Elements of Chronological Calculus-1

(Lecture 3)

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Plan of previous lecture

1. Time-Optimal Problem
2. Smooth manifolds
3. Tangent space and tangent vector
4. Ordinary differential equations on manifolds

Plan of this lecture

1. Points, Diffeomorphisms, and Vector Fields
2. Seminorms and $C^\infty(M)$-Topology
3. Families of Functionals and Operators
4. ODEs with discontinuous right-hand side
5. Definition of the right chronological exponential
6. Formal series expansion
7. Estimates and convergence of the series
8. Left chronological exponential

Points, Diffeomorphisms, and Vector Fields

- We identify points, diffeomorphisms, and vector fields on the manifold M with functionals and operators on the algebra $C^\infty(M)$ of all smooth real-valued functions on M.
- Addition, multiplication, and product with constants are defined in the *algebra* $C^\infty(M)$, as usual, pointwise: if $a, b \in C^\infty(M)$, $q \in M$, $\alpha \in \mathbb{R}$, then

$$(a+b)(q) = a(q) + b(q),$$
$$(a \cdot b)(q) = a(q) \cdot b(q),$$
$$(\alpha \cdot a)(q) = \alpha \cdot a(q).$$

- Any *point* $q \in M$ defines a *linear functional*

$$\widehat{q} : C^\infty(M) \to \mathbb{R}, \qquad \widehat{q}a = a(q), \; a \in C^\infty(M).$$

- The functionals $\widehat{q}$ are homomorphisms of the algebras $C^\infty(M)$ and $\mathbb{R}$:

$$
\begin{aligned}
\widehat{q}(a+b) &= \widehat{q}a + \widehat{q}b, && a,\ b \in C^\infty(M),\\
\widehat{q}(a \cdot b) &= (\widehat{q}a) \cdot (\widehat{q}b), && a,\ b \in C^\infty(M),\\
\widehat{q}(\alpha \cdot a) &= \alpha \cdot \widehat{q}a, && \alpha \in \mathbb{R},\ a \in C^\infty(M).
\end{aligned}
$$

- So to any point $q \in M$, there corresponds a nontrivial *homomorphism of algebras* $\widehat{q} : C^\infty(M) \to \mathbb{R}$. It turns out that there exists an inverse correspondence.

Proposition 1

Let $\varphi : C^\infty(M) \to \mathbb{R}$ be a nontrivial homomorphism of algebras. Then there exists a point $q \in M$ such that $\varphi = \widehat{q}$.

Proof.

[AS] A.A. Agrachev, Yu.L. Sachkov, *Control theory from the geometric viewpoint.* Springer-Verlag, 2004. □

- Not only the manifold M can be reconstructed as a set from the algebra $C^\infty(M)$. One can recover topology on M from the weak topology in the space of functionals on $C^\infty(M)$:

$$\lim_{n\to\infty} q_n = q \quad \text{if and only if} \quad \lim_{n\to\infty} \widehat{q}_n a = \widehat{q} a \quad \forall a \in C^\infty(M).$$

- Moreover, the smooth structure on M is also recovered from $C^\infty(M)$, actually, "by definition": a real function on the set $\{\widehat{q} \mid q \in M\}$ is smooth if and only if it has a form $\widehat{q} \mapsto \widehat{q} a$ for some $a \in C^\infty(M)$.
- Any *diffeomorphism* $P : M \to M$ defines an *automorphism of the algebra $C^\infty(M)$*:

$$\widehat{P} : C^\infty(M) \to C^\infty(M), \qquad \widehat{P} \in \mathrm{Aut}(C^\infty(M)),$$
$$(\widehat{P} a)(q) = a(P(q)), \qquad q \in M, \quad a \in C^\infty(M),$$

i.e., $\widehat{P}$ acts as a change of variables in a function a.

- Conversely, any automorphism of $C^\infty(M)$ has such a form.

Proposition 2

Any automorphism $A : C^\infty(M) \to C^\infty(M)$ *has a form of* $\widehat{P}$ *for some* $P \in \operatorname{Diff} M$.

Proof.

Let $A \in \operatorname{Aut}(C^\infty(M))$. Take any point $q \in M$. Then the composition

$$\widehat{q} \circ A : C^\infty(M) \to \mathbb{R}$$

is a nonzero homomorphism of algebras, thus it has the form $\widehat{q_1}$ for some $q_1 \in M$. We denote $q_1 = P(q)$ and obtain

$$\widehat{q} \circ A = \widehat{q_1} = \widehat{P(q)} = \widehat{q} \circ \widehat{P} \qquad \forall q \in M,$$

i.e.,

$$A = \widehat{P},$$

and P is the required diffeomorphism. □

- Now we characterize *tangent vectors* to M as *functionals* on $C^\infty(M)$.
- Tangent vectors to M are velocity vectors to curves in M, and points of M are identified with linear functionals on $C^\infty(M)$; thus we should obtain linear functionals on $C^\infty(M)$, but not homomorphisms into $\mathbb{R}$.
- To understand, which functionals on $C^\infty(M)$ correspond to tangent vectors to M, take a smooth curve $q(t)$ of points in M. Then the corresponding curve of functionals $\widehat{q}(t) = \widehat{q(t)}$ on $C^\infty(M)$ satisfies the multiplicative rule

$$\widehat{q}(t)(a \cdot b) = \widehat{q}(t)a \cdot \widehat{q}(t)b, \qquad a,\ b \in C^\infty(M).$$

- We differentiate this equality at $t = 0$ and obtain that the velocity vector to the curve of functionals

$$\xi \stackrel{\text{def}}{=} \left.\frac{d\,\widehat{q}}{d\,t}\right|_{t=0}, \qquad \xi : C^\infty(M) \to \mathbb{R},$$

satisfies the Leibniz rule:

$$\xi(ab) = \xi(a)b(q(0)) + a(q(0))\xi(b).$$

- Consequently, to each tangent vector $v \in T_qM$ we should put into correspondence a linear functional
$$\xi : C^\infty(M) \to \mathbb{R}$$
such that
$$\xi(ab) = (\xi a)b(q) + a(q)(\xi b), \qquad a,\ b \in C^\infty(M). \tag{1}$$
- But there is a linear functional $\xi = \widehat{v}$ naturally related to any tangent vector $v \in T_qM$, the directional derivative along v:
$$\widehat{v}a = \left.\frac{d}{d\,t}\right|_{t=0} a(q(t)), \qquad q(0) = q, \quad \dot{q}(0) = v,$$
and such functional satisfies Leibniz rule (1).
- Now we show that this rule characterizes exactly directional derivatives.

Proposition 3

Let $\xi : C^\infty(M) \to \mathbb{R}$ *be a linear functional that satisfies Leibniz rule* (1) *for some point* $q \in M$*. Then* $\xi = \widehat{v}$ *for some tangent vector* $v \in T_qM$*.*

Proof.

- Notice first of all that any functional ξ that meets Leibniz rule (1) is local, i.e., it depends only on values of functions in an arbitrarily small neighborhood $O_q \subset M$ of the point q:

$$\tilde{a}|_{O_q} = a|_{O_q} \quad \Rightarrow \quad \xi\tilde{a} = \xi a, \qquad a,\ \tilde{a} \in C^\infty(M).$$

- Indeed, take a cut function $b \in C^\infty(M)$ such that $b|_{M\setminus O_q} \equiv 1$ and $b(q) = 0$. Then $(\tilde{a} - a)b = \tilde{a} - a$, thus

$$\xi(\tilde{a} - a) = \xi((\tilde{a} - a)b) = \xi(\tilde{a} - a)\, b(q) + (\tilde{a} - a)(q)\, \xi b = 0.$$

- So the statement of the proposition is local, and we prove it in coordinates.
- Let $(x_1, \ldots, x_n)$ be local coordinates on M centered at the point q. We have to prove that there exist $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ such that $\xi = \sum_{i=1}^n \alpha_i \left.\frac{\partial}{\partial x_i}\right|_0$.

- First of all,
$$\xi(1) = \xi(1\cdot 1) = (\xi 1)\cdot 1 + 1\cdot(\xi 1) = 2\xi(1),$$
thus $\xi(1) = 0$. By linearity, $\xi(\text{const}) = 0$.
- In order to find the action of ξ on an arbitrary smooth function, we expand it by the Hadamard Lemma:
$$a(x) = a(0) + \sum_{i=1}^{n} \int_0^1 \frac{\partial a}{\partial x_i}(tx)x_i\,dt = a(0) + \sum_{i=1}^{n} b_i(x)x_i,$$
where $b_i(x) = \int_0^1 \frac{\partial a}{\partial x_i}(tx)\,dt$ are smooth functions.
- Now
$$\xi a = \sum_{i=1}^{n} \xi(b_i x_i) = \sum_{i=1}^{n} ((\xi b_i)x_i(0) + b_i(0)(\xi x_i)) = \sum_{i=1}^{n} \alpha_i \frac{\partial a}{\partial x_i}(0),$$
where we denote $\alpha_i = \xi x_i$ and make use of the equality $b_i(0) = \dfrac{\partial a}{\partial x_i}(0)$. □

- So *tangent vectors* $v \in T_q M$ can be identified with directional derivatives $\widehat{v} : C^\infty(M) \to \mathbb{R}$, i.e., *linear functionals that meet Leibniz rule* (1).
- Now we characterize *vector fields* on M. A smooth vector field on M is a family of tangent vectors $v_q \in T_q M$, $q \in M$, such that for any $a \in C^\infty(M)$ the mapping $q \mapsto v_q a$, $q \in M$, is a smooth function on M.
- To a smooth vector field $V \in \operatorname{Vec} M$ there corresponds a *linear operator*

$$\widehat{V} : C^\infty(M) \to C^\infty(M)$$

that satisfies the Leibniz rule

$$\widehat{V}(ab) = (\widehat{V}a)b + a(\widehat{V}b), \qquad a, \ b \in C^\infty(M),$$

the directional derivative (Lie derivative) along V.

- A linear operator on an algebra meeting the Leibniz rule is called a *derivation* of the algebra, so the Lie derivative $\widehat{V}$ is a derivation of the algebra $C^\infty(M)$.

- We show that the correspondence between smooth vector fields on M and derivations of the algebra $C^\infty(M)$ is invertible.

Proposition 4

Any derivation of the algebra $C^\infty(M)$ is the directional derivative along some smooth vector field on M.

Proof.

Let $D : C^\infty(M) \to C^\infty(M)$ be a derivation. Take any point $q \in M$. We show that the linear functional

$$d_q \stackrel{\text{def}}{=} \widehat{q} \circ D : C^\infty(M) \to \mathbb{R}$$

is a directional derivative at the point q, i.e., satisfies Leibniz rule (1):

$$\begin{aligned} d_q(ab) &= \widehat{q}(D(ab)) = \widehat{q}((Da)b + a(Db)) = \widehat{q}(Da)b(q) + a(q)\widehat{q}(Db) = \\ &(d_q a)b(q) + a(q)(d_q b), \qquad a, \ b \in C^\infty(M). \end{aligned}$$

□

- So we can identify points $q \in M$, diffeomorphisms $P \in \operatorname{Diff} M$, and vector fields $V \in \operatorname{Vec} M$ with nontrivial homomorphisms $\widehat{q} : C^\infty(M) \to \mathbb{R}$, automorphisms $\widehat{P} : C^\infty(M) \to C^\infty(M)$, and derivations $\widehat{V} : C^\infty(M) \to C^\infty(M)$ respectively.
- For example, we can write a point $P(q)$ in the operator notation as $\widehat{q} \circ \widehat{P}$.
- Moreover, in the sequel we omit hats and write $q \circ P$. This does not cause ambiguity: if q is to the right of P, then q is a point, P a diffeomorphism, and $P(q)$ is the value of the diffeomorphism P at the point q. And if q is to the left of P, then q is a homomorphism, P an automorphism, and $q \circ P$ a homomorphism of $C^\infty(M)$.
- Similarly, $V(q) \in T_qM$ is the value of the vector field V at the point q, and $q \circ V : C^\infty(M) \to \mathbb{R}$ is the directional derivative along the vector $V(q)$.

Seminorms and $C^\infty(M)$-Topology

- We introduce seminorms and topology on the space $C^\infty(M)$.
- By Whitney's Theorem, a smooth manifold M can be properly embedded into a Euclidean space $\mathbb{R}^N$ for sufficiently large N. Denote by h_i, $i = 1, \ldots, N$, the smooth vector field on M that is the orthogonal projection from $\mathbb{R}^N$ to M of the constant basis vector field $\frac{\partial}{\partial x_i} \in \mathrm{Vec}(\mathbb{R}^N)$. So we have N vector fields $h_1, \ldots, h_N \in \mathrm{Vec}\, M$ that span the tangent space T_qM at each point $q \in M$.
- We define the family of seminorms $\| \cdot \|_{s,K}$ on the space $C^\infty(M)$ in the following way:

$$\|a\|_{s,K} = \sup\left\{|h_{i_l} \circ \cdots \circ h_{i_1} a(q)| \mid q \in K,\ 1 \le i_1, \ldots, i_l \le N,\ 0 \le l \le s\right\}, \qquad a \in C^\infty(M), \quad s \ge 0, \quad K \Subset M.$$

- This family of seminorms defines a topology on $C^\infty(M)$.

- A local base of this topology is given by the subsets

$$\left\{ a \in C^\infty(M) \mid \|a\|_{n,K_n} < \frac{1}{n} \right\}, \qquad n \in \mathbb{N},$$

  where K_n, $n \in \mathbb{N}$, is a chained system of compacta that cover M:

$$K_n \subset K_{n+1}, \qquad \bigcup_{n=1}^{\infty} K_n = M.$$

- This topology on $C^\infty(M)$ does not depend on embedding of M into $\mathbb{R}^N$. It is called the *topology of uniform convergence of all derivatives on compacta*, or just *$C^\infty(M)$-topology*.
- This topology turns $C^\infty(M)$ into a Fréchet space (a complete, metrizable, locally convex topological vector space).
- A sequence of functions $a_k \in C^\infty(M)$ converges to $a \in C^\infty(M)$ as $k \to \infty$ if and only if

$$\lim_{k\to\infty} \|a_k - a\|_{s,K} = 0 \quad \forall\ s \geq 0,\ K \Subset M.$$

- For vector fields $V \in \operatorname{Vec} M$, we define the seminorms

$$\|V\|_{s,K} = \sup \{ \|Va\|_{s,K} \mid \|a\|_{s+1,K} = 1 \} , \qquad s \geq 0, \quad K \Subset M. \tag{2}$$

- One can prove that any vector field $V \in \operatorname{Vec} M$ has finite seminorms $\|V\|_{s,K}$, and that there holds an estimate of the action of a diffeomorphism $P \in \operatorname{Diff} M$ on a function $a \in C^\infty(M)$:

$$\|Pa\|_{s,K} \leq C_{s,P} \|a\|_{s,P(K)}, \qquad s \geq 0, \quad K \Subset M. \tag{3}$$

- Thus vector fields and diffeomorphisms are linear *continuous* operators on the topological vector space $C^\infty(M)$.

Families of Functionals and Operators

- In the sequel we will often consider *one-parameter families* of points, diffeomorphisms, and vector fields that satisfy various regularity properties (e.g. differentiability or absolute continuity) with respect to the parameter.
- Since we treat points as functionals, and diffeomorphisms and vector fields as operators on $C^\infty(M)$, we can introduce regularity properties for them in the weak sense, via the corresponding properties for one-parameter families of functions

$$t \mapsto a_t, \qquad a_t \in C^\infty(M), \quad t \in \mathbb{R}.$$

- So we start from definitions for families of functions.
- *Continuity* and *differentiability* of a family of functions a_t w.r.t. parameter t are defined in a standard way since $C^\infty(M)$ is a topological vector space.

- A family of functions a_t is called *measurable* w.r.t. t if the real function $t \mapsto a_t(q)$ is measurable for any $q \in M$. A measurable family a_t is called *locally integrable* if

$$\int_{t_0}^{t_1} \|a_t\|_{s,K}\, dt < \infty \qquad \forall\ s \geq 0, \quad K \Subset M, \quad t_0,\ t_1 \in \mathbb{R}.$$

- A family a_t is called *absolutely continuous* w.r.t. t if

$$a_t = a_{t_0} + \int_{t_0}^{t} b_\tau \, d\tau$$

  for some locally integrable family of functions b_t.

- A family a_t is called *Lipschitzian* w.r.t. t if

$$\|a_t - a_\tau\|_{s,K} \leq C_{s,K}|t-\tau| \qquad \forall s \geq 0, \quad K \Subset M, \quad t,\ \tau \in \mathbb{R},$$

  and *locally bounded* w.r.t. t if

$$\|a_t\|_{s,K} \leq C_{s,K,I}, \qquad \forall\ s \geq 0, \quad K \Subset M, \quad I \Subset \mathbb{R}, \quad t \in I,$$

  where $C_{s,K}$ and $C_{s,K,I}$ are some constants depending on s, K, and I.

- Now we can define regularity properties of families of functionals and operators on $C^\infty(M)$.
- A family of linear functionals or linear operators on $C^\infty(M)$

$$t \mapsto A_t, \qquad t \in \mathbb{R},$$

has some regularity property (i.e., is *continuous*, *differentiable*, *measurable*, *locally integrable*, *absolutely continuous*, *Lipschitzian*, *locally bounded* w.r.t. t) if the family

$$t \mapsto A_t a, \qquad t \in \mathbb{R},$$

has the same property for any $a \in C^\infty(M)$.
- A locally bounded w.r.t. t family of vector fields

$$t \mapsto V_t, \qquad V_t \in \operatorname{Vec} M, \quad t \in \mathbb{R},$$

is called a *nonautonomous vector field*, or simply a *vector field*, on M.
- An absolutely continuous w.r.t. t family of diffeomorphisms

$$t \mapsto P^t, \qquad P^t \in \operatorname{Diff} M, \quad t \in \mathbb{R},$$

is called a *flow* on M.

- So, for a nonautonomous vector field V_t, the family of functions $t \mapsto V_t a$ is locally integrable for any $a \in C^\infty(M)$.
- Similarly, for a flow P^t, the family of functions $(P^t a)(q) = a(P^t(q))$ is absolutely continuous w.r.t. t for any $a \in C^\infty(M)$.
- Integrals of measurable locally integrable families, and derivatives of differentiable families are also defined in the weak sense:

$$\int_{t_0}^{t_1} A_t \, dt \ : a \mapsto \int_{t_0}^{t_1} (A_t a) \, dt, \qquad a \in C^\infty(M),$$

$$\frac{d}{d\,t} A_t \ : a \mapsto \frac{d}{d\,t}(A_t a), \qquad a \in C^\infty(M).$$

- One can show that if A_t and B_t are continuous families of operators on $C^\infty(M)$ which are differentiable at t_0, then the family $A_t \circ B_t$ is continuous, moreover, differentiable at t_0, and satisfies the Leibniz rule:

$$\left.\frac{d}{d\,t}\right|_{t_0} (A_t \circ B_t) = \left(\left.\frac{d}{d\,t}\right|_{t_0} A_t \right) \circ B_{t_0} + A_{t_0} \circ \left(\left.\frac{d}{d\,t}\right|_{t_0} B_t \right).$$

- If families A_t and B_t of operators are absolutely continuous, then the composition $A_t \circ B_t$ is absolutely continuous as well, the same is true for composition of functionals with operators.
- For an absolute continuous family of functions a_t, the family $A_t a_t$ is also absolutely continuous, and the Leibniz rule holds for it as well.

ODEs with discontinuous right-hand side

- We consider a *nonautonomous ordinary differential equation* of the form

$$\dot{q} = V_t(q), \qquad q(0) = q_0, \tag{4}$$

  where V_t is a nonautonomous vector field on M, and study the flow determined by this field.
- We denote by $\dot{q}$ the derivative $\frac{d\,q}{d\,t}$, so equation (4) reads in the expanded form as

$$\frac{d\,q(t)}{d\,t} = V_t(q(t)).$$

- To obtain local solutions to the Cauchy problem (4) on a manifold M, we reduce it to a Cauchy problem in a Euclidean space.
- Choose local coordinates $x = (x^1, \dots, x^n)$ in a neighborhood O_{q_0} of the point q_0:

$$\Phi : O_{q_0} \subset M \to O_{x_0} \subset \mathbb{R}^n, \qquad \Phi : q \mapsto x,$$
$$\Phi(q_0) = x_0.$$

- In these coordinates, the field V_t reads

$$(\Phi_* V_t)(x) = \widetilde{V}_t(x) = \sum_{i=1}^{n} v_i(t,x)\frac{\partial}{\partial x^i}, \qquad x \in O_{x_0}, \quad t \in \mathbb{R}, \tag{5}$$

and problem (4) takes the form

$$\dot{x} = \widetilde{V}_t(x), \quad x(0) = x_0, \qquad x \in O_{x_0} \subset \mathbb{R}^n. \tag{6}$$

- Since the nonautonomous vector field $V_t \in \operatorname{Vec} M$ is locally bounded, the components $v_i(t,x)$, $i = 1, \ldots, n$, of its coordinate representation (5) are:
  (1) measurable and locally bounded w.r.t. t for any fixed $x \in O_{x_0}$,
  (2) smooth w.r.t. x for any fixed $t \in \mathbb{R}$,
  (3) differentiable in x with locally bounded partial derivatives:

$$\left|\frac{\partial v_i}{\partial x}(t,x)\right| \leq C_{I,K}, \qquad t \in I \Subset \mathbb{R},\ x \in K \Subset O_{x_0},\ i = 1, \ldots, n.$$

- By the classical Carathéodory Theorem, the Cauchy problem (6) has a unique solution, i.e., a vector-function $x(t, x_0)$, Lipschitzian w.r.t. t and smooth w.r.t. x_0, and such that:
  (1) ODE (6) is satisfied for almost all t,
  (2) initial condition holds: $x(0, x_0) = x_0$.
- Then the pull-back of this solution from $\mathbb{R}^n$ to M

$$q(t, q_0) = \Phi^{-1}(x(t, x_0)),$$

  is a solution to problem (4) in M.
- The mapping $q(t, q_0)$ is Lipschitzian w.r.t. t and smooth w.r.t. q_0, it satisfies almost everywhere the ODE and the initial condition in (4).
- For any $q_0 \in M$, the solution $q(t, q_0)$ to the Cauchy problem (4) can be continued to a maximal interval $t \in J_{q_0} \subset \mathbb{R}$ containing the origin and depending on q_0.
- We will assume that the solutions $q(t, q_0)$ are defined for all $q_0 \in M$ and all $t \in \mathbb{R}$, i.e., $J_{q_0} = \mathbb{R}$ for any $q_0 \in M$. Then the nonautonomous field V_t is called *complete*.
- This holds, e.g., when all the fields V_t, $t \in \mathbb{R}$, vanish outside of a common compactum in M (in this case we say that the nonautonomous vector field V_t has a *compact support*).

Definition of the right chronological exponential

- The Cauchy problem $\dot{q} = V_t(q)$, $q(0) = q_0$, rewritten as a linear equation for Lipschitzian w.r.t. t families of functionals on $C^\infty(M)$:
$$\dot{q}(t) = q(t) \circ V_t, \qquad q(0) = q_0, \tag{7}$$
is satisfied for the family of functionals
$$q(t, q_0) \,:\, C^\infty(M) \to \mathbb{R}, \qquad q_0 \in M, \quad t \in \mathbb{R}$$
constructed in the previous subsection.
- We prove later that this Cauchy problem has no other solutions.
- Thus the flow defined as
$$P^t \;:\; q_0 \mapsto q(t, q_0) \tag{8}$$
is a unique solution of the operator Cauchy problem $\dot{P}^t = P^t \circ V_t$, $P^0 = \mathrm{Id}$ (where Id is the identity operator), in the class of Lipschitzian flows on M.
- The flow P^t determined in (8) is called the *right chronological exponential* of the field V_t and is denoted as $P^t = \overrightarrow{\exp} \displaystyle\int_0^t V_\tau \, d\tau$.

Formal series expansion

- We rewrite differential equation in (7) as an integral one:

$$q(t) = q_0 + \int_0^t q(\tau) \circ V_\tau \, d\tau \tag{9}$$

then substitute this expression for $q(t)$ into the right-hand side

$$\begin{aligned} &= q_0 + \int_0^t \left(q_0 + \int_0^{\tau_1} q(\tau_2) \circ V_{\tau_2} \, d\tau_2 \right) \circ V_{\tau_1} \, d\tau_1 \\ &= q_0 \circ \left(\mathrm{Id} + \int_0^t V_\tau \, dt \right) + \iint\limits_{0 \le \tau_2 \le \tau_1 \le t} q(\tau_2) \circ V_{\tau_2} \circ V_{\tau_1} \, d\tau_2 \, d\tau_1, \end{aligned}$$

repeat this procedure iteratively, and obtain the decomposition:

$$
q(t) = q_0 \circ \left(\mathrm{Id} + \int_0^t V_\tau \, d\tau + \iint\limits_{\Delta_2(t)} V_{\tau_2} \circ V_{\tau_1} \, d\tau_2 \, d\tau_1 + \ldots + \right.
$$

$$
\left. \int\limits_{\Delta_n(t)} \cdots \int V_{\tau_n} \circ \cdots \circ V_{\tau_1} \, d\tau_n \, \ldots \, d\tau_1 \right) +
$$

$$
\int\limits_{\Delta_{n+1}(t)} \cdots \int q(\tau_{n+1}) \circ V_{\tau_{n+1}} \circ \cdots \circ V_{\tau_1} \, d\tau_{n+1} \, \ldots \, d\tau_1 . \qquad (10)
$$

- Here

$$
\Delta_n(t) = \{(\tau_1, \ldots, \tau_n) \in \mathbb{R}^n \mid 0 \leq \tau_n \leq \cdots \leq \tau_1 \leq t\}
$$

is the n-dimensional simplex.

- Purely formally passing in (10) to the limit $n \to \infty$, we obtain a formal series for the solution $q(t)$ to problem (7):

$$q_0 \circ \left(\mathrm{Id} + \sum_{n=1}^{\infty} \int_{\Delta_n(t)} \cdots \int V_{\tau_n} \circ \cdots \circ V_{\tau_1} \, d\tau_n \, \ldots \, d\tau_1 \right),$$

thus for the solution P^t to our Cauchy problem:

$$\mathrm{Id} + \sum_{n=1}^{\infty} \int_{\Delta_n(t)} \cdots \int V_{\tau_n} \circ \cdots \circ V_{\tau_1} \, d\tau_n \, \ldots \, d\tau_1. \qquad (11)$$

Estimates and convergence of the series

- Unfortunately, series (11) never converges on $C^\infty(M)$ in the weak sense (if $V_t \not\equiv 0$): there always exists a smooth function on M, on which it diverges.
- Although, one can show that series (11) gives an asymptotic expansion for the chronological exponential $P^t = \overrightarrow{\exp} \int_0^t V_\tau \, d\tau$.
- There holds the following bound of the remainder term: denote the m-th partial sum of series (11) as $S_m(t) = \mathrm{Id} + \sum_{n=1}^{m-1} \underset{\Delta_n(t)}{\int \cdots \int} V_{\tau_n} \circ \cdots \circ V_{\tau_1} \, d\tau_n \ldots d\tau_1$, then for any $a \in C^\infty(M)$, $s \geq 0$, $K \Subset M$

$$\left\| \left(\overrightarrow{\exp} \int_0^t V_\tau \, d\tau - S_m(t) \right) a \right\|_{s,K} \leq C e^{C \int_0^t \|V_\tau\|_{s,K'} \, d\tau} \frac{1}{m!} \left(\int_0^t \|V_\tau\|_{s+m-1,K'} \, d\tau \right)^m \|a\|_{s+m,K'} = O(t^m), \qquad t \to 0, \tag{12}$$

where $K' \Subset M$ is some compactum containing K, see [AS].

- It follows from estimate (12) that

$$\left\| \left(\overrightarrow{\exp} \int_0^t \varepsilon V_\tau \, d\tau - S_m^\varepsilon(t) \right) a \right\|_{s,K} = O(\varepsilon^m), \qquad \varepsilon \to 0,$$

where $S_m^\varepsilon(t)$ is the m-th partial sum of series (11) for the field εV_t.

- Thus we have an asymptotic series expansion:

$$\overrightarrow{\exp} \int_0^t V_\tau \, d\tau \approx \mathrm{Id} + \sum_{n=1}^{\infty} \int \cdots \int_{\Delta_n(t)} V_{\tau_n} \circ \cdots \circ V_{\tau_1} \, d\tau_n \ldots d\tau_1. \tag{13}$$

- In the sequel we will use terms of the zeroth, first, and second orders of the series obtained:

$$\overrightarrow{\exp} \int_0^t V_\tau \, d\tau \approx \mathrm{Id} + \int_0^t V_\tau \, d\tau + \iint_{0 \le \tau_2 \le \tau_1 \le t} V_{\tau_2} \circ V_{\tau_1} \, d\tau_2 \, d\tau_1 + \cdots \, .$$

- We prove now that the asymptotic series converges to the chronological exponential on any normed subspace $L \subset C^\infty(M)$ where V_t is well-defined and bounded:

$$V_t L \subset L, \qquad \|V_t\| = \sup\{\|V_t a\| \mid a \in L, \ \|a\| \leq 1\} < \infty. \tag{14}$$

- We apply operator series (13) to any $a \in L$ and bound terms of the series obtained:

$$a + \sum_{n=1}^{\infty} \int \cdots \int_{\Delta_n(t)} V_{\tau_n} \circ \cdots \circ V_{\tau_1} \, a \, d\tau_n \, \ldots \, d\tau_1 . \tag{15}$$

$$
\begin{aligned}
&\left\| \underset{\Delta_n(t)}{\int \cdots \int} V_{\tau_n} \circ \cdots \circ V_{\tau_1} \, a \, d\tau_n \, \ldots \, d\tau_1 \right\| \\
&\quad \leq \underset{0 \leq \tau_n \leq \cdots \leq \tau_1 \leq t}{\int \cdots \int} \|V_{\tau_n}\| \cdot \cdots \cdot \|V_{\tau_1}\| \, d\tau_n \, \ldots \, d\tau_1 \cdot \|a\| \\
&\quad = \underset{0 \leq \tau_{\sigma(n)} \leq \cdots \leq \tau_{\sigma(1)} \leq t}{\int \cdots \int} \|V_{\tau_n}\| \cdot \cdots \cdot \|V_{\tau_1}\| \, d\tau_n \, \ldots \, d\tau_1 \cdot \|a\| \\
&\quad = \frac{1}{n!} \int_0^t \ldots \int_0^t \|V_{\tau_n}\| \cdot \cdots \cdot \|V_{\tau_1}\| \, d\tau_n \, \ldots \, d\tau_1 \cdot \|a\| \\
&\quad = \frac{1}{n!} \left(\int_0^t \|V_\tau\| \, d\tau \right)^n \cdot \|a\|.
\end{aligned}
$$

- So series (15) is majorized by the exponential series, thus the operator series (13) converges on L.
- Series (15) can be differentiated termwise, thus it satisfies the same ODE as the function $P^t a$:

$$\dot{a}_t = V_t a_t, \qquad a_0 = a.$$

- Consequently,

$$P^t a = a + \sum_{n=1}^{\infty} \int \cdots \int_{\Delta_n(t)} V_{\tau_n} \circ \cdots \circ V_{\tau_1}\, a \, d\tau_n \ldots d\tau_1 .$$

- So in the case (14) the asymptotic series converges to the chronological exponential and there holds the bound

$$\|P^t a\| \leq e^{\int_0^t \|V_\tau\| \, d\tau} \|a\|, \qquad a \in L.$$

- Moreover, one can show that the bound and convergence hold not only for locally bounded, but also for integrable on $[0, t]$ vector fields: $\displaystyle\int_0^t \|V_\tau\| \, d\tau < \infty$.

- Notice that conditions (14) are satisfied for any finite-dimensional V_t-invariant subspace $L \subset C^\infty(M)$. In particular, this is the case when $M = \mathbb{R}^n$, L is the space of linear functions, and V_t is a linear vector field on $\mathbb{R}^n$.
- If M, V_t, and a are real analytic, then series (15) converges for sufficiently small t.

Left chronological exponential

- Consider the inverse operator $Q^t = (P^t)^{-1}$ to the right chronological exponential $P^t = \overrightarrow{\exp} \int_0^t V_\tau \, d\tau$.
- We find an ODE for the flow Q^t by differentiation of the identity
$$P^t \circ Q^t = \mathrm{Id}\,.$$
- Leibniz rule yields $\dot{P}^t \circ Q^t + P^t \circ \dot{Q}^t = 0$, thus, in view of the ODE for the flow P^t,
$$P^t \circ V_t \circ Q^t + P^t \circ \dot{Q}^t = 0.$$
- We multiply this equality by Q^t from the left and obtain
$$V_t \circ Q^t + \dot{Q}^t = 0.$$

  That is, the flow Q^t is a solution of the Cauchy problem
$$\frac{d}{d\,t} Q^t = -V_t \circ Q^t, \qquad Q^0 = \mathrm{Id}, \tag{16}$$
  which is dual to the Cauchy problem for P^t: $\frac{d}{d\,t} P^t = P^t \circ V_t$, $P^0 = \mathrm{Id}$.

- The flow Q^t is called the *left chronological exponential* and is denoted as

$$Q^t = \overleftarrow{\exp} \int_0^t (-V_\tau)\, d\tau.$$

- We find an asymptotic expansion for the left chronological exponential in the same way as for the right one, by successive substitutions into the right-hand side:

$$\begin{aligned} Q^t &= \mathrm{Id} + \int_0^t (-V_\tau) \circ Q^\tau \, d\tau \\ &= \mathrm{Id} + \int_0^t (-V_\tau)\, d\tau + \iint\limits_{\Delta_2(t)} (-V_{\tau_1}) \circ (-V_{\tau_2}) \circ Q^{\tau_2}\, d\tau_2\, d\tau_1 = \cdots \\ &= \mathrm{Id} + \sum_{n=1}^{m-1} \int\limits_{\Delta_n(t)} \cdots \int (-V_{\tau_1}) \circ \cdots \circ (-V_{\tau_n})\, d\tau_n \ldots d\tau_1 \\ &\quad + \int\limits_{\Delta_m(t)} \cdots \int (-V_{\tau_1}) \circ \cdots \circ (-V_{\tau_m}) \circ Q^{\tau_m}\, d\tau_m \ldots d\tau_1 . \end{aligned}$$

- For the left chronological exponential holds an estimate of the remainder term as (12) for the right one, and the series obtained is asymptotic:

$$\overleftarrow{\exp} \int_0^t (-V_\tau)\, d\tau \approx \mathrm{Id} + \sum_{n=1}^{\infty} \int \cdots \int_{\Delta_n(t)} (-V_{\tau_1}) \circ \cdots \circ (-V_{\tau_n})\, d\tau_n \ldots d\tau_1 .$$

- Notice that the reverse arrow in the left chronological exponential $\overleftarrow{\exp}$ corresponds to the reverse order of the operators $(-V_{\tau_1}) \circ \cdots \circ (-V_{\tau_n})$, $\tau_n \leq \ldots \leq \tau_1$.
- The right and left chronological exponentials satisfy the corresponding differential equations:

$$\frac{d}{d\,t} \overrightarrow{\exp} \int_0^t V_\tau\, d\tau = \overrightarrow{\exp} \int_0^t V_\tau\, d\tau \circ V_t,$$

$$\frac{d}{d\,t} \overleftarrow{\exp} \int_0^t (-V_\tau)\, d\tau = -V_t \circ \overleftarrow{\exp} \int_0^t (-V_\tau)\, d\tau.$$

The directions of arrows correlate with the direction of appearance of the operators V_t and $(-V_t)$ in the right-hand side of these ODEs.

- If the initial value is prescribed at a moment of time $t_0 \neq 0$, then the lower limit of integrals in the chronological exponentials is t_0.
- There holds the following obvious rule for composition of flows:

$$\overrightarrow{\exp} \int_{t_0}^{t_1} V_\tau \, d\tau \circ \overrightarrow{\exp} \int_{t_1}^{t_2} V_\tau \, d\tau = \overrightarrow{\exp} \int_{t_0}^{t_2} V_\tau \, d\tau.$$

- There hold the identities

$$\overrightarrow{\exp} \int_{t_0}^{t_1} V_\tau \, d\tau = \left(\overrightarrow{\exp} \int_{t_1}^{t_0} V_\tau \, d\tau \right)^{-1} = \overleftarrow{\exp} \int_{t_1}^{t_0} (-V_\tau) \, d\tau. \qquad (17)$$

- We saw that equation (7) for Lipschitzian families of functionals has a solution $q(t) = q_0 \circ \overrightarrow{\exp} \int_0^t V_\tau \, d\tau$. We can prove now that this equation has no other solutions.

Proposition 5

Let V_t be a complete nonautonomous vector field on M. Then Cauchy problem (7) has a unique solution in the class of Lipschitzian families of functionals on $C^\infty(M)$.

Proof.

Let a Lipschitzian family of functionals q_t be a solution to problem (7). Then

$$\frac{d}{d\,t}\left(q_t \circ (P^t)^{-1}\right) = \frac{d}{d\,t}\left(q_t \circ Q^t\right) = q_t \circ V_t \circ Q^t - q_t \circ V_t \circ Q^t = 0,$$

thus $q_t \circ Q^t \equiv \text{const}$. But $Q^0 = \mathrm{Id}$, consequently, $q_t \circ Q^t \equiv q_0$, hence

$$q_t = q_0 \circ P^t = q_0 \circ \overrightarrow{\exp} \int_0^t V_\tau \, d\tau$$

is a unique solution of Cauchy problem (7). □

Similarly, the both operator equations $\dot{P}^t = P^t \circ V_t$ and $\dot{Q}^t = -V_t \circ Q^t$ have no other solutions in addition to the chronological exponentials.

Plan of this lecture

1. Points, Diffeomorphisms, and Vector Fields
2. Seminorms and $C^\infty(M)$-Topology
3. Families of Functionals and Operators
4. ODEs with discontinuous right-hand side
5. Definition of the right chronological exponential
6. Formal series expansion
7. Estimates and convergence of the series
8. Left chronological exponential