})$ the composition

$$G_{\widetilde{q}} \circ G_q \ : \ G_q^{-1}(G_q(W_0) \cap G_{\widetilde{q}}(\widetilde{W_0})) \to G_{\widetilde{q}}^{-1}(G_q(W_0) \cap G_{\widetilde{q}}(\widetilde{W_0}))$$

  is a diffeomorphism.
- Thus the orbit $\mathcal{O}_{q_0}$ is a smooth manifold.
- Moreover, $\mathcal{O}_{q_0} \subset M$ is an immersed submanifold of dimension $m = \dim \Pi_{q_0}$.
- *9)* It follows from item 5) above that the smooth manifold $\mathcal{O}_{q_0}$ has a tangent space

$$T_q\mathcal{O}_{q_0} = \Pi_q = \operatorname{span}(\mathcal{P}_*\mathcal{F})(q), \qquad q \in \mathcal{O}_{q_0}.$$

- The Orbit theorem is proved.

Theorem (*Orbit theorem*, Nagano–Sussmann)

Let $\mathcal{F} \subset \mathrm{Vec}(M)$, and let $q_0 \in M$.

(1) *The orbit $\mathcal{O}_{q_0}$ is a connected immersed submanifold of M.*

(2) *For any $q \in \mathcal{O}_{q_0}$*

$$T_q\mathcal{O}_{q_0} = \mathrm{span}(\mathcal{P}_*\mathcal{F})(q) = \mathrm{span}\{(P_* V)(q) \mid P \in \mathcal{P}, \quad V \in \mathcal{F}\},$$

$$\mathcal{P} = \{e^{t_N f_N} \circ \cdots \circ e^{t_1 f_1} \mid t_i \in \mathbb{R}, \quad f_i \in \mathcal{F}, \quad N \in \mathbb{N}\}.$$

Corollary: Orbit and Lie algebra of the system

Corollary

For any $q_0 \in M$ *and any* $q \in \mathcal{O}_{q_0}$ *we have* $\mathrm{Lie}_q(\mathcal{F}) \subset T_q\mathcal{O}_{q_0}$*, where*

$$\mathrm{Lie}_q(\mathcal{F}) = \mathrm{span}\{[f_N, [\dots, [f_2, f_1] \dots]](q) \mid f_i \in \mathcal{F},\ N \in \mathbb{N}\} \subset T_q M.$$

- *Proof.* Let $q_0 \in M$, $q \in \mathcal{O}_{q_0}$.
- Take any $f \in \mathcal{F}$. Then $\varphi(t) = e^{tf}(q) \in \mathcal{O}_{q_0}$, thus $\dot{\varphi}(0) = f(q) \in T_q\mathcal{O}_{q_0}$. It follows that $\mathcal{F}(q) \subset T_q\mathcal{O}_{q_0}$.
- Further, take any $f_1, f_2 \in \mathcal{F}$, then $\varphi(t) = e^{-tf_2} \circ e^{-tf_1} \circ e^{tf_2} \circ e^{tf_1}(q) \in \mathcal{O}_{q_0}$. Thus

  $$\left.\frac{d}{dt}\right|_{t=0} \varphi(\sqrt{t}) = [f_1, f_2](q) \in T_q\mathcal{O}_{q_0}.$$

  It follows that $[\mathcal{F}, \mathcal{F}](q) \subset T_q\mathcal{O}_{q_0}$.
- We prove similarly that $[[\mathcal{F}, \mathcal{F}], \mathcal{F}](q) \subset T_q\mathcal{O}_{q_0}$, and by induction that $\mathrm{Lie}_q(\mathcal{F}) \subset T_q\mathcal{O}_{q_0}$.

$\square$

Analytic and non-analytic cases

- In the analytic case the inclusion $\mathrm{Lie}_q(\mathcal{F}) \subset T_q\mathcal{O}_{q_0}$ turns into an equality.

Proposition

Let M and $\mathcal{F}$ be real-analytic. Then for any $q_0 \in M$ and any $q \in \mathcal{O}_{q_0}$

$$\mathrm{Lie}_q(\mathcal{F}) = T_q\mathcal{O}_{q_0}.$$

- But in a smooth non-analytic case the inclusion $\mathrm{Lie}_q(\mathcal{F}) \subset T_q\mathcal{O}_{q_0}$ may become strict.
- Example: Orbit of non-analytic system.
  - let $M = \mathbb{R}^2_{x,y}$, $\mathcal{F} = \{f_1, f_2\}$, $f_1 = \frac{\partial}{\partial x}$, $f_2 = a(x)\frac{\partial}{\partial y}$, where $a \in C^\infty(\mathbb{R})$, $a(x) = 0$ for $x \leq 0$, $a(x) > 0$ for $x > 0$.
  - It is easy to see that $\mathcal{O}_q = \mathbb{R}^2$ for any $q = (x, y) \in \mathbb{R}^2$.
  - Although, for $x < 0$ we have

$$\mathrm{Lie}_q(\mathcal{F}) = \mathrm{span}(f_1(q)) \neq T_q\mathcal{O}_q.$$

Corollary: Rashevskii-Chow theorem

- A system $\mathcal{F} \subset \mathrm{Vec}(M)$ is called *completely nonholonomic* (*full-rank*, *bracket-generating*) if $\mathrm{Lie}_q(\mathcal{F}) = T_qM \quad \forall q \in M$.

Theorem (Rashevskii-Chow)

If $\mathcal{F} \subset \mathrm{Vec}(M)$ is full-rank and M is connected, then $\mathcal{O}_q = M \qquad \forall\, q \in M$.

Proof.

- Take any $q \in M$ and any $q_1 \in \mathcal{O}_q$.
- We have $T_{q_1}\mathcal{O}_q \supset \mathrm{Lie}_{q_1}(\mathcal{F}) = T_{q_1}M$, thus $\dim \mathcal{O}_q = \dim M$, i.e., $\mathcal{O}_q$ is open in M.
- On the other hand, any orbit is closed as a complement to the union of all other orbits.
- Thus any orbit is a connected component of M. Since M is connected, each orbit coincides with M.

$\square$

Corollary: Lie algebra rank condition

Corollary (Lie algebra rank condition, LARC)

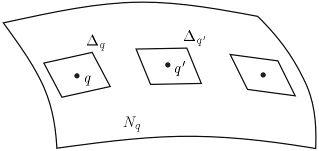
If a manifold M is connected, and a system $\mathcal{F} \subset \mathrm{Vec}(M)$ is symmetric and completely nonholonomic, then it is globally controllable on M.

Distributions

- A *distribution* on a smooth manifold M is a smooth mapping

$$\Delta\colon q \mapsto \Delta_q \subset T_qM, \quad q \in M,$$

  where the vector subspaces Δ_q have the same dimension called the *rank* of Δ.
- An immersed submanifold $N \subset M$ is called an *integral manifold* of a distribution Δ if $\quad \forall q \in N \quad T_qN = \Delta_q$.
- A distribution Δ on M is called *integrable* if for any point $q \in M$ there exists an integral manifold $N_q \ni q$.
- Denote by $\bar{\Delta} = \{f \in \mathrm{Vec}(M) \mid f(q) \in \Delta_q \quad \forall q \in M\}$ the set of vector fields tangent to Δ.
- A distribution Δ is called *holonomic* if $[\bar{\Delta}, \bar{\Delta}] \subset \bar{\Delta}$.



Corollary: Frobenius theorem

Theorem (Frobenius)

A distribution is integrable iff it is holonomic.

Proof:

- *Necessity*. Take any $f, g \in \bar{\Delta}$. Let $q \in M$, and let $N_q \ni q$ be the integral manifold of Δ through q.
- Then
$$\varphi(t) = e^{-tg} \circ e^{-tf} \circ e^{tg} \circ e^{tf}(q) \in N_q,$$
thus
$$\left.\frac{d}{dt}\right|_{t=0} \varphi(\sqrt{t}) = [f, g](q) \in T_q N_q = \Delta_q.$$
- So $[f, g] \in \bar{\Delta}$, and the inclusion $[\bar{\Delta}, \bar{\Delta}] \subset \bar{\Delta}$ follows.

Frobenius theorem

- *Sufficiency*. We consider only the analytic case.
- We have
$$[\bar{\Delta}, \bar{\Delta}] \subset \bar{\Delta}, \qquad [[\bar{\Delta}, \bar{\Delta}], \bar{\Delta}] \subset [\bar{\Delta}, \bar{\Delta}] \subset \bar{\Delta}.$$
- Inductively $\mathrm{Lie}_q(\bar{\Delta}) \subset \bar{\Delta}_q = \Delta_q$.
- The reverse inclusion is obvious, thus $\mathrm{Lie}_q(\bar{\Delta}) = \Delta_q$, $q \in M$.
Denote $N_q = \mathcal{O}_q(\bar{\Delta})$ and prove that N_q is an integral manifold of Δ:
$$T_{q'} N_q = T_{q'}(\mathcal{O}_q(\bar{\Delta})) = \mathrm{Lie}_{q'}(\bar{\Delta}) = \Delta_{q'}, \quad q' \in N_q.$$
- So $N_q \ni q$ is the integral manifold of Δ, and Δ is integrable. $\square$

Corollary: Frobenius condition

- Consider a *local frame* of Δ:

$$\Delta_q = \mathrm{span}(f_1(q), \ldots, f_k(q)), \quad q \in S \subset M, \quad f_1, \ldots, f_k \in \mathrm{Vec}(S), \quad k = \dim \Delta_q,$$

where S is an open subset of M.

- Then the inclusion $[\bar{\Delta}, \bar{\Delta}] \subset \bar{\Delta}$ takes the form

$$[f_i, f_j](q) = \sum_{l=1}^{k} c_{ij}^l(q) f_l(q), \qquad q \in S, \quad c_{ij}^l \in C^\infty(S).$$

- This equality is called the *Frobenius condition*.

Example:
The sub-Riemannian problem on the group of motions of the plane

- The control system has the following form:

$$\mathcal{F} = \{u_1 f_1 + u_2 f_2 \mid (u_1, u_2) \in \mathbb{R}^2\} \subset \mathrm{Vec}(\mathbb{R}^2 \times S^1),$$
$$f_1 = \cos\theta \frac{\partial}{\partial x} + \sin\theta \frac{\partial}{\partial y}, \qquad f_2 = \frac{\partial}{\partial \theta}.$$

- The system is symmetric: $\mathcal{F} = -\mathcal{F}$.
- Compute its Lie algebra:

$$[f_1, f_2] = \sin\theta \frac{\partial}{\partial x} - \cos\theta \frac{\partial}{\partial y} =: f_3,$$
$$\mathrm{Lie}_q(\mathcal{F}) = \mathrm{span}(f_1(q), f_2(q), f_3(q)) = T_q(\mathbb{R}^2 \times S^1).$$

- The system $\mathcal{F}$ is completely nonholonomic, thus controllable.

Example: Orbits of different dimensions

- Let

$$M = \mathbb{R}_x, \qquad \mathcal{F} = \left\{ x \frac{\partial}{\partial x} \right\} \subset \mathrm{Vec}(M).$$

- We have:

$$\begin{aligned} x_0 > 0 \quad &\Rightarrow \quad \mathcal{O}_{x_0} = \{x > 0\}, \\ x_0 = 0 \quad &\Rightarrow \quad \mathcal{O}_{x_0} = \{x = 0\}, \\ x_0 < 0 \quad &\Rightarrow \quad \mathcal{O}_{x_0} = \{x < 0\}, \end{aligned}$$

- Thus the system has two one-dimensional orbits and one zero-dimensional orbit.

Example: More orbits of different dimensions

- Let

$$M = \mathbb{R}^3_{x,y,z}, \qquad \mathcal{F} = \left\{ x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x},\ y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y},\ z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z} \right\} \subset \mathrm{Vec}(M).$$

- Then for any point $q \in \mathbb{R}^3$

$$\mathcal{O}_q = \{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = |q|^2\},$$

- This is a sphere for $q \neq 0$ and a point for $q = 0$.
- An orbit of a control system is a generalisation of a trajectory of a vector field to the case of more than one vector field.

Exercises

1. Prove formula (1).
2. Let $N \subset M$ be an immersed submanifold. Prove that if a vector field $f \in \mathrm{Vec}(M)$ satisfies the condition $f(q) \in T_q N$ for all $q \in N$, then $e^{tf}(q) \in N$ for all $q \in N$, $|t| < \varepsilon$.
3. Study integrability of the distribution $\Delta = \operatorname{span}(f_1, f_2)$, $f_1 = z\frac{\partial}{\partial x} + x\frac{\partial}{\partial z}$, $f_2 = z\frac{\partial}{\partial y} + y\frac{\partial}{\partial z}$, $(x, y, z) \in \mathbb{R}^3$, $z \neq 0$. If it is integrable, describe its integral manifolds.
4. Prove that the mappings $t_i \mapsto e^{t_i f_i}(q)$ are continuous in the topology of $M^{\mathcal{F}}$; see item 7) of the proof of the Orbit Theorem.
5. Fill the gaps in item 8) of the proof of the Orbit Theorem.