Sub-Riemannian structures on Lie groups
(Lecture 8)

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«Geometric control theory, nonholonomic geometry, and their applications»

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7. The Ox Forgotten, Leaving the Man Alone:

Riding on the animal, he is at last back in his home,
Where lo! the ox is no more; the man alone sits serenely.
Though the red sun is high up in the sky, he is still quietly dreaming,
Under a straw-thatched roof are his whip and rope idly lying.

Pu-ming, "The Ten Oxherding Pictures"

Reminder: Plan of the previous lecture

1. Proof of Pontryagin maximum principle for sub-Riemannian problems

Outline of this lecture

1. Sub-Riemannian structures, minimizers, spheres
2. Sub-Riemannian problem on the group of Euclidean plane motions SE(2)
3. Sub-Riemannian problem on the group of hyperbolic plane motions SH(2)

Basic definitions

- a smooth manifold M,
- distribution $\Delta = \{\Delta_q \subset T_q M \mid q \in M\}$, $\dim \Delta_q \equiv \text{const}$,
- scalar product in Δ:

$$g = \{g_q - \text{scalar product in } \Delta_q \mid q \in M\}$$

- SR manifold (M, Δ, g), SR structure (Δ, g) on M
- horizontal (admissible) curve $q \in \text{Lip}([0, t_1], M)$:

$$\dot{q}(t) \in \Delta_{q(t)} \text{ for a.e. } t \in [0, t_1],$$

- length $l(q(\cdot)) = \int_0^{t_1} (g(\dot{q}(t), \dot{q}(t))^{1/2} \, dt$,
- SR distance $d(q_0, q_1) = \inf\{l(q(\cdot)) \mid q(\cdot) \text{ horizontal curve},$

$$q(0) = q_0, \ q(t_1) = q_1\},$$

- SR minimizer $q(t)$, $t \in [0, t_1]$: horizontal curve s.t. $l(q(\cdot)) = d(q(0), q(t_1))$,
- sphere $S_R(q_0) = \{q \in M \mid d(q, q_0) = R\}$,
  ball $B_R(q_0) = \{q \in M \mid d(q, q_0) \leq R\}$,
- geodesic: horizontal curve whose short arcs are minimizers,
- cut time along geodesic $q(t)$:

$$t_{\text{cut}}(q(\cdot)) = \sup\{t > 0 \mid q(s),\ s \in [0, t],\ \text{minimizer }\},$$

- cut point $q(t_1)$, $t_1 = t_{\text{cut}}(q(\cdot))$,
- cut locus

$$\text{Cut}_{q_0} = \{q_1 \in M \mid q_1 \text{ cut point for some geodesic } q(\cdot),\ q(0) = q_0\}$$

- first conjugate time along geodesic $q(t)$:

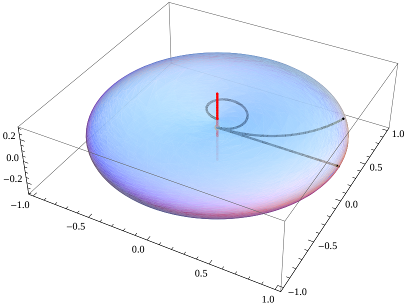
$$t^1_{\text{conj}}(q(\cdot)) = \sup\{t > 0 \mid q(s),\ s \in [0,t],\ \text{locally optimal}\ \},$$

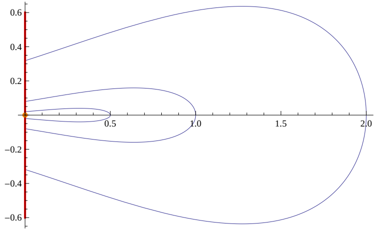
- $q(\cdot)$ locally optimal if $\exists$ a neighborhood of $O \supset \{q(t)\}$ including $q(\cdot)$ is the minimizer on $(O, \Delta|_O, g|_O)$,
- the first conjugate point along the geodesic $q(t)$: $q(t_1)$, $t_1 = t^1_{\text{conj}}(q(\cdot))$,
- the first caustic:

$$\text{Conj}_{q_0} = \{q_1 \in M \mid q_1 \text{ the first conjugate point for some geodesic } q(\cdot),\ q(0) = q_0\}.$$

Example: Heisenberg Group

- $M = \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \mid (a, b, c) \in \mathbb{R}^3 \right\}$
- $X_1 = \frac{\partial}{\partial x} - \frac{y}{2}\frac{\partial}{\partial z}, \quad X_2 = \frac{\partial}{\partial y} + \frac{x}{2}\frac{\partial}{\partial z}, \; x = a, \; y = b, \; z = c - ab/2$
- $\Delta_q = \text{span}(X_1(q), X_2(q)), \quad g(X_i, X_j) = \delta_{ij}$





Optimal control problem

- SR manifold (M, Δ, g)
- Orthonormal frame:

$$\Delta_q = \operatorname{span}(X_1(q), \dots, X_k(q)), \quad g(X_i, X_j) = \delta_{ij}, \; i, k = 1, \dots, k,$$

- Minimizer $q(t)$ — solution of the problem

$$\dot{q} = \sum_{i=1}^{k} u_i X_1(q), \qquad q \in M, \quad u_i \in \mathbb{R},$$

$$q(0) = q_0, \qquad q(t_1) = q_1,$$

$$l = \int_0^{t_1} \left(\sum_{i=1}^{k} u_i^2(t) \right)^{1/2} dt \to \min$$

$$\Leftrightarrow \quad J = \frac{1}{2} \int_0^{t_1} \sum_{i=1}^{k} u_i^2(t)\, dt \to \min .$$

Theorem 1 (Rashevskii-Chow)

Let M be connected and for all $q \in M$

$$\operatorname{span}(X_i(q),\ [X_i, X_j](q),\ [[X_i, X_j], X_l](q),\ \dots) = T_q M. \tag{1}$$

Then for $\forall\ q_0, q_1 \in M\ \exists$ a horizontal curve $q(t)$, $t \in [0, t_1]$, so $q(0) = q_0$, $q(t_1) = q_1$.

Further, the condition of full rank (1) is assumed to be satisfied.

Theorem 2 (Filippov)

A minimizer connecting points $q_0, q_1 \in M$ exists if one of the following conditions is satisfied:

- *q_1 is sufficiently close to q_0,*
- *the balls $B_R(q_0)$ are compact,*
- *(Δ, g) is left-invariant on the Lie group M.*

Pontryagin's Maximum Principle

- $h_i(\lambda) = \langle \lambda, X_i(q) \rangle$, $\lambda \in T^*M$.

Theorem 3 (Pontryagin)

If $q(t)$, $t \in [0, t_1]$, is a length miminizer corresponding to control $u(t)$, then $\exists \lambda \in \mathrm{Lip}([0, t_1], T^*M)$, $\lambda(t) \in T^*_{q(t)}M$, s.t.:

(N) *either* $\dot{\lambda}(t) = \vec{H}(\lambda(t))$, $H(\lambda) = \frac{1}{2}\sum_{i=1}^{k} h_i^2(\lambda)$, $u_i(t) = h_i(\lambda(t))$,

(A) *or* $h_1(\lambda(t)) = \cdots = h_k(\lambda(t)) \equiv 0$, $\dot{\lambda}(t) = \sum_{i=1}^{k} u_i(t)\vec{h}_i(\lambda(t))$.

(N) $\Rightarrow$ $\lambda(t)$ is a normal extremal, $q(t)$ is a normal extremal trajectory,

(A) $\Rightarrow$ $\lambda(t)$ is an abnormal extremal, $q(t)$ is an abnormal extremal trajectory.

Optimality of normal geodesics

- $q(t)$ — normal extremal trajectory $\Rightarrow$
  $q(t)$ — geodesic (strong Legendre condition)

- $\lambda(t)$ — normal extremal $\Rightarrow$
  $\lambda(t) = e^{t\vec{H}}(\lambda_0), \quad H(\lambda(t)) \equiv \text{const}$

- $\lambda_0 \in C = \{H(\lambda) \equiv 1/2\} \cap T^*_{q_0}M$

- Exponential map $\text{Exp} : C \times \mathbb{R}_+ \to M$, $\text{Exp}(\lambda, t) = q(t) = \pi \circ e^{t\vec{H}}(\lambda)$.

- q_1 is a Maxwell point on $q(t)$:
  $\exists \widetilde{q}(t) \not\equiv q(t)$, $\widetilde{q}(0) = q(0)$, $\widetilde{q}(t_1) = q(t_1) = q_1$.

Theorem 4

Let $q(t) = \text{Exp}(\lambda, t)$ be a normal geodesic that does not contain abnormal arcs. If t_1 is the cut time, then $q(t_1)$ is the first Maxwell point or the first conjugate point.

Smoothness of Spheres

Theorem 5

If $\Delta_{q_0} \neq T_{q_0}M$, then any sphere $S_R(q_0)$ is not a smooth manifold (if $S_R(q_0) \neq \emptyset$).

Theorem 6

Let $q_1 \in S_R(q_0)$. Suppose that:

(1) *q_1 is connected to q_0 by a unique normal minimizer $q(t)$,*

(2) *q_1 is not a conjugate point along $q(t)$.*

Then $S_R(q_0)$ is a smooth manifold in the neighborhood of q_1.

Corollary 1

Reasons for cut points and singularities of spheres:

(1) *abnormal shortest paths,*

(2) *Maxwell points,*

(3) *conjugate points.*

Group of Euclidean plane motions

$$\mathrm{SE}(2) = \left\{ \begin{pmatrix} \cos\theta & -\sin\theta & x \\ \sin\theta & \cos\theta & y \\ 0 & 0 & 1 \end{pmatrix} \mid (x, y) \in \mathbb{R}^2,\ \theta \in S^1 \right\}$$

$$X_1(q) = \cos\theta \frac{\partial}{\partial x} + \sin\theta \frac{\partial}{\partial y}, \qquad X_2(q) = \frac{\partial}{\partial \theta}.$$

$$M = \mathrm{SE}(2), \qquad \Delta = \mathrm{span}(X_1, X_2), \qquad g(X_i, X_j) = \delta_{ij}.$$

$$\dot q = u_1 X_1 + u_2 X_2, \qquad q = (x, y, \theta) \in \mathrm{SE}(2), \quad (u_1, u_2) \in \mathbb{R}^2,$$

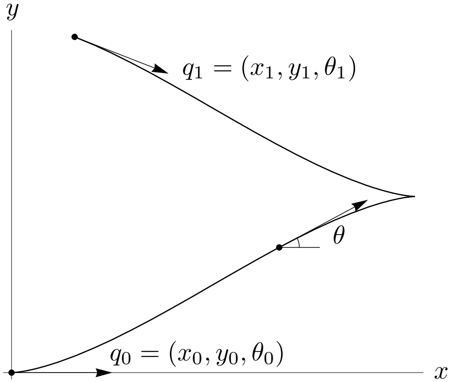
$$q(0) = q_0 = \mathrm{Id} = (0, 0, 0), \qquad q(t_1) = q_1,$$

$$I = \int_0^{t_1} \sqrt{\dot x^2 + \dot y^2 + \dot\theta^2}\, dt = \int_0^{t_1} \sqrt{u_1^2 + u_2^2}\, dt \to \min.$$

Contact sub-Riemannian structure on SE(2)

- $X_3 = [X_1, X_2] = \sin\theta \frac{\partial}{\partial x} - \cos\theta \frac{\partial}{\partial y}$
- $\mathrm{span}(X_1(q), X_2(q), X_3(q)) = T_q M \quad \Rightarrow \quad$ complete controllability
- Growth vector $(2, 3) \quad \Rightarrow \quad$ contact distribution
- A.A. Agrachev's invariants: $\kappa = \chi$
- The only left-invariant contact sub-Riemannian structure on SE(2), up to dilations and local isometries

Problem on optimal motion of a mobile robot on a plane



$$I = \int_0^{t_1} \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{\theta}^2}\, dt \to \min$$

Pontryagin's maximum principle

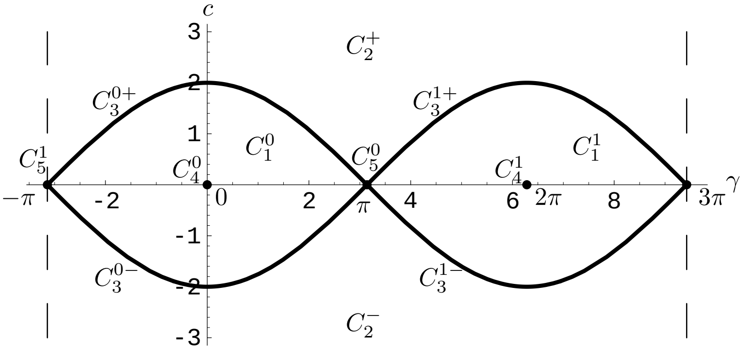
- Abnormal extremal trajectories are constant.
- Normal extremals:

$$\dot{\gamma} = c, \quad \dot{c} = -\sin\gamma, \qquad (\gamma, c) \in C \cong (2S^1_\gamma) \times \mathbb{R}_c,$$
$$\dot{x} = \sin\frac{\gamma}{2}\cos\theta, \quad \dot{y} = \sin\frac{\gamma}{2}\sin\theta, \quad \dot{\theta} = -\cos\frac{\gamma}{2}.$$

- The energy integral $E = \frac{c^2}{2} - \cos\gamma \in [-1, +\infty)$
- $\gamma(t)$, $c(t)$, $q(t)$: parameterization by Jacobi functions sn, cn, dn, E.

Partition of the pendulum phase cylinder $C = \cup_{i=1}^{5} C_i$

- $C_1 = \{\lambda \in C \mid E \in (-1, 1)\}$ $\Rightarrow$ pendulum oscillations,
- $C_2 = \{\lambda \in C \mid E \in (1, +\infty)\}$ $\Rightarrow$ pendulum rotations,
- $C_3 = \{\lambda \in C \mid E = 1,\ c \neq 0\}$ $\Rightarrow$ critical motion,
- $C_4 = \{\lambda \in C \mid E = -1\}$ $\Rightarrow$ stable equilibrium,
- $C_5 = \{\lambda \in C \mid E = 1,\ c = 0\}$ $\Rightarrow$ unstable equilibrium.

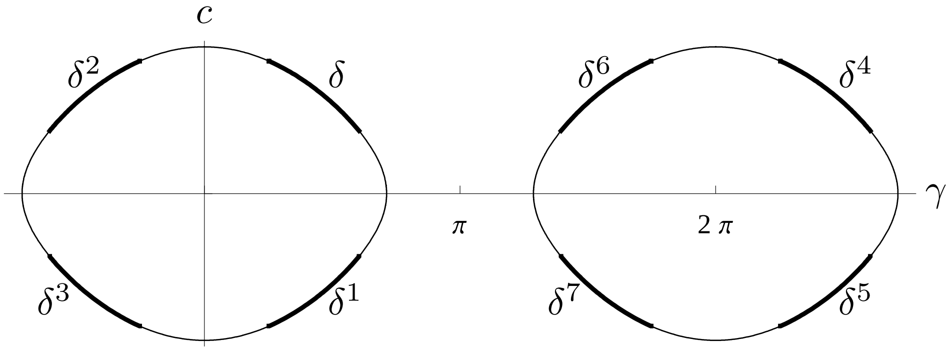


Reflections of ε^i in the phase cylinder of a pendulum $\ddot{\gamma} = -\sin\gamma$

- $\varepsilon^i : C \to C, \quad \varepsilon^i_* \vec{H}_v = \pm \vec{H}_v, \quad \vec{H}_v = c\frac{\partial}{\partial\gamma} - \sin\gamma\frac{\partial}{\partial c} \in \operatorname{Vec} C,$
- Symmetry group of a parallelepiped

$$G = \{\mathrm{Id}, \varepsilon^1, \dots, \varepsilon^7\} = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2.$$

- Action of reflections $\varepsilon^i : \delta \mapsto \delta^i$ on the pendulum trajectory:



First Maxwell time corresponding to symmetries

Symmetries of the exponential map:

$$\mathrm{Exp} \circ \varepsilon^i(\lambda, t) = \varepsilon^i \circ \mathrm{Exp}(\lambda, t), \qquad (\lambda, t) \in C \times \mathbb{R}_+, \quad \varepsilon^i \in G.$$

$$t_{\mathrm{Max}}(\lambda) = \min\{t > 0 \mid \exists \varepsilon^i \in G \ : \ \varepsilon^i(\lambda, t) \neq (\lambda, t), \quad \mathrm{Exp} \circ \varepsilon^i(\lambda, t) = \mathrm{Exp}(\lambda, t)\}$$

Theorem 7

- $E = -1 \quad \Rightarrow \quad t_{\mathrm{Max}}(\lambda) = \pi,$
- $E \in (-1, 1) \quad \Rightarrow \quad t_{\mathrm{Max}}(\lambda) = 2K(k), \ k = \sqrt{(E+1)/2},$
- $E = 1 \quad \Rightarrow \quad t_{\mathrm{Max}}(\lambda) = +\infty,$
- $E > 1 \quad \Rightarrow \quad t_{\mathrm{Max}}(\lambda) = 2kp_1(k), \ k = \sqrt{2/(E+1)},$

$p_1(k) = \min\{p > 0 \mid \mathrm{cn}(p, k)) - p) - \mathrm{dn}(p, k)\,\mathrm{sn}(p, k) = 0\}.$

Estimates of the first conjugate time

Theorem 8

- $E \in [-1, 1] \quad \Rightarrow \quad t^1_{\text{conj}}(\lambda) = +\infty,$
- $E > 1 \quad \Rightarrow \quad t^1_{\text{conj}}(\lambda) \in [t_{\text{Max}}(\lambda), 4kK],$
- $\forall \lambda \in C \quad t^1_{\text{conj}}(\lambda) \geq t_{\text{Max}}(\lambda).$

Proof method:
Homotopy invariance of the Maslov index (number of conjugate points)

Global structure of the exponential map

- $\mathrm{Exp} : C \times \mathbb{R}_+ = N \to M$: non-opt. geodes. for $t > t_{\mathrm{Max}}(\lambda)$,
- $\widehat{N} = \{(\lambda, t) \in C \times \mathbb{R}_+ \mid t \leq t_{\mathrm{Max}}(\lambda)\}$, $\widehat{M} = M \setminus \{q_0\}$,
  $\mathrm{Exp} : \widehat{N} \to \widehat{M}$ surjective, not injective (Maxwell points),
- $\widetilde{M} = \{q \in M \mid \varepsilon^i(q) \neq q\} =$
  $= \{q \in M \mid \sin\theta \neq 0,\ R_i(q) \neq 0\} = \cup_{i=1}^{8} M_i$,
  $\widetilde{N} = \mathrm{Exp}^{-1}(\widetilde{M}) =$
  $= \{(\lambda, t) \in N \mid t < t_{\mathrm{Max}}(\lambda),\ \sin(\gamma_{t/2}/2) \neq 0\} = \cup_{i=1}^{8} D_i$,
  $\mathrm{Exp} : \widetilde{N} \to \widetilde{M}$: neither Maxwell point nor conjugate points.

Theorem 9

$\mathrm{Exp} : D_i \to M_i$ — *diffeomorphism*, $i = 1, \ldots, 8$.
$\mathrm{Exp} : \widetilde{N} \to \widetilde{M}$ — *diffeomorphism.*

Diffeomorphic stratifications and the cut set

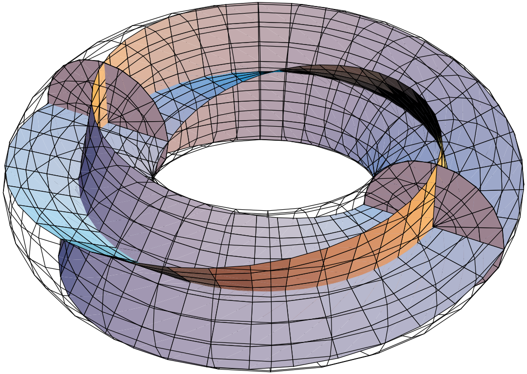
- $\mathrm{Cut}, \mathrm{Max} \subset M' = \widehat{M} \setminus \widetilde{M} = \{q \in M \mid \sin\theta R_1(q) R_2(q) = 0\}$,
- $N' = \widehat{N} \setminus \widetilde{N}$,
- $\mathrm{Exp} \, : \, N' \to M'$,
- Stratifications: $N' = \sqcup_{i=1}^{58} N'_i, \; M' = \cup_{i=1}^{58} M'_i$,
- $\mathrm{Exp} \, : \, N'_i \to M'_i$ — diffeomorphism, $i = 1, \dots, 58$
- $\mathrm{Max} = \cup\{M'_i \mid \exists M'_j = M'_i, \; j \neq i\}$,
- $\mathrm{Cut} = \mathrm{Max} \cup (\; Cut \cap \mathrm{Conj})$,
- $\mathrm{Cut} = \mathrm{Cut}_{\mathrm{loc}} \cup \mathrm{Cut}_{\mathrm{glob}}$,
- $\mathrm{Cut}_{\mathrm{glob}} = \{q \in M \mid \theta = \pi\}, \; d(q_0, \mathrm{Cut}_{\mathrm{glob}}) = \pi$,
- $\mathrm{Cut}_{\mathrm{loc}} \subset \{R_2 = 0\}, \; \mathrm{cl}(\mathrm{Cut}_{\mathrm{loc}}) \ni q_0$,
- $\mathrm{Cut}_{\mathrm{loc}} = \{q \in M \mid \theta \in (-\pi, \pi), \;\; R_2 = 0, \;\; |R_1| > R_1^1(|\theta|)\}$,

  $R_1 = y \cos\frac{\theta}{2} - x \sin\frac{\theta}{2}, \qquad R_2 = x \cos\frac{\theta}{2} + x \sin\frac{\theta}{2}$,

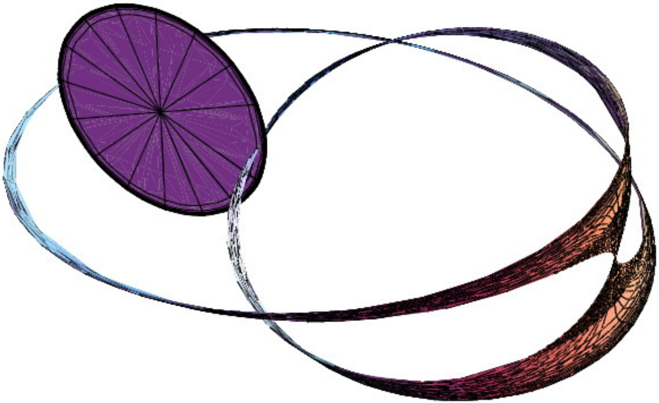
  $R_1^1(\theta) = 2(p_1(k) - \mathrm{E}(p_1(k), k))$,

  $k = k_1(\theta)$ — inverse function to $\theta = k \, \mathrm{sn}(p_1(k), k)$.

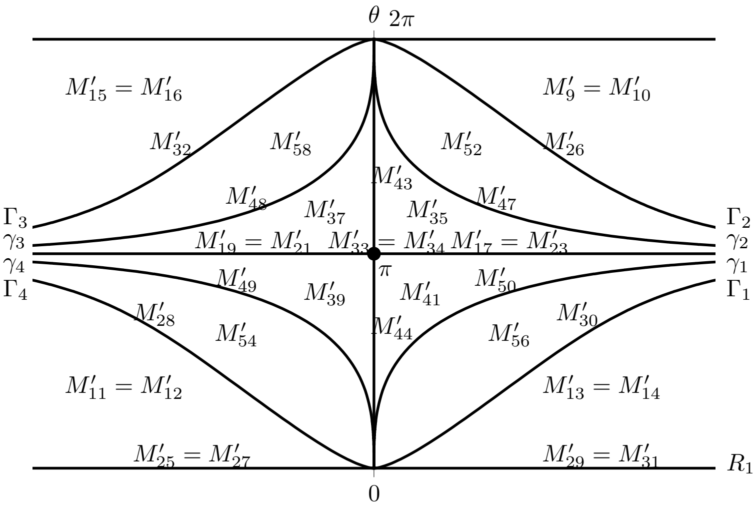
Set $M' \supset$ Cut $\supset$ Max



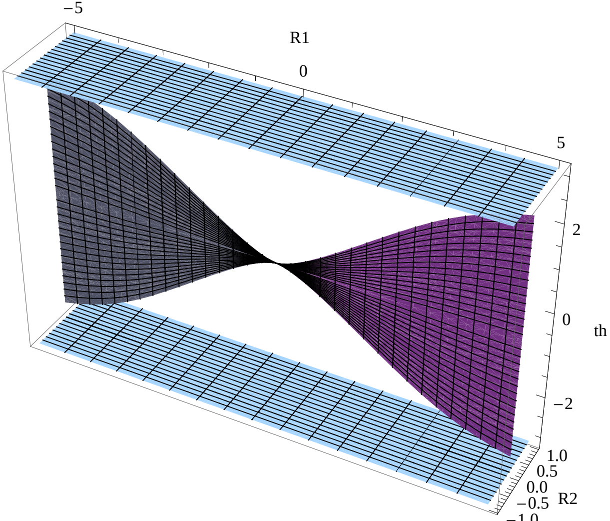
Cut locus: global location



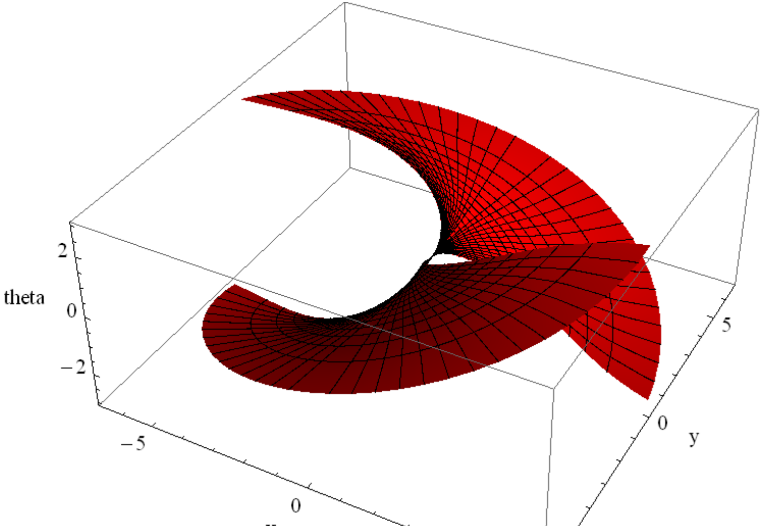
Stratification of the Mobius strip $R_2(q) = 0$



Cut locus
in straightening coordinates (R_1, R_2, θ)



Local component of cut locus in original coordinates (x, y, θ)



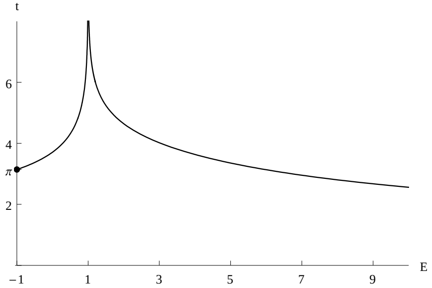
Optimal synthesis in a problem $q(0) = q_1$, $q(t_1) = q_0 = (0, 0, 0)$

- $q_1 \in \widehat{M} = M \setminus \{q_0\}$
- $\mathrm{Exp} : \widehat{N} \to \widehat{M}$ surjective
- $\mathrm{Exp}^{-1}(q) = \begin{cases} \{(\lambda, t)\}, & \text{if } q \in \widehat{M} \setminus \mathrm{Max}, \\ \{(\lambda', t) \neq (\lambda'', t)\}, & \text{if } q \in \mathrm{Max} \end{cases}$
- $\mathrm{Exp}^{-1}(q_1) = (\lambda, t), \qquad \lambda = (\gamma, c) \in (2S^1) \times \mathbb{R}, \ t > 0$
- $\ddot{\gamma}_s = -\sin\gamma_s, \ (\gamma_0, \dot{\gamma}_0) = (\gamma, c), \ s \in [0, t]$
- $u_1(q_1) = -\sin(\gamma_t/2), \ u_2(q_1) = \cos(\gamma_t/2)$
- the optimal synthesis $q_1 \mapsto (u_1, u_2)$ is two-valued on Max, single-valued on $\widehat{M} \setminus \mathrm{Max}$.

Cut time

Theorem 10

- $t_{\mathrm{cut}}(\lambda) = t_{\mathrm{Max}}(\lambda), \quad \lambda \in C,$
- $t_{\mathrm{cut}} \circ \varepsilon^i = t_{\mathrm{cut}}, \quad \varepsilon^i \in G,$
- $\vec{H}_v t_{\mathrm{cut}} = 0,$
- $t_{\mathrm{cut}} \; : \; C \to (0, +\infty]$ *is continuous,* $\quad t_{\mathrm{cut}}|_{E \neq \pm 1}$ *is smooth.*



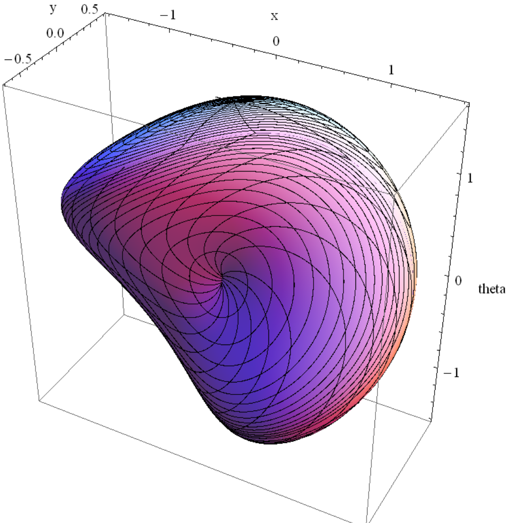
Sub-Riemannian spheres

- $R \in (0, \pi) \quad \Rightarrow \quad S_R \cong S^2$,
- $R = \pi \quad \Rightarrow \quad S_R \cong S^2/\{N = S\}$,
- $R > \pi \quad \Rightarrow \quad S_R \cong T^2$.

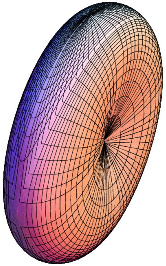
Singularities of spheres:

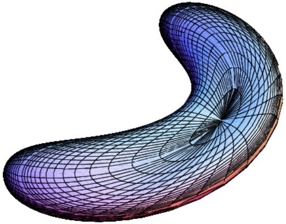
$$S_R \cap \mathrm{Cut} = (S_R \cap \mathrm{Max}) \cup (S_R \cap \mathrm{Cut} \cap \mathrm{Conj}).$$

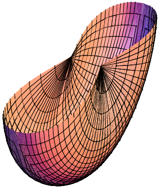
Sub-Riemannian sphere of radius $< \pi$ in the original coordinates (x, y, θ)

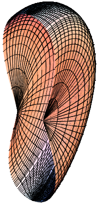


Sub-Riemannian sphere of radius $< \pi$ in rectifying coordinates (R_1, R_2, θ)

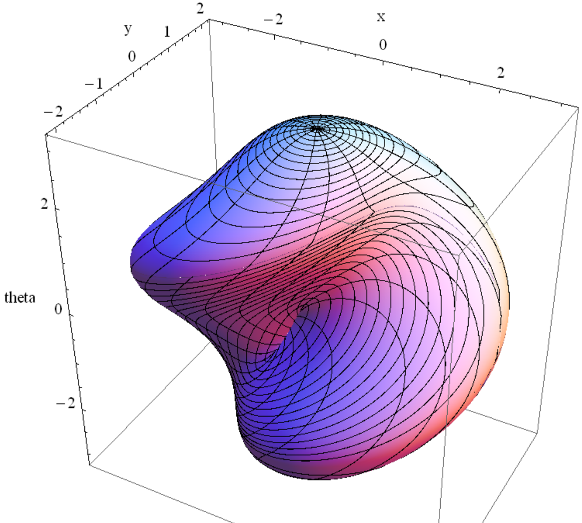




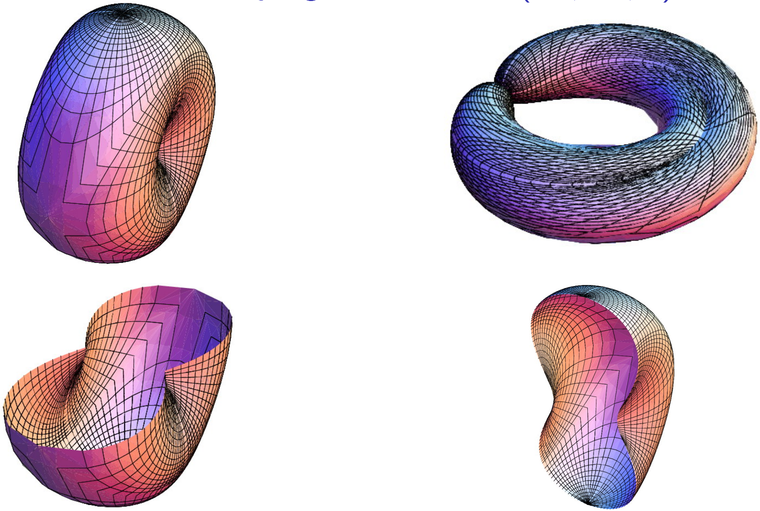




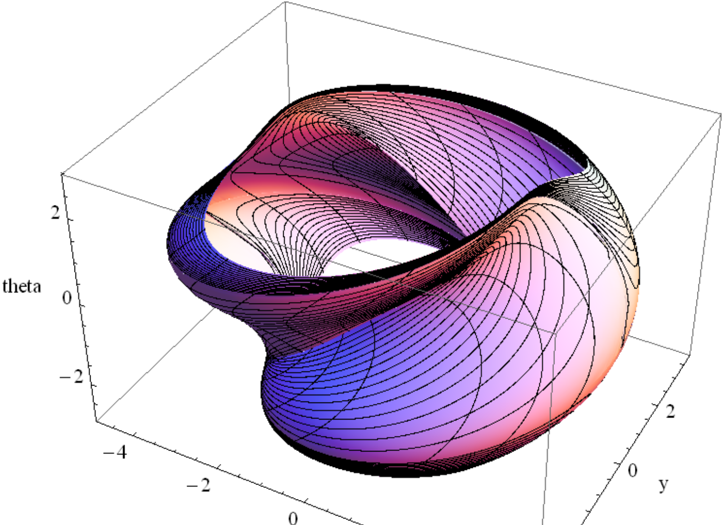
Sub-Riemannian sphere of radius π in original coordinates (x, y, θ)



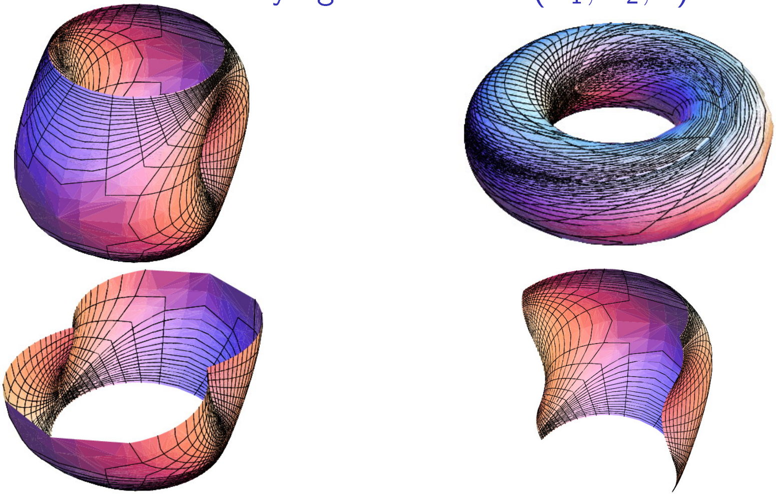
Sub-Riemannian sphere of radius π in rectifying coordinates (R_1, R_2, θ)



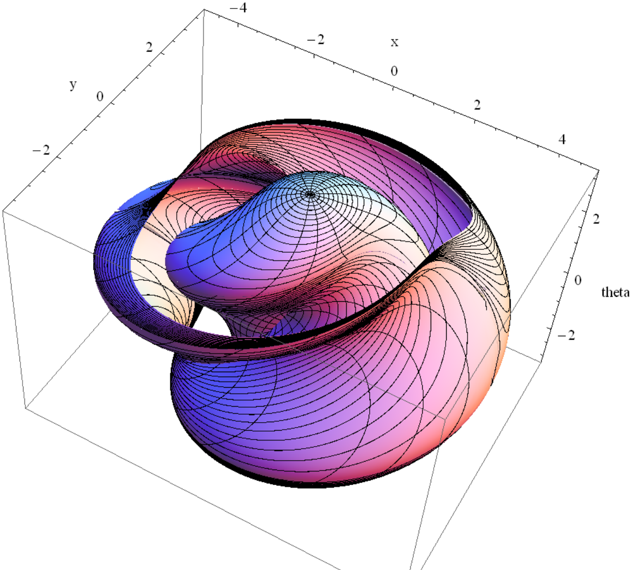
Sub-Riemannian sphere of radius $> \pi$ in original coordinates (x, y, θ)



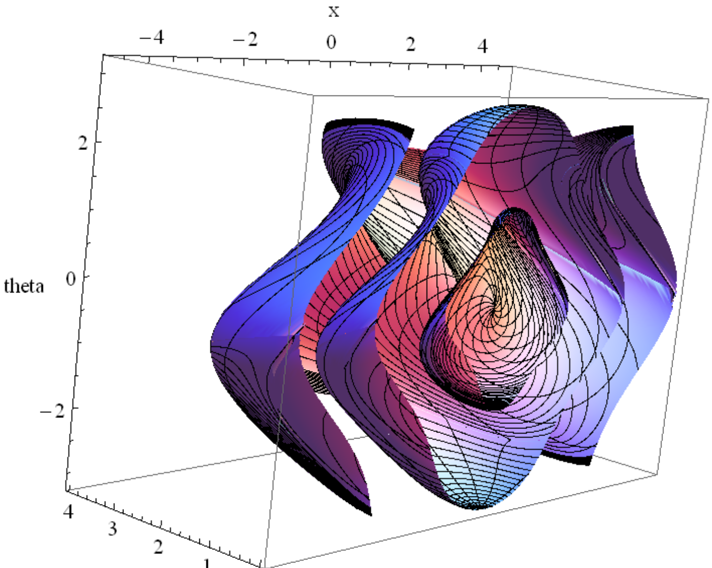
Sub-Riemannian sphere of radius $> \pi$ in rectifying coordinates (R_1, R_2, θ)



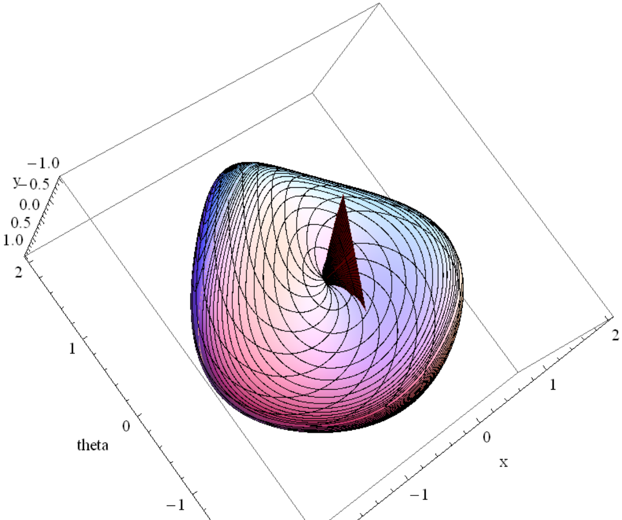
Matryoshka of radius spheres π and $> \pi$



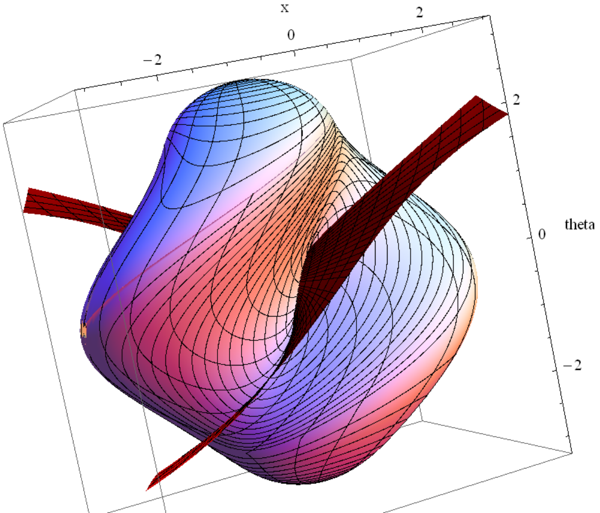
Matryoshka of hemispheres of radius $< \pi$, π and $> \pi$



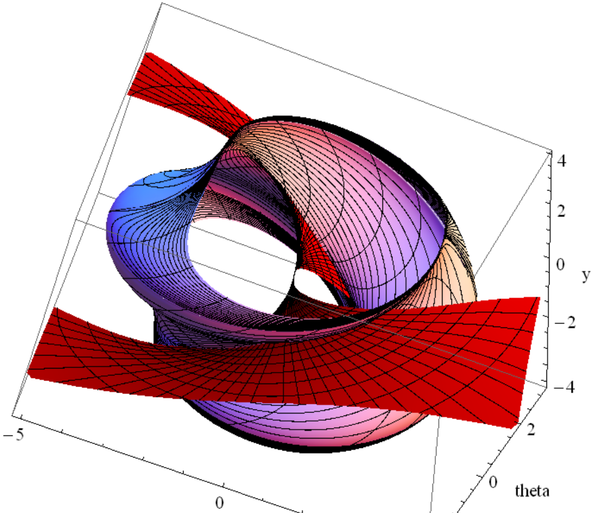
Sub-Riemannian sphere of radius $< \pi$ and the set locus



Sub-Riemannian sphere of radius π and cut locus



Sub-Riemannian sphere of radius $> \pi$ and the cut locus



Group of hyperbolic motions of the plane: Statement of the problem

$$\mathrm{SH}(2) = \left\{ \begin{pmatrix} \operatorname{ch} z & \operatorname{sh} z & x \\ \operatorname{sh} z & \operatorname{ch} z & y \\ 0 & 0 & 1 \end{pmatrix} \mid (x, y, z) \in \mathbb{R}^3 \right\}$$

Left-invariant frame:

$$X_1(q) = \operatorname{ch} z \frac{\partial}{\partial x} + \operatorname{sh} z \frac{\partial}{\partial y}, \qquad X_2(q) = \frac{\partial}{\partial z}.$$

$$M = \mathrm{SH}(2), \qquad \Delta = \operatorname{span}(X_1, X_2), \qquad g(X_i, X_j) = \delta_{ij}.$$

$$\dot q = u_1 X_1 + u_2 X_2, \qquad q = (x, y, \theta) \in \mathrm{SH}(2), \quad (u_1, u_2) \in \mathbb{R}^2,$$

$$q(0) = q_0 = \mathrm{Id} = (0, 0, 0), \qquad q(t_1) = q_1,$$

$$l = \int_0^{t_1} \sqrt{\dot x^2 + \dot y^2 + \dot\theta^2}\, dt = \int_0^{t_1} \sqrt{u_1^2 + u_2^2}\, dt \to \min.$$

Contact sub-Riemannian structure on SH(2)

- $X_3 = [X_1, X_2] = -\operatorname{sh} z \frac{\partial}{\partial x} - \operatorname{ch} z \frac{\partial}{\partial y}$
- $\operatorname{span}(X_1(q), X_2(q), X_3(q)) = T_q M \quad \Rightarrow \quad$ complete controllability
- Growth vector $(2, 3) \quad \Rightarrow \quad$ contact distribution
- A.A. Agrachev invariants: $\kappa = -\chi$
- The only left-invariant contact sub-Riemannian structure on SH(2), up to dilations and local isometries

Pontryagin's maximum principle

- Anormal extremal trajectories are constant.
- Normal extremals:

$$\dot{\gamma} = c, \quad \dot{c} = -\sin\gamma, \qquad (\gamma, c) \in C \cong (2S^1_\gamma) \times \mathbb{R}_c,$$
$$\dot{x} = \cos\frac{\gamma}{2}\,\mathrm{ch}\,z, \quad \dot{y} = \cos\frac{\gamma}{2}\,\mathrm{sh}\,z, \quad \dot{z} = \sin\frac{\gamma}{2}.$$

- $\gamma(t)$, $c(t)$, $q(t)$: parameterization by Jacobi functions sn, cn, dn, E
- Symmetry group of Exp:

$$G = \{\mathrm{Id}, \varepsilon^1, \dots, \varepsilon^7\}.$$

First Maxwell Time and Conjugate Times

Theorem 11

- $E = -1 \quad \Rightarrow \quad t_{\text{Max}}(\lambda) = 2\pi$,
- $E \in (-1, 1) \quad \Rightarrow \quad t_{\text{Max}}(\lambda) = 4K(k),\ k = \sqrt{(E+1)/2}$,
- $E = 1 \quad \Rightarrow \quad t_{\text{Max}}(\lambda) = +\infty$,
- $E > 1 \quad \Rightarrow \quad t_{\text{Max}}(\lambda) = 4kK(k),\ k = \sqrt{2/(E+1)}$.

Theorem 12

- $t^n_{\text{Max}}(\lambda) \leq t^n_{\mathrm{conj}}(\lambda) \leq t^{n+1}_{\text{Max}}(\lambda)$ *for any* $\lambda \in C$, $n \in \mathbb{N}$.
- *A generalization of Rolle's theorem is valid: between successive Maxwell points there is one conjugate point.*

Global structure of the exponential map

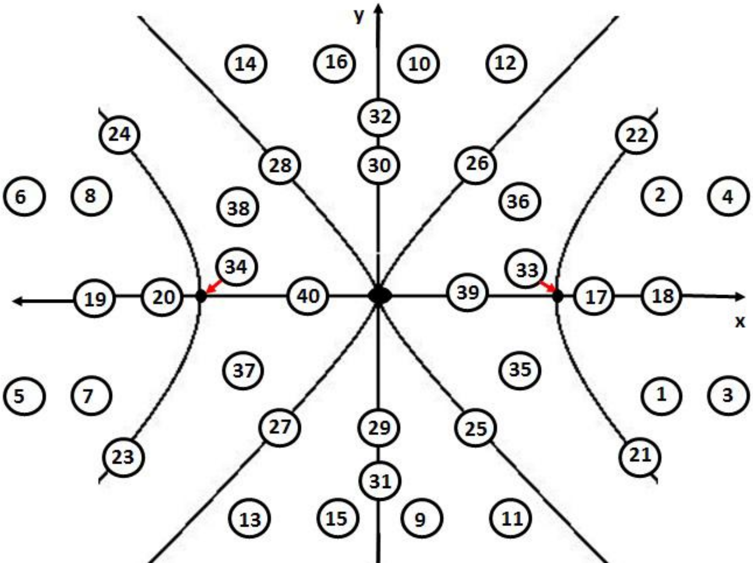
Diffeomorphic stratifications in the preimage and image of Exp:

- $\widehat{N} = \{(\lambda, t) \in C \times \mathbb{R}_+ \mid t \leq t_{\text{Max}}(\lambda)\} =$
  $= \cup_{i=1}^{2} D_i \cup (\cup_{i=1}^{40} N_i'),$
- $\widehat{M} = M \setminus \{q_0\} =$
  $= \cup_{i=1}^{2} M_i \cup (\cup_{i=1}^{40} M_i').$

Theorem 13

- $\text{Exp} : D_i \to M_i$ — *diffeomorphism,* $i = 1, 2.$
- $\text{Exp} : N_i' \to M_i'$ — *diffeomorphism,* $i = 1, \dots, 40.$

Stratification of plane $M' = \{z = 0\}$

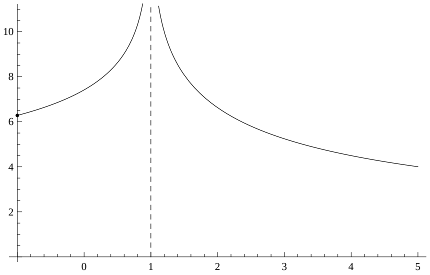


Cut time

Theorem 14

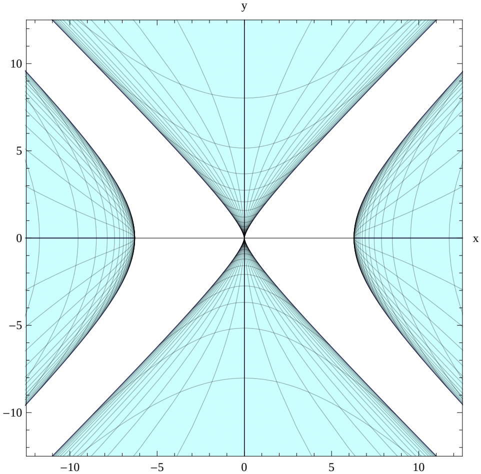
- $t_{\mathrm{cut}}(\lambda) = t_{\mathrm{Max}}(\lambda)$, $\lambda \in C$,
- $t_{\mathrm{cut}} \circ \varepsilon^i = t_{\mathrm{cut}}$, $\varepsilon^i \in G$,
- $\vec{H}_v t_{\mathrm{cut}} = 0$,
- $t_{\mathrm{cut}} \; : \; C \to (0, +\infty]$ *is continuous,* $t_{\mathrm{cut}}|_{E \neq \pm 1}$ *is smooth*

Graph of the function $t_{\mathrm{cut}} = t_{\mathrm{cut}}(E)$:



Cut locus

- $Cut = \mathrm{Max} \cup (\mathrm{Cut} \cap \mathrm{Conj}) = \mathrm{Cut}_{\mathrm{loc}} \cup \mathrm{Cut}_{\mathrm{glob}}$,
- $\mathrm{cl}(\mathrm{Cut}_{\mathrm{loc}}) \ni q_0, \ d(q_0, \mathrm{Cut}_{\mathrm{glob}}) = 2\pi$,
- $\mathrm{Cut} \subset \{z = 0\}$.

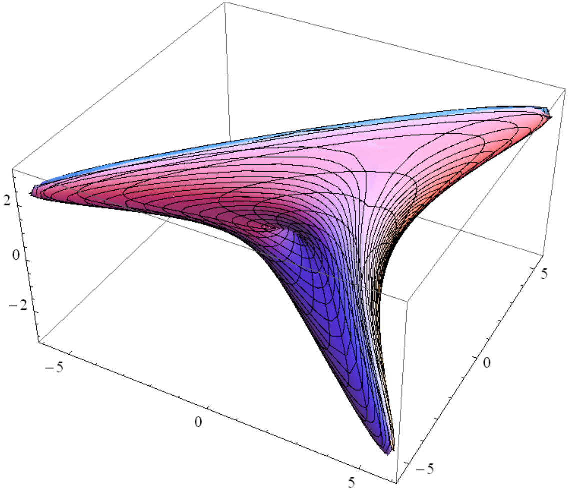


Sub-Riemannian spheres

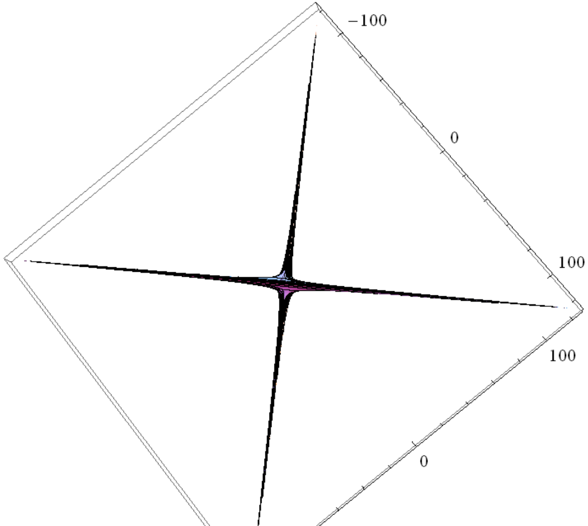
- $R > 0 \quad \Rightarrow \quad S_R \cong S^2,$
- Singularities of spheres:

$$S_R \cap \mathrm{Cut} = (S_R \cap \mathrm{Max}) \cup (S_R \cap \mathrm{Cut} \cap \mathrm{Conj}).$$

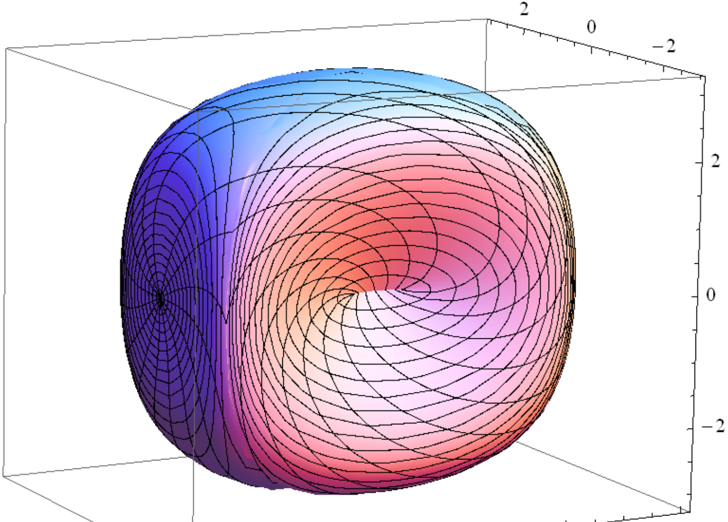
Sub-Riemannian sphere of radius $< 2\pi$ in the original coordinates (x, y, z)



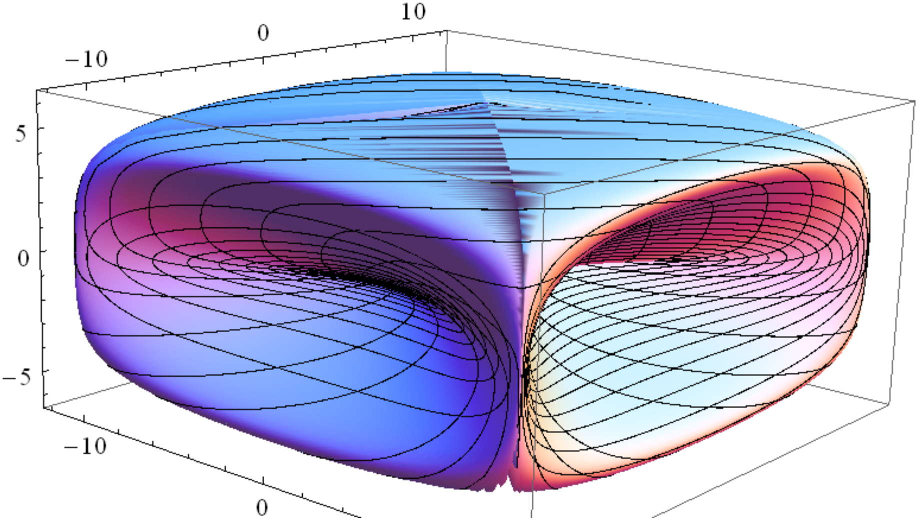
Sub-Riemannian sphere of radius 2π in original coordinates (x, y, z)



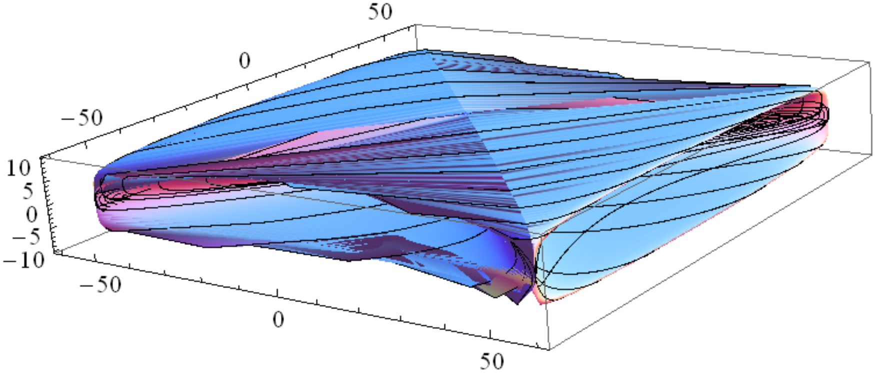
Sub-Riemannian sphere of radius $< 2\pi$ in rectifying coordinates (R_1, R_2, z)



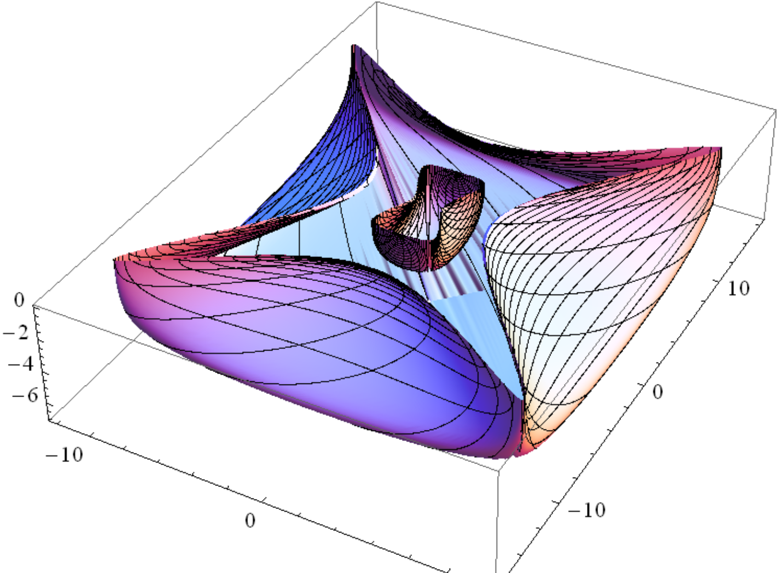
Sub-Riemannian sphere of radius 2π in rectifying coordinates (R_1, R_2, z)



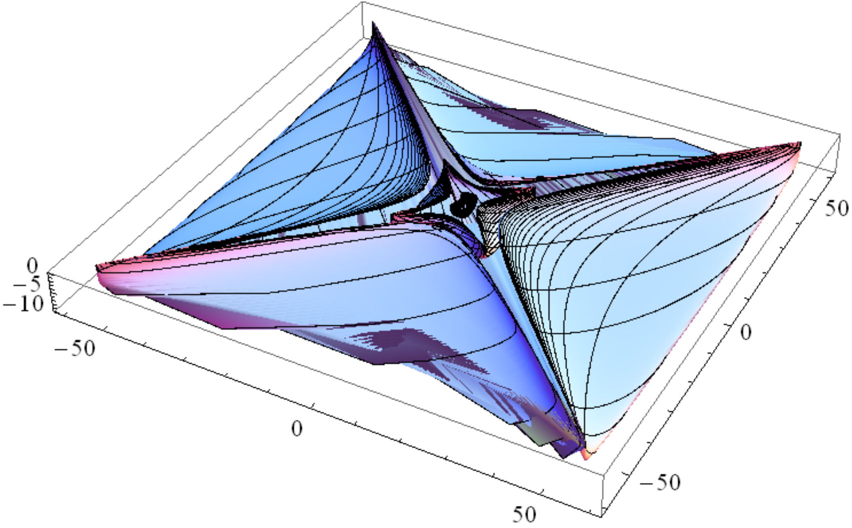
Sub-Riemannian sphere of radius $> 2\pi$ in rectifying coordinates (R_1, R_2, z)



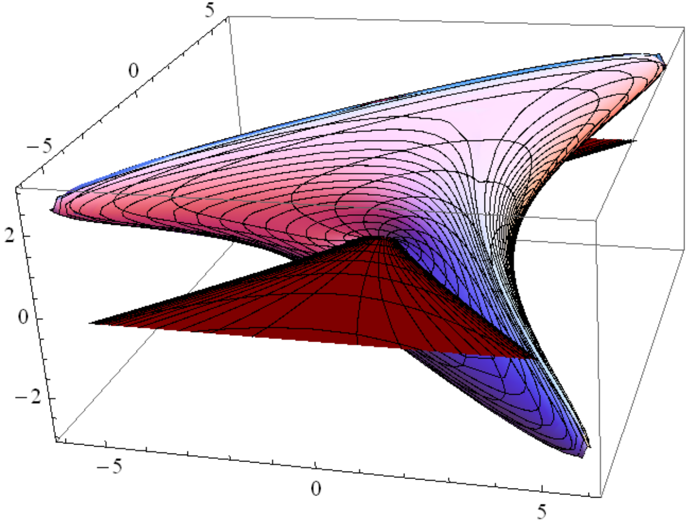
Matryoshka hemispheres of radius π and 2π



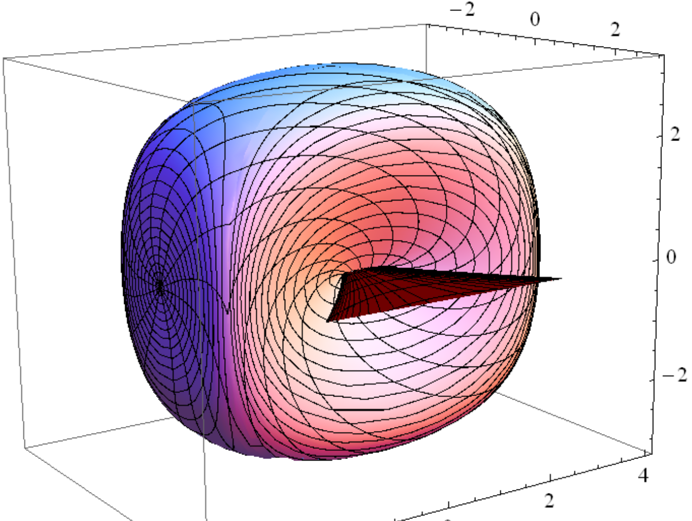
Matryoshka of hemispheres of radii π, 2π and 3π



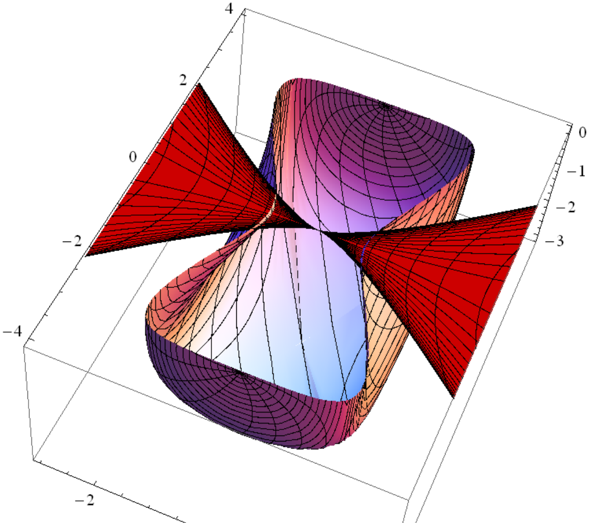
Sphere of radius π and cut locus in original coordinates (x, y, z)



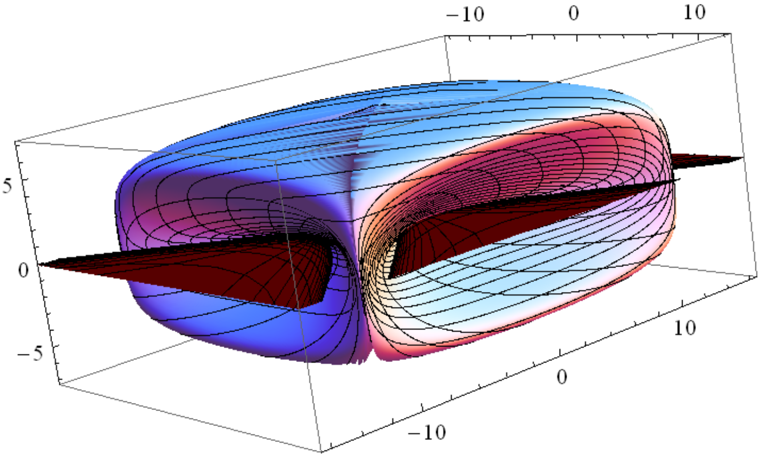
Sphere of radius π and the cut locus in rectifying coordinates (R_1, R_2, z)



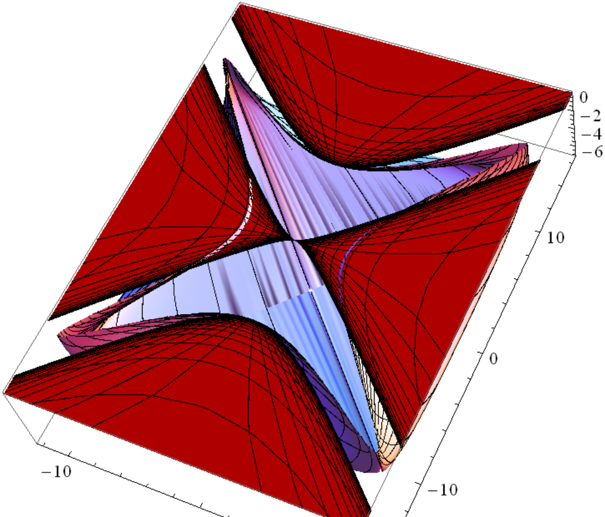
A half-sphere of radius π and the cut locus in straightening coordinates (R_1, R_2, z)



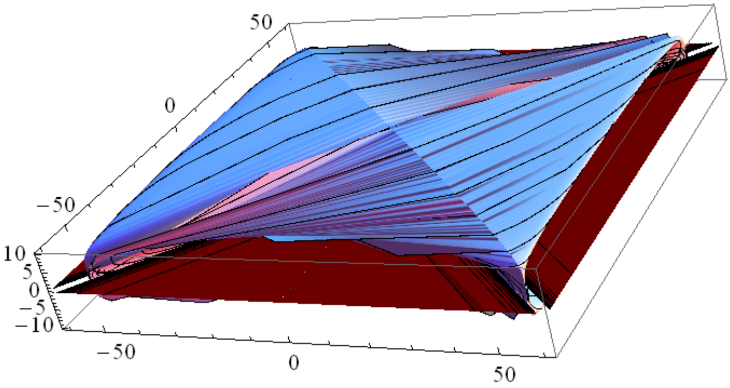
A sphere of radius 2π and the cut locus in straightening coordinates (R_1, R_2, z)



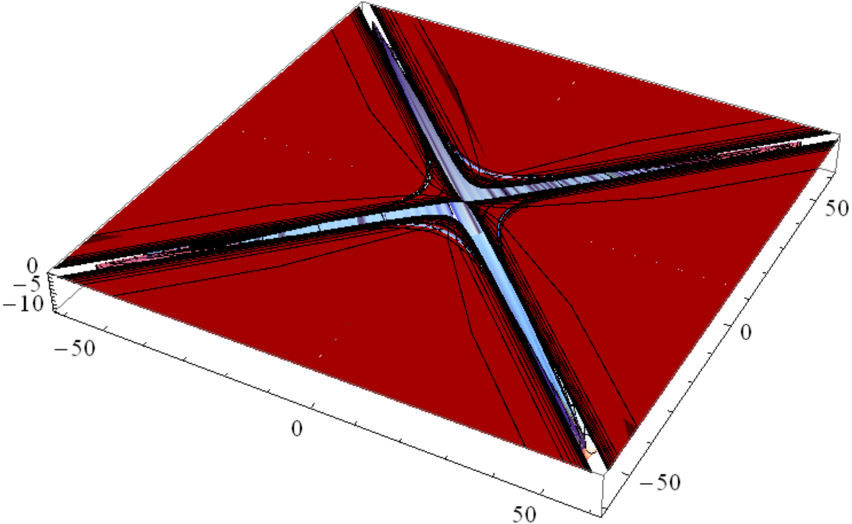
A half-sphere of radius 2π and the cut locus in straightening coordinates (R_1, R_2, z)



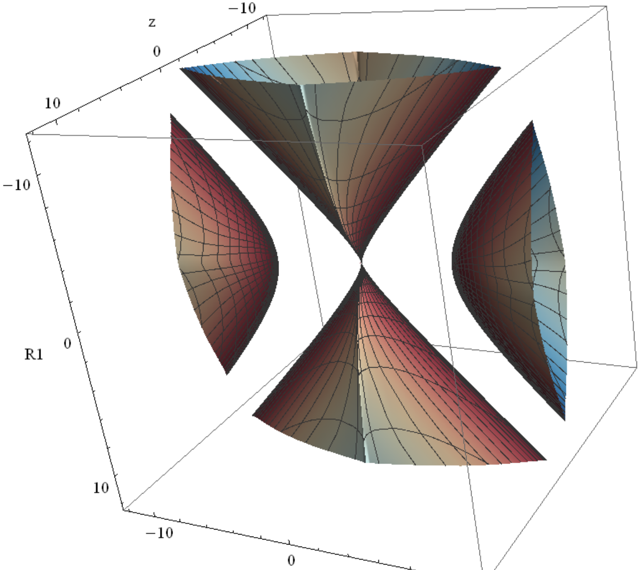
Sphere of radius 3π and the cut locus in straightening coordinates (R_1, R_2, z)



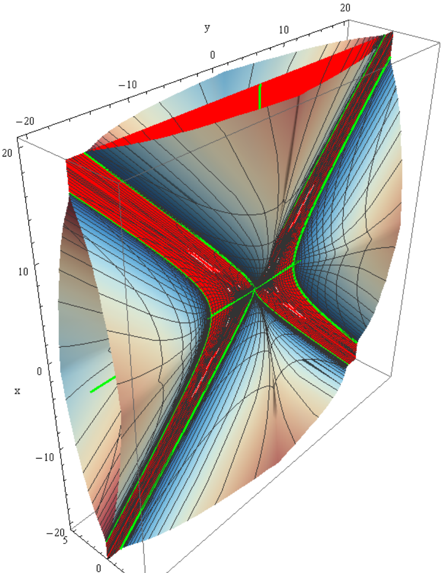
Half-sphere of radius 3π and the cut locus in straightening coordinates (R_1, R_2, z)



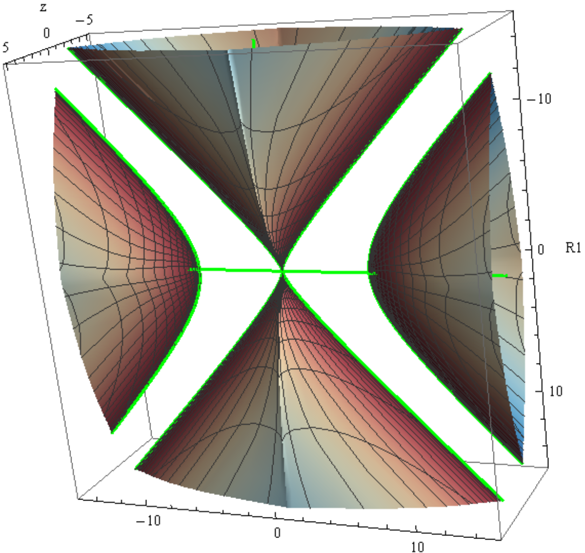
First caustic



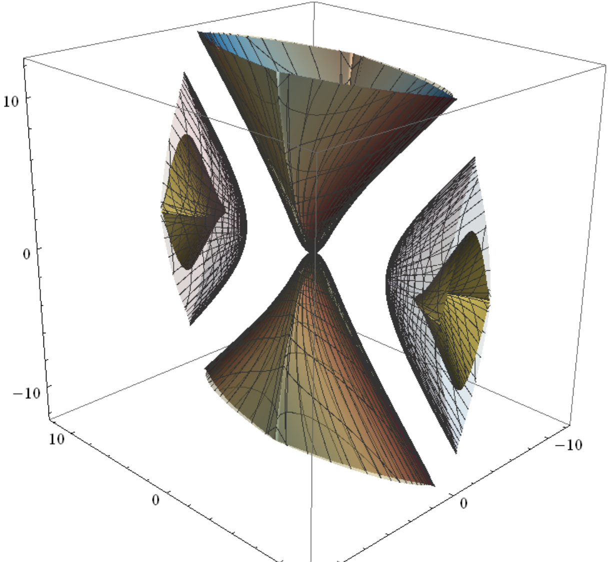
First caustic and the cut locus



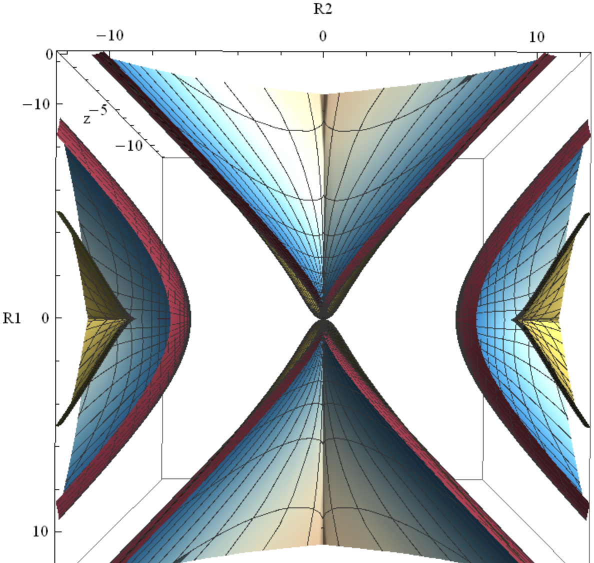
Return edges of the first caustic



First and second caustics



First and second caustics in section



Exercises

1. Solve the sub-Riemannian problems on the groups SE(2) and SH(2) independently.