Optimal Control Problem: Statement and existence of solutions. Lebesgue measure and integral

(Lecture 1)

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Plan of course

1. Statement of the optimal control problem
2. Measurable sets and functions, Carathéodory differential equations
3. Sufficient Filippov conditions for the existence of an optimal control
4. Differential equations on smooth manifolds
5. Elements of chronological calculus of R.V.Gamkrelidze—A.A.Agrachev
6. Differential forms
7. Elements of symplectic geometry
8. Proof of the Pontryagin maximum principle on manifolds: geometric form, optimal control problems with different boundary conditions.
9. Examples of optimal syntheses.

Plan of lecture

1. Optimal Control Problem Statement
2. Lebesgue measurable sets and functions
3. Lebesgue integral
4. Carathéodory ODEs
5. Reduction of Optimal Control Problem to Study of Attainable Sets
6. Filippov's theorem: Compactness of Attainable Sets
7. Time-Optimal Problem

Optimal Control Problem Statement

Control system:

$$\dot{q} = f_u(q), \qquad q \in M, \quad u \in U \subset \mathbb{R}^m. \tag{1}$$

- M a smooth manifold
- U an arbitrary subset of $\mathbb{R}^m$
- right-hand side of (1):

$$q \mapsto f_u(q) \text{ is a smooth vector field on } M \text{ for any fixed } u \in U, \tag{2}$$

$$(q, u) \mapsto f_u(q) \text{ is a continuous mapping for } q \in M,\ u \in \overline{U}, \tag{3}$$

  and moreover, in any local coordinates on M

$$(q, u) \mapsto \frac{\partial f_u}{\partial q}(q) \text{ is a continuous mapping for } q \in M,\ u \in \overline{U}. \tag{4}$$

- *Admissible controls* are measurable locally bounded mappings

$$u \,:\, t \mapsto u(t) \in U,$$

  i.e., $u \in L_\infty([0, t_1], U)$.

- Substitute such a control $u = u(t)$ for control parameter into system (1)
- $\Rightarrow$ nonautonomous ODE $\dot{q} = f_u(q)$
- By Carathéodory's Theorem, for any point $q_0 \in M$, the Cauchy problem

$$\dot{q} = f_u(q), \qquad q(0) = q_0, \tag{5}$$

  has a unique solution $q_u(t)$.
- In order to compare admissible controls one with another on a segment $[0, t_1]$, introduce a *cost functional*:

$$J(u) = \int_0^{t_1} \varphi(q_u(t), u(t))\, dt \tag{6}$$

  with an integrand

$$\varphi : M \times U \to \mathbb{R}$$

  satisfying the same regularity assumptions as the right-hand side f, see (2)–(4).

- Take any pair of points $q_0, q_1 \in M$.
- Consider the following *optimal control problem*:

Problem 1

Minimize the functional J among all admissible controls $u = u(t)$, $t \in [0, t_1]$, for which the corresponding solution $q_u(t)$ of Cauchy problem (5) satisfies the boundary condition

$$q_u(t_1) = q_1 . \tag{7}$$

- This problem can also be written as follows:

$$\dot{q} = f_u(q), \qquad q \in M, \quad u \in U \subset \mathbb{R}^m, \tag{8}$$

$$q(0) = q_0, \qquad q(t_1) = q_1, \tag{9}$$

$$J(u) = \int_0^{t_1} \varphi(q(t), u(t))\, dt \to \min . \tag{10}$$

- Two types of problems: with fixed terminal time t_1 and free t_1.
- A solution u of this problem is called an *optimal control*, and the corresponding curve $q_u(t)$ is an *optimal trajectory*.

Example: Euler elasticae

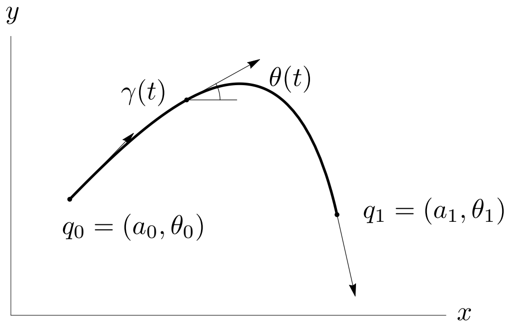
Given:

- uniform elastic rod of length l in the plane
- the rod has fixed endpoints and tangents at endpoints

Find:

- the profile of the rod.

Example: Euler elasticae



$$\dot{x} = \cos\theta, \qquad q = (x, y, \theta) \in \mathbb{R}^2 \times S^1,$$

$$\dot{y} = \sin\theta, \qquad u \in \mathbb{R},$$

$$\dot{\theta} = u,$$

$$q(0) = q_0, \qquad q(t_1) = q_1,$$

$t_1 = l$ is the length of the rod,

$$J = \frac{1}{2}\int_0^{t_1} u^2\, dt \to \min.$$

Definition of Lebesgue measure in $I = [0, 1]$: H. Lebesgue, 1902 [1]

- Measure of intervals:

$$m(\emptyset) := 0, \qquad m(|a, b|) := b - a, \quad b \geq a, \quad | = [\text{ or }].$$

- Measure of elementary sets: $m'(\sqcup_{i=1}^{\infty} |a_i, b_i|) := \sum_{i=1}^{\infty} m(|a_i, b_i|)$
- Outer measure: $\mu^*(A) := \inf \left\{ \sum_{i=1}^{\infty} m(P_i) \mid A \subset \cup_{i=1}^{\infty} P_i, \ P_i \text{ intervals} \right\}.$
- Lebesgue measure:
    - $A \subset I$ is called *measurable* if

$$\forall \, \varepsilon > 0 \ \exists \text{ elementary set } B \subset I : \ \mu^*(A \triangle B) < \varepsilon, \qquad A \triangle B := (A \setminus B) \cup (B \setminus A).$$

    - A measurable $\Rightarrow$ *Lebesgue measure* $\mu(A) := \mu^*(A)$.

[1] A.N. Kolmogorov, S.V. Fomin, "Elements of theory of functions and functional analysis"

Properties of Lebesgue measure

1. System of measurable sets is closed w.r.t. $\cup_{i=1}^{\infty}$, $\cap_{i=1}^{\infty}$, $\setminus$, $\triangle$
2. σ-additivity: $\quad A_i$ measurable $\quad \Rightarrow \quad \mu(\sqcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i)$.
3. Continuity: $\quad A_1 \supset A_2 \supset \cdots$ measurable $\quad \Rightarrow \quad \mu(\cap_{i=1}^{\infty} A_i) = \lim_{i\to\infty} \mu(A_i)$.
4. Open, closed sets are measurable.
5. There exist non-measurable sets (G. Vitali, 1905)
6. $A \subset \mathbb{R}$ is measurable if $\forall\ A \cap I_n$ is measurable, $I_n = (n, n+1]$, $n \in \mathbb{Z}$,
7. $\mu(A) := \sum_{n=-\infty}^{+\infty} \mu(A \cap I_n) \in [0, +\infty]$.
8. $\mu(A) = 0 \quad \Leftrightarrow \quad \forall \varepsilon > 0\ \exists$ intervals: $\cup_{i=1}^{\infty} P_i \supset A$, $\sum_{i=1}^{\infty} m(P_i) < \varepsilon$.
9. A property P holds *almost everywhere* (a.e.) on a set X if $\exists\ A \subset X$, $\mu(A) = 0$, s.t. P holds on $X \setminus A$.
10. $f : \mathbb{R} \to \mathbb{R}^m$ is *measurable* if $f^{-1}(O)$ is measurable for any open $O \subset \mathbb{R}^m$.

Theorem 2

Let $B, B' \subset \mathbb{R}^3$ be balls of different radii. Then there exist decompositions

$$B = X_1 \sqcup \cdots \sqcup X_n, \qquad B' = X'_1 \sqcup \cdots \sqcup X'_n$$

such that

$$\exists f_i \in \mathrm{SE}(3) \ : \ f_i(X_i) = X'_i, \qquad i = 1, \ldots, n.$$

- Sets X_i, X'_i are not measurable.
- $n \geq 5$.
- B, B' can be replaced by any bounded subsets in $\mathbb{R}^3$ with nonempty interior.
- Similar theorem for $\mathbb{R}^2$ instead of $\mathbb{R}^3$ fails.
  Reason: SE(2) is solvable, while SE(3) is not:
  $[\mathfrak{se}(3), \mathfrak{se}(3)] = \mathfrak{so}(3)$, $[\mathfrak{so}(3), \mathfrak{so}(3)] = \mathfrak{so}(3) \neq \{0\}$.

Lebesgue integral: Definition

- Let $\mu(X) < +\infty$. A function $f : X \to \mathbb{R}$ is simple if it is measurable and takes not more than countable number of values.
- Th.: A function $f(x)$ taking not more than countable number of values y_1, y_2, $\ldots$ is measurable iff al sets $f^{-1}(y_n)$ are measurable.
- Th.: A function $f(x)$ is measurable iff it is a uniform limit of simple measurable functions.
- Let f be a simple measurable function taking values $y_1, y_2, \ldots$. Let $A \subset X$ be measurable. Then
$$\int_A f(x)d\mu := \sum_n y_n \mu(f^{-1}(y_n)).$$
A function f is called integrable on A if this series absolutely converges.
- A measurable function f is called *integrable* on $A \subset X$ if there exist a sequence of simple integrable on A functions $\{f_n\}$ that converges uniformly to f. Then
$$\int_A f(x)d\mu := \lim_{n\to\infty} \int_A f_n(x)d\mu.$$

Lebesgue integral: Properties

1. $\int_A 1 d\mu = \mu(A)$.
2. Linearity: $\int_A (af(x) + bg(x))d\mu = a\int_A f(x)d\mu + b\int_A g(x)d\mu$.
3. $f(x)$ bounded on $A \quad \Rightarrow \quad f(x)$ integrable on A.
4. Monotonicity: $f(x) \leq g(x) \quad \Rightarrow \quad \int_A f(x)d\mu \leq \int_A g(x)d\mu$.
5. $\mu(A) = 0 \quad \Rightarrow \quad \int_A f(x)d\mu = 0$.
6. $f(x) = g(x)$ a.e. $\quad \Rightarrow \quad \int_A f(x)d\mu = \int_A g(x)d\mu$.
7. $g(x)$ integrable on A and $|f(x)| \leq g(x)$ a.e. $\quad \Rightarrow \quad f(x)$ integrable on A.
8. Functions f and $|f|$ are integrable or non-integrable simultaneously.
9. σ-additivity: if $A = \sqcup_n A_n$ then $\int_A f(x)d\mu = \sum_n \int_{A_n} f(x)d\mu$.
10. Absolute continuity: f in integrable on $A \quad \Rightarrow \quad \forall\varepsilon > 0\ \exists\delta > 0$ s.t. $\left|\int_E f(x)d\mu\right| < \varepsilon$ for any measurable $E \subset A$, $\mu(E) < \varepsilon$.
11. $\mu(X) = \infty$, $X = \cup_n X_n$, $X_n \subset X_{n+1}$, $\mu(X_n) < \infty \quad \Rightarrow$ $\int_X f(x)d\mu := \lim_{n\to\infty} \int_{X_n} f(x)d\mu$.

Spaces of integrable functions

$f : X \to \mathbb{R}$ measurable.

1. $L_p(X, \mu) = \{f \mid \|f\|_p < \infty\}$, $\|f\|_p = \left(\int_X |f(x)|^p d\mu\right)^{1/p}$, $p \in [1, +\infty)$.
2. $L_\infty(X, \mu) = \{f \mid \|f\|_\infty < \infty\}$, $\|f\|_\infty = \operatorname{ess\,sup}_{x \in X} |f(x)|$.
3. $1 \leq p_1 < p_2 \leq \infty \quad \Rightarrow \quad L_{p_1} \supsetneq L_{p_2}$.
4. L_p, $p \in [1, +\infty]$, are Banach spaces (= complete normed spaces).
5. L_2 is a Hilbert space (= complete Euclidean infinite-dimensional space), $(f, g) = \int_X f(x)g(x)d\mu$.

Carathéodory ODEs: C. Carathéodory, 1873–1950 [2]

- Carathéodory conditions: let for a domain $D \subset \mathbb{R}^{1+n}_{t,x}$
  1. $f(t,x)$ is defined and continuous in x for almost all t
  2. $f(t,x)$ is measurable in t for any x
  3. $|f(t,x)| \leq m(t)$, where $m(t)$ is Lebesgue integrable on any segment
- Carathéodory ODE: $\dot{x} = f(t,x)$, where $f : D \to \mathbb{R}^n$ satisfies conditions 1–3.
- Solution to Carathéodory ODE: $x : |a,b| \to \mathbb{R}^n$, $x(t) = x(t_0) + \int_{t_0}^{t} f(s, x(s))ds$, $t_0 \in |a,b|$.
- Existence: Solutions exist on sufficiently small segments $[t_0, t_0 + \varepsilon]$, $\varepsilon > 0$.
- Uniqueness: If $|f(t,x) - f(t,y)| \leq l(t)|x - y|$, $l(t)$ Lebesgue integrable, then a solution is unique.
- Extension: Any solution in compact D can be extended in both sides up to ∂D.

[2] A.F. Filippov, "Differential equations with discontinuous right-hand side"

Optimal Control Problem Statement

$$\dot{q} = f_u(q), \qquad q \in M, \quad u \in U \subset \mathbb{R}^m, \tag{11}$$

$$q(0) = q_0, \tag{12}$$

$$q(t_1) = q_1, \tag{13}$$

$$J(u) = \int_0^{t_1} \varphi(q, u)dt \to \min. \tag{14}$$

$q = q_u(\cdot)$ — solution to Cauchy problem (11), (12) corresponding to an admissible control $u(\cdot)$.

Attainable sets

- Fix an initial point $q_0 \in M$.
- *Attainable set* of control system (11) for time $t \geq 0$ from q_0 with measurable locally bounded controls is defined as follows:

$$\mathcal{A}_{q_0}(t) = \{q_u(t) \mid u \in L_\infty([0, t], U)\} \,.$$

- Similarly, one can consider the attainable sets for time not greater than t:

$$\mathcal{A}^t_{q_0} = \bigcup_{0 \leq \tau \leq t} \mathcal{A}_{q_0}(\tau)$$

and for arbitrary nonnegative time:

$$\mathcal{A}_{q_0} = \bigcup_{0 \leq \tau < \infty} \mathcal{A}_{q_0}(\tau).$$

Extended system

- Optimal control problems on M can be reduced to the study of attainable sets of some auxiliary control systems on the extended state space
$$\widehat{M} = \mathbb{R} \times M = \{\widehat{q} = (y, q) \mid y \in \mathbb{R},\ q \in M\}.$$
- Consider the following extended control system on $\widehat{M}$:
$$\frac{d\,\widehat{q}}{d\,t} = \widehat{f}_u(\widehat{q}), \qquad \widehat{q} \in \widehat{M},\ u \in U, \tag{15}$$
with the right-hand side
$$\widehat{f}_u(\widehat{q}) = \begin{pmatrix} \varphi(q, u) \\ f_u(q) \end{pmatrix}, \qquad q \in M, \quad u \in U,$$
where φ is the integrand of the cost functional J, see (14).
- Denote by $\widehat{q}_u(t)$ the solution of the extended system (15) with the initial conditions
$$\widehat{q}_u(0) = \begin{pmatrix} y(0) \\ q(0) \end{pmatrix} = \begin{pmatrix} 0 \\ q_0 \end{pmatrix}.$$

Reduction to Study of Attainable Sets

Theorem 3

Let $q_{\widetilde{u}}(t)$, $t \in [0, t_1]$, be an optimal trajectory in the problem (11)–(14) *with the fixed terminal time t_1. Then $\widehat{q}_{\widetilde{u}}(t_1) \in \partial \widehat{\mathcal{A}}_{(0,q_0)}(t_1)$.*

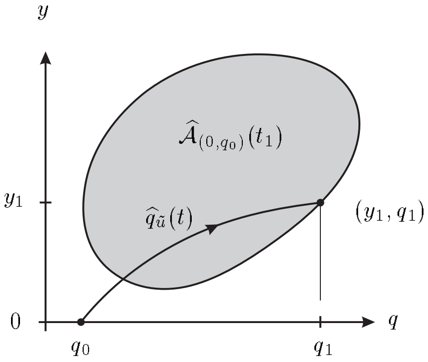


Figure: $q_{\widetilde{u}}(t)$ optimal

Proof.

- Solutions $\widehat{q}_u(t)$ of the extended system are expressed through solutions $q_u(t)$ of the original system (11) as

$$\widehat{q}_u(t) = \begin{pmatrix} J_t(u) \\ q_u(t) \end{pmatrix}, \qquad J_t(u) = \int_0^t \varphi(q_u(\tau), u(\tau))\, d\tau.$$

- Thus attainable sets of the extended system (15) have the form

$$\widehat{\mathcal{A}}_{(0,q_0)}(t) = \{(J_t(u), q_u(t)) \mid u \in L_\infty([0,t], U)\}.$$

- The set $\widehat{\mathcal{A}}_{(0,q_0)}(t_1)$ should not intersect the ray $\left\{(y, q_1) \in \widehat{M} \mid y < J_{t_1}(\widetilde{u})\right\}$.
- Indeed, suppose that there exists a point $(y, q_1) \in \widehat{\mathcal{A}}_{(0,q_0)}(t_1), \quad y < J_{t_1}(\widetilde{u})$.
- Then the trajectory of the extended system $\widehat{q}_u(t)$ that steers $(0, q_0)$ to (y, q_1):

$$\widehat{q}_u(0) = \begin{pmatrix} 0 \\ q_0 \end{pmatrix}, \qquad \widehat{q}_u(t_1) = \begin{pmatrix} y \\ q_1 \end{pmatrix},$$

gives a trajectory $q_u(t)$, $q_u(0) = q_0$, $q_u(t_1) = q_1$, with $J_{t_1}(u) = y < J_{t_1}(\widetilde{u})$, a contradiction to optimality of $\widetilde{u}$. □

Existence of optimal trajectories for problems with fixed t_1

Theorem 4

Let $q_1 \in \mathcal{A}_{q_0}(t_1)$. If $\widehat{\mathcal{A}}_{(0,q_0)}(t_1)$ is compact, then there exists an optimal trajectory in the problem (11)–(14) with the fixed terminal time t_1.

Proof.

- The intersection $\widehat{\mathcal{A}}_{(0,q_0)}(t_1) \cap \{(y, q_1) \in \widehat{M}\}$ is nonempty and compact.
- Denote $\widetilde{J} = \min\{y \in \mathbb{R} \mid (y, q_1) \in \widehat{\mathcal{A}}_{(0,q_0)}(t_1)\}$.
- $(\widetilde{J}, q_1) \in \widehat{\mathcal{A}}_{(0,q_0)}(t_1)$.
- There exists an admissible control $\widetilde{u}$ such that $q_{\widetilde{u}}$ steers q_0 to q_1 for time t_1 with the cost $\widetilde{J}$.
- The trajectory $q_{\widetilde{u}}$ is optimal.

□

Existence of optimal trajectories for problems with free t_1

Theorem 5

Let $q_1 \in \mathcal{A}_{q_0}$. Let $\widehat{\mathcal{A}}^t_{(0,q_0)}$, $t > 0$, be compact. Let there extist $\overline{u} \in L_\infty[0, \overline{t}_1]$ that steers q_0 to q_1 such that for any $u \in L_\infty[0, t_1]$ that steers q_0 to q_1:

$$t_1 > \overline{t}_1 \quad \Rightarrow \quad J(u) > J(\overline{u}).$$

Then there exists an optimal trajectory in the problem (11)–(14) *with the free t_1.*

Proof.

- Denote $I^t = \left\{ y \in \mathbb{R} \mid (y, q_1) \in \widehat{\mathcal{A}}^t_{(0,q_0)} \right\}$, $J^t = \min I^t$.
- Since $q_1 \in \mathcal{A}_{q_0}(t_1)$ for some $t_1 > 0$, then $I^{t_1} \neq \emptyset$.
- Let $T = \max(t_1, \overline{t}_1)$. We have $I^T \neq \emptyset$. Denote $\widetilde{J} = J^T$.
- There exists $\widetilde{u} \in L_\infty[0, \widetilde{t}_1]$ that steers q_0 to q_1 with the cost $\widetilde{J} = J(\widetilde{u})$.
- The control $\widetilde{u}$ is optimal in the problem with the free t_1.

Compactness of attainable sets

Theorem 6 (Filippov)

Let the space of control parameters $U \Subset \mathbb{R}^m$ *be compact. Let there exist a compact* $K \Subset M$ *such that* $f_u(q) = 0$ *for* $q \notin K$, $u \in U$. *Moreover, let the velocity sets*

$$f_U(q) = \{f_u(q) \mid u \in U\} \subset T_q M, \qquad q \in M,$$

be convex. Then the attainable sets $\mathcal{A}_{q_0}(t)$ *and* $\mathcal{A}^t_{q_0}$ *are compact for all* $q_0 \in M$, $t > 0$.

Remark 1

The condition of convexity of the velocity sets $f_U(q)$ is natural: the flow of the ODE

$$\dot{q} = \alpha(t) f_{u_1}(q) + (1 - \alpha(t)) f_{u_2}(q), \qquad 0 \leq \alpha(t) \leq 1,$$

can be approximated by flows of the systems of the form

$$\dot{q} = f_v(q), \quad \text{where} \quad v(t) \in \{u_1(t), u_2(t)\}.$$

Sketch of the proof of Filippov's Theorem: 1/5

- All nonautonomous vector fields $f_u(q)$ with admissible controls u have a common compact support, thus are complete.
- Under hypotheses of the theorem, velocities $f_u(q)$, $q \in M$, $u \in U$, are uniformly bounded, thus all trajectories $q(t)$ of control system (11) starting at q_0 are Lipschitzian with the same Lipschitz constant.
- Embed the manifold M into a Euclidean space $\mathbb{R}^N$, then the space of continuous curves $q(t)$ becomes endowed with the uniform topology of continuous mappings from $[0, t_1]$ to $\mathbb{R}^N$.
- The set of trajectories $q(t)$ of control system (11) starting at q_0 is uniformly bounded:

$$\|q(t)\| \leq C$$

and equicontinous:

$$\forall \varepsilon > 0 \ \exists \delta > 0 \ \forall q(\cdot) \ \forall |t_1 - t_2| < \delta \quad \|q(t_1) - q(t_2)\| < \varepsilon.$$

Sketch of the proof of Filippov's Theorem: 2/5

Theorem 7 (Arzelà–Ascoli)

Consider a family of mappins $\mathcal{F} \subset C([0, t_1], M)$, where M is a complete metric space. If $\mathcal{F}$ is uniformly bounded and equicontinuous, then it is precompact:

$$\forall \{q_n\} \subset \mathcal{F} \; \exists \text{ a converging subsequence } q_{n_k} \to q \in C([0, t_1], M).$$

- Thus the set of admissible trajectories is precompact in the topology of uniform convergence.
- For any sequence $q_n(t)$ of admissible trajectories:

  $$\dot{q}_n(t) = f_{u_n}(q_n(t)), \qquad 0 \le t \le t_1, \quad q_n(0) = q_0,$$

  there exists a uniformly converging subsequence, we denote it again by $q_n(t)$:

  $$q_n(\cdot) \to q(\cdot) \text{ in } C([0, t_1], M) \text{ as } n \to \infty.$$

- Now we show that $q(t)$ is an admissible trajectory of control system (11).

Sketch of the proof of Filippov's Theorem: 3/5

- Fix a sufficiently small $\varepsilon > 0$.
- Then in local coordinates

$$\frac{1}{\varepsilon}(q_n(t+\varepsilon) - q_n(t)) = \frac{1}{\varepsilon}\int_t^{t+\varepsilon} f_{u_n}(q_n(\tau))\,d\tau$$

$$\in \operatorname{conv} \bigcup_{\tau\in[t,t+\varepsilon]} f_U(q_n(\tau)) \subset \operatorname{conv} \bigcup_{q\in O_{q(t)}(c\varepsilon)} f_U(q),$$

where c is the doubled Lipschitz constant of admissible trajectories.

- We pass to the limit $n \to \infty$ and obtain

$$\frac{1}{\varepsilon}(q(t+\varepsilon) - q(t)) \in \operatorname{conv} \bigcup_{q\in O_{q(t)}(c\varepsilon)} f_U(q).$$

- Now let $\varepsilon \to 0$. If t is a point of differentiability of $q(t)$, then

$$\dot{q}(t) \in f_U(q)$$

since $f_U(q)$ is convex.

Sketch of the proof of Filippov's Theorem: 4/5

- In order to show that $q(t)$ is an admissible trajectory of control system (11), we should find a measurable selection $u(t) \in U$ that generates $q(t)$.
- We do this via the lexicographic order on the set $U = \{(u_1, \dots, u_m)\} \subset \mathbb{R}^m$.
- The set

$$V_t = \{v \in U \mid \dot{q}(t) = f_v(q(t))\}$$

  is a compact subset of U, thus of $\mathbb{R}^m$.
- There exists a vector $v^{\min}(t) \in V_t$ minimal in the sense of lexicographic order. To find $v^{\min}(t)$, we minimize the first coordinate on V_t:

$$v_1^{\min} = \min\{ v_1 \mid v = (v_1, \dots, v_m) \in V_t \},$$

  then minimize the second coordinate on the compact set found at the first step:

$$v_2^{\min} = \min\{ v_2 \mid v = (v_1^{\min}, v_2, \dots, v_m) \in V_t \}, \quad \dots,$$
$$v_m^{\min} = \min\{ v_m \mid v = (v_1^{\min}, \dots, v_{m-1}^{\min}, v_m) \in V_t \}.$$

Sketch of the proof of Filippov's Theorem: 5/5

- The control $v^{\min}(t) = (v_1^{\min}(t), \ldots, v_m^{\min}(t))$ is measurable, thus $q(t)$ is an admissible trajectory of system (11) generated by this control.
- The proof of compactness of the attainable set $\mathcal{A}_{q_0}(t)$ is complete.
- Compactness of $\mathcal{A}_{q_0}^t$ is proved similarly. $\square$

Discussion on completeness

- In Filippov's theorem, the hypothesis of common compact support of the vector fields in the right-hand side is essential to ensure the uniform boundedness of velocities and completeness of vector fields.
- On a manifold, sufficient conditions for completeness of a vector field cannot be given in terms of boundedness of the vector field and its derivatives: a constant vector field is not complete on a bounded domain in $\mathbb{R}^n$.
- Nevertheless, one can prove compactness of attainable sets for many systems without the assumption of common compact support. If for such a system we have a priori bounds on solutions, then we can multiply its right-hand side by a cut-off function, and obtain a system with vector fields having compact support.
- We can apply Filippov's theorem to the new system. Since trajectories of the initial and new systems coincide in a domain of interest for us, we obtain a conclusion on compactness of attainable sets for the initial system.

A priori bound in $\mathbb{R}^n$

- For control systems on $M = \mathbb{R}^n$, there exist well-known sufficient conditions for completeness of vector fields.
- If the right-hand side grows at infinity not faster than a linear field, i.e.,

$$|f_u(x)| \ \leq C(1 + |x|), \qquad x \in \mathbb{R}^n, \quad u \in U, \tag{16}$$

  for some constant C, then the nonautonomous vector fields $f_u(x)$ are complete (here $|x| = \sqrt{x_1^2 + \cdots + x_n^2}$ is the norm of a point $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$).
- These conditions provide an a priori bound for solutions: any solution $x(t)$ of the control system

$$\dot{x} = f_u(x), \qquad x \in \mathbb{R}^n, \quad u \in U, \tag{17}$$

  with the right-hand side satisfying (16) admits the bound

$$|x(t)| \leq e^{2Ct} \left(|x(0)| + 1\right), \qquad t \geq 0.$$

Compactness of attainable sets in $\mathbb{R}^n$

- Filippov's theorem plus the previous remark imply the following sufficient condition for compactness of attainable sets for systems in $\mathbb{R}^n$.

Corollary 8

Let system (17) *have a compact space of control parameters* $U \Subset \mathbb{R}^m$ *and convex velocity sets* $f_U(x)$, $x \in \mathbb{R}^n$.
Suppose moreover that the right-hand side of the system satisfies a sublinear bound of the form (16).
Then the attainable sets $\mathcal{A}_{x_0}(t)$ *and* $\mathcal{A}^t_{x_0}$ *are compact for all* $x_0 \in \mathbb{R}^n$, $t > 0$.

Time-optimal problem

- Given a pair of points $q_0 \in M$ and $q_1 \in \mathcal{A}_{q_0}$, the *time-optimal problem* consists in minimizing the time of motion from q_0 to q_1 via admissible controls of control system (11):

$$\min_u \{t_1 \mid q_u(t_1) = q_1\}. \tag{18}$$

- That is, we consider the optimal control problem with the integrand $\varphi(q, u) \equiv 1$ and free terminal time t_1.
- Reduction of optimal control problems to the study of attainable sets and Filippov's Theorem yield the following existence result.

Corollary 9

Under the hypotheses of Filippov's Theorem 6, *time-optimal problem* (11), (18) *has a solution for any points* $q_0 \in M$, $q_1 \in \mathcal{A}_{q_0}$.

Example of a time-optimal problem: Stopping a train

Given:

- material point of mass $m > 0$ with coordinate $x \in \mathbb{R}$
- force F bounded by the absolute value by $F_{\max} > 0$
- initial position x_0 and initial velocity $\dot{x}_0$ of the material point

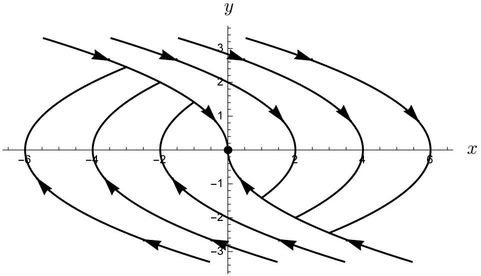
Find:

- force F that steers the point to the origin with zero velocity, for a minimal time.

$$
\begin{aligned}
&\dot{x}_1 = x_2, \qquad (x_1, x_2) \in \mathbb{R}^2,\\
&\dot{x}_2 = u, \qquad |u| \le 1,\\
&(x_1, x_2)(0) = (x_0, \dot{x}_0), \qquad (x_1, x_2)(t_1) = (0, 0),\\
&t_1 \to \min .
\end{aligned}
$$

Example: Stopping a train

- Trajectories of the system with a constant control $u \neq 0$ are the parabolas $\frac{x_2^2}{2} = u x_1 + C$:



- Now it is visually obvious that $(0,0) \in \mathcal{A}_{(x_1,x_2)}$ for any $(x_1, x_2) \in \mathbb{R}^2$.
- The set of control parameters $U = [-1, 1]$ is compact, the set of admissible velocity vectors $f(x, U) = \{(x_2, u) \mid u \in [-1, 1]\}$ is convex for any $x \in \mathbb{R}^2$, and the right-hand side of the control system has sublinear growth: $|f(x, u)| \leq C(|x| + 1)$.
- All hypotheses of the Filippov theorem are satisfied, thus optimal control exists.

Plan of lecture

1. Optimal Control Problem Statement
2. Lebesgue measurable sets and functions
3. Lebesgue integral
4. Carathéodory ODEs
5. Reduction of Optimal Control Problem to Study of Attainable Sets
6. Filippov's theorem: Compactness of Attainable Sets
7. Time-Optimal Problem