Proof of Pontryagin maximum principle for sub-Riemannian problems
(Lecture 7)

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6. Coming Home on the Ox's Back:

Riding on the animal, he leisurely wends his way home:
Enveloped in the evening mist, how tunefully the flute vanishes away!
Singing a ditty, beating time, his heart is filled with a joy indescribable!
That he is now one of those who know, need it be told?

Pu-ming, "The Ten Oxherding Pictures"

Reminder: Plan of the previous lecture

1. Sub-Riemannian problems
2. The sub-Riemannian problem on the Heisenberg group.

Plan of this lecture

1. Proof of Pontryagin maximum principle for sub-Riemannian problems

Optimal control problem

At this lecture we prove Pontryagin maximum principle for the sub-Riemannian optimal control problem:

$$\dot{q} = \sum_{i=1}^{k} u_i f_i(q) =: f_u(q), \qquad q \in M, \quad u = (u_1, \ldots, u_k) \in \mathbb{R}^k,$$

$$q(0) = q_0, \qquad q(t_1) = q_1,$$

$$l = \int_0^{t_1} \left(\sum_{i=1}^{k} u_i^2 \right)^{1/2} dt \to \min .$$

Statement of PMP for SR problem

Theorem 1 (PMP for SR problems)

Let $\overline{q} \in \mathrm{Lip}([0, t_1], M)$ *be a SR minimizer for which the corresponding control* $\overline{u}(t)$ *satisfies the condition* $\sum_{i=1}^{k} \overline{u}_i^2(t) \equiv \mathrm{const}$. *Then there exists a curve* $\lambda_t \in \mathrm{Lip}([0, t_1], T^*M)$, $\pi(\lambda_t) = \overline{q}(t)$, *such that for almost all* $t \in [0, t_1]$

$$\dot{\lambda}_t = \sum_{i=1}^{k} \overline{u}_i(t) \vec{h}_i(\lambda_t), \qquad (1)$$

and one of the conditions hold:

(N) $\quad h_i(\lambda_t) \equiv \overline{u}_i(t), \quad i = 1, \dots, k,$ *or*

(A) $\quad h_i(\lambda_t) \equiv 0, \quad i = 1, \dots, k, \qquad \lambda_t \neq 0 \qquad \forall t \in [0, t_1].$

- In conditions (N), (A) corresponding to the normal and abnormal cases, as always, $h_i(\lambda) = \langle \lambda, f_i \rangle$, $i = 1, \dots, k$

Reduction to Theorems 2, 3

Theorem 1 follows from the next two theorems.

Theorem 2

Let the hypotheses of Theorem 1 hold. For any $t \in [0, t_1]$, *let* $P_t : M \to M$ *denote the flow of the nonautonomous vector field* $f_{\overline{u}(t)} = \sum_{i=1}^{k} \overline{u}_i(t) f_i$ *from the time* 0 *to the time* t. *Then there exists* $\lambda_0 \in T^*_{q_0}M$ *such that the curve*

$$\lambda_t = (P_t^{-1})^*(\lambda_0) \in T^*_{\overline{q}(t)}M \tag{2}$$

satisfies one of conditions (N), (A) *of Theorem* 1.

Theorem 3

Let the hypotheses of Theorems 1 *and* 2 *hold. Then ODE* (1) *follows from identity* (2).

Flow of nonautonomous vector field

- In Theorem 2, the flow $P_t : M \to M$ of the nonautonomous field $f_{\overline{u}(t)}$ from the time 0 to the time t is given as follows:

$$P_t(q) = \overline{q}(t), \qquad q \in M, \quad t \in [0, t_1],$$
$$\frac{d}{dt}\overline{q}(t) = \sum_{i=1}^{k} \overline{u}_i(t) f_i(\overline{q}(t)), \qquad \overline{q}(0) = q.$$

- Further, in Theorem 2 we use the mapping $(P_t^{-1})^* : T^*_{q_0}M \to T^*_{\overline{q}(t)}M$, recall the necessary definition. If $F : M \to N$ is a smooth mapping between smooth manifolds and $q \in M$, then there is defined the differential

$$F_{*q} : T_q M \to T_{F(q)} N,$$

and the dual mapping of cotangent spaces:

$$F_q^* = (F_{*q})^* : T^*_{F(q)} N \to T^*_q M,$$
$$\langle F_q^*(\lambda), v \rangle = \langle \lambda, F_{q*}(v) \rangle, \qquad v \in T_q M, \quad \lambda \in T^*_{F(q)} N.$$

Reduction to the study of attainable sets

- Replace the length $l = \int_0^{t_1} (\sum_{i=1}^{k} u_i^2)^{1/2} \, dt$ by the energy $J = \frac{1}{2} \int_0^{t_1} \sum_{i=1}^{k} u_i^2 \, dt$.
- In order to include the functional J into dynamics of the system, introduce a new variable equal to the running value of the cost functional along a trajectory $q_u(t)$: $y(t) = \frac{1}{2} \int_0^{t} \sum_{i=1}^{k} u_i^2 \, dt$.
- Respectively, we introduce an extended state $\widehat{q} = \begin{pmatrix} y \\ q \end{pmatrix} \in \mathbb{R} \times M$ that satisfies an *extended control system*

$$\frac{d\widehat{q}}{dt} = \begin{pmatrix} \dot{y} \\ \dot{q} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \sum_{i=1}^{k} u_i^2 \\ f(q, u) \end{pmatrix} =: \widehat{f}(\widehat{q}, u).$$

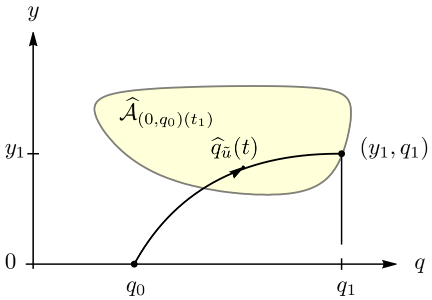
- The boundary conditions for this system are

$$\widehat{q}(0) = \begin{pmatrix} 0 \\ q_0 \end{pmatrix}, \qquad \widehat{q}(t_1) = \begin{pmatrix} J \\ q_1 \end{pmatrix}.$$

Reduction to the study of attainable sets

- A trajectory $q_{\tilde{u}}(t)$ is optimal for the optimal control problem with fixed time t_1 if and only if the corresponding trajectory $\widehat{q}_{\tilde{u}}(t)$ of the extended system comes to a point (y_1, q_1) of the attainable set $\widehat{\mathscr{A}_{(0,q_0)}}(t_1)$ such that

$$\widehat{\mathscr{A}_{(0,q_0)}}(t_1) \cap \{(y, q_1) \mid y < y_1\} = \emptyset.$$



Proof of Theorem 2: 1/11

- The curve $\overline{q}(t)$ is a minimizer of the length functional $l = \int_0^{t_1} \left(\sum_{i=1}^{k} u_i^2 \right)^{1/2} dt$ of constant velocity, thus it is a minimizer of the energy functional $J(u) = \frac{1}{2} \int_0^{t_1} \sum_{i=1}^{k} u_i^2(t)\, dt$ for a fixed t_1.
- Take any control $u(\,\cdot\,) = \overline{u}(\,\cdot\,) + v(\,\cdot\,) \in L^\infty([0, t_1], \mathbb{R}^k)$ and consider the corresponding Cauchy problem

$$\dot{q}(t) = f_{u(t)}(q(t)) = \sum_{i=1}^{k} u_i(t) f_i(q(t)), \qquad q(0) = q_0.$$

- Recall that $P_t : M \to M$ is the flow of the nonautonomous vector field $f_{\overline{u}(t)}$ from the time 0 to the time t.
- Consider the curve $x(t) = P_t^{-1}(q(t))$ and derive an ODE for $x(t)$.

Proof of Theorem 2: 2/11

- We differentiate the identity $q(t) = P_t(x(t))$ and get

$$\dot{q}(t) = f_{\overline{u}(t)}(P_t(x(t))) + (P_t)_*\dot{x}(t),$$

whence

$$\begin{aligned}\dot{x}(t) &= (P_t^{-1})_*[\dot{q}(t) - f_{\overline{u}(t)}(P_t(x(t)))] \\ &= (P_t^{-1})_*[(f_{u(t)} - f_{\overline{u}(t)})(P_t(x(t)))] \\ &= [(P_t^{-1})_*(f_{u(t)-\overline{u}(t)})](x(t)) \\ &= [(P_t^{-1})_* f_{v(t)}](x(t)).\end{aligned}$$

- We denote the nonautonomous vector field $g_v^t = (P_t^{-1})_* f_v$ and get the required ODE

$$\dot{x}(t) = g_{v(t)}^t(x(t)), \qquad x(0) = P_0^{-1}(q_0) = q_0. \tag{3}$$

- Notice that f_v is linear in v, thus g_v^t is linear in v.

Proof of Theorem 2: 3/11

- For any $v \in L^\infty([0,t_1],\mathbb{R}^k)$, consider a mapping

$$\mathbb{R} \ni s \mapsto \begin{pmatrix} x(t_1;\overline{u}+sv) \\ J(\overline{u}+sv) \end{pmatrix} \in M \times \mathbb{R},$$

where $x(t_1;\overline{u}+sv)$ is the solution to Cauchy problem (3) corresponding to the control $\overline{u}+sv$, and $J(\overline{u}+sv)$ is the corresponding energy.

Lemma 4

There exists a covector $\overline{\lambda} \in (T_{q_0}M \oplus \mathbb{R})^*$, $\overline{\lambda} \neq 0$, such that for any $v \in L^\infty([0,t_1],\mathbb{R}^k)$ there holds the equality

$$\left\langle \overline{\lambda}, \left(\left.\frac{\partial x(t_1;\overline{u}+sv)}{\partial s}\right|_{s=0}, \left.\frac{\partial J(\overline{u}+sv)}{\partial s}\right|_{s=0} \right) \right\rangle = 0. \tag{4}$$

Proof of Theorem 2: 4/11, Proof of Lemma 4

- Denote

$$\Phi(v) = \left(\frac{\partial x(t_1; \overline{u} + sv)}{\partial s}\bigg|_{s=0}, \frac{\partial J(\overline{u} + sv)}{\partial s}\bigg|_{s=0} \right),$$

$$\Phi : L^\infty([0, t_1], \mathbb{R}^k) \to T_{q_0}M \oplus \mathbb{R}.$$

- We compute the derivatives in the definition of the mapping Φ. It is easy to see that

$$\frac{\partial J(\overline{u} + sv)}{\partial s}\bigg|_{s=0} = \int_0^{t_1} \sum_{i=1}^{k} \overline{u}_i(t) v_i(t)\, dt. \tag{5}$$

Indeed, this follows from the expansion

$$J(\overline{u} + sv) = \frac{1}{2}\int_0^{t_1} |\overline{u} + sv|^2\, dt$$

$$= \frac{1}{2}\int_0^{t_1} \left(|\overline{u}|^2 + 2s\sum_{i=1}^{k} \overline{u}_i(t) v_i(t) + s^2 |v|^2 \right) dt.$$

Proof of Theorem 2: 5/11, Proof of Lemma 4

- Further, we show that

$$\left.\frac{\partial x(t_1;\overline{u}+sv)}{\partial s}\right|_{s=0} = \int_0^{t_1} g^t_{v(t)}(q_0)\,dt = \int_0^{t_1} \sum_{i=1}^k ((P_t^{-1})_* f_i)(q_0) v_i(t)\,dt. \qquad (6)$$

- The ODE $\dot{x}(t;\overline{u}+sv) = g^t_{sv}(x(t;\overline{u}+sv))$ implies in local coordinates that

$$x(t_1;\overline{u}+sv) = q_0 + \int_0^{t_1} g^t_{sv(t)}(x(t;\overline{u}+sv))\,dt$$
$$= q_0 + s\int_0^{t_1} g^t_{v(t)}(x(t;\overline{u}+sv))\,dt$$

since $g^t_{sv(t)} = s g^t_{v(t)}$, whence

$$\left.\frac{\partial x(t_1;\overline{u}+sv)}{\partial s}\right|_{s=0} = \int_0^{t_1} g^t_{v(t)}(x(t;\overline{u}))\,dt$$
$$= \int_0^{t_1} g^t_{v(t)}(q_0)\,dt = \int_0^{t_1} \sum_{i=1}^k ((P_t^{-1})_* f_i)(q_0) v_i(t)\,dt.$$

Proof of Theorem 2: 6/11, Proof of Lemma 4

- One can see from (5), (6) that the mapping Φ is linear in v. We show that it is not surjective.
- By contradiction, let $\operatorname{Im}\Phi = T_{q_0}M \oplus \mathbb{R}$, then there exist $v^0, \dots, v^n \in L^\infty([0, t_1], \mathbb{R}^k)$ such that $\Phi(v^0), \dots, \Phi(v^n)$ are linearly independent, i.e., the vectors

$$\begin{pmatrix} \left.\frac{\partial x(t_1; \overline{u} + s v^0)}{\partial s}\right|_{s=0} \\ \left.\frac{\partial J(\overline{u} + s v^0)}{\partial s}\right|_{s=0} \end{pmatrix}, \quad \dots, \quad \begin{pmatrix} \left.\frac{\partial x(t_1; \overline{u} + s v^n)}{\partial s}\right|_{s=0} \\ \left.\frac{\partial J(\overline{u} + s v^n)}{\partial s}\right|_{s=0} \end{pmatrix}$$

are linearly independent.
- Consider the mapping

$$F \ : \ (s_0, \dots, s_n) \mapsto \begin{pmatrix} x\left(t_1; \overline{u} + \sum_{i=0}^{n} s_i v^i\right) \\ J\left(\overline{u} + \sum_{i=0}^{n} s_i v^i\right) \end{pmatrix}, \qquad \mathbb{R}^{n+1} \to M \times \mathbb{R}.$$

Proof of Theorem 2: 7/11, Proof of Lemma 4

- The mapping F is smooth near the point $0 \in \mathbb{R}^{n+1}$ and has a nondegenerate Jacobian at this point.
- Thus there exists a neighbourhood $O_0 \subset \mathbb{R}^{n+1}$ such that the restriction $F|_{O_0}$ is a diffeomorphism.
- Consequently,

$$F(0) = \begin{pmatrix} x(t_1; \overline{u}) \\ J(\overline{u}) \end{pmatrix} = \begin{pmatrix} q_0 \\ J(\overline{u}) \end{pmatrix} \in \operatorname{int} F(O_0).$$

- Thus there exists a control $v(\cdot) = \sum_{i=0}^{n} s_i v^i(\cdot)$ for which

$$x(t_1; \overline{u} + v) = q_0, \qquad J(\overline{u} + v) < J(\overline{u}).$$

Proof of Theorem 2: 8/11, Proof of Lemma 4

- Consider the corresponding trajectory $t \mapsto q(t; \overline{u} + v)$. We have

$$q(0; \overline{u} + v) = q_0,$$
$$q(t_1; \overline{u} + v) = P_{t_1}(x(t_1; \overline{u} + v)) = P_{t_1}(q_0) = q_1.$$

- So the curve $q(t; \overline{u} + v)$ connects the points q_0 and q_1 with a lesser value of the functional J than the optimal trajectory $\overline{q}(t) = q(t; \overline{u})$.
- The contradiction obtained completes the proof of Lemma 4.

Proof of Theorem 2: 9/11

- We continue the proof of Theorem 2.
- By the previous lemma, there exists a covector $0 \neq \overline{\lambda} \in (T_{q_0}M \oplus \mathbb{R})^*$ such that for any $v \in L^\infty([0, t_1], \mathbb{R}^k)$ we have

$$\left\langle \overline{\lambda}, \left(\left.\frac{\partial x(t_1; \overline{u} + sv)}{\partial s}\right|_{s=0}, \left.\frac{\partial J(\overline{u} + sv)}{\partial s}\right|_{s=0} \right) \right\rangle = 0.$$

- It is obvious that if this condition holds for some covector $\overline{\lambda}$, then it also holds for any covector $\alpha\overline{\lambda}$, $\alpha \neq 0$.
- Consequently, we can choose a covector $\overline{\lambda}$ of the form

$$\overline{\lambda} = (\lambda_0, -1) \qquad \text{or} \qquad \overline{\lambda} = (\lambda_0, 0), \quad \lambda_0 \neq 0.$$

- Thus there exists a covector $\lambda_0 \in T^*_{q_0}M$ such that for any $v \in L^\infty([0, t_1], \mathbb{R}^k)$

$$\left.\frac{\partial J(\overline{u} + sv)}{\partial s}\right|_{s=0} - \left\langle \lambda_0, \left.\frac{\partial x(t_1; \overline{u} + sv)}{\partial s}\right|_{s=0} \right\rangle = 0 \qquad (7)$$

or

$$0 = \left\langle \lambda_0, \left.\frac{\partial x(t_1; \overline{u} + sv)}{\partial s}\right|_{s=0} \right\rangle, \qquad \lambda_0 \neq 0. \qquad (8)$$

- Consider the case (7).
- Equalities (5) and (6) imply that for any $v \in L^\infty([0, t_1], \mathbb{R}^k)$

$$\int_0^{t_1} \sum_{i=1}^k \overline{u}_i(t) v_i(t)\, dt = \int_0^{t_1} \sum_{i=1}^k \langle \lambda_0, ((P_t^{-1})_* f_i)(q_0) \rangle\, v_i(t)\, dt$$

$$= \int_0^{t_1} \sum_{i=1}^k \langle \lambda_t, f_i(\overline{q}(t)) \rangle v_i(t)\, dt = \int_0^{t_1} \sum_{i=1}^k h_i(\lambda_t) v_i(t)\, dt.$$

Proof of Theorem 2: 11/11

- Since the functions $v_i \in L^\infty[0, t_1]$ are arbitrary, we get in case (7)
  (*N*) $\quad \overline{u}_i(t) = h_i(\lambda_t), \quad i = 1, \dots, k.$
- Similarly, in case (8) we get the condition
  (*A*) $\quad 0 = h_i(\lambda_t), \quad i = 1, \dots, k; \qquad \lambda_0 \neq 0,$

  exercise.
- Theorem 2 is proved.

Proof of Theorem 3: 1/7

- Now we prove Theorem 3.
- Recall: we should show that the curve $\lambda_t = (P_t^{-1})^* \lambda_0 \in T^*_{\overline{q}(t)} M$ satisfies the ODE $\dot{\lambda}_t = \sum_{i=1}^{k} \overline{u}_i(t) \vec{h}_i(\lambda_t)$.
- Now we prove this for the flow of an autonomous vector field.

Lemma 5

Let $X \in \mathrm{Vec}(M)$, $P_t = e^{tX}$. Then the curve $\lambda_t = (P_t^{-1})^* \lambda_0$ satisfies the ODE $\dot{\lambda}_t = \vec{h}_X(\lambda_t)$.

- We set $\varphi(t) = (P_t^{-1})^*(\lambda_0)$, then we have to prove that
$$\dot{\varphi}(t) = \vec{h}_X(\varphi(t)) \in T_{\varphi(t)}(T^*M).$$
- A function $a \in C^\infty(T^*M)$ is called *constant on fibers of T^*M* if it has the form $a = \alpha \circ \pi$ for some function $\alpha \in C^\infty(M)$. Notation: $a \in C^\infty_{\mathrm{cst}}(T^*M)$.
- A function $h_Y \in C^\infty(T^*M)$ is called *linear on fibers of T^*M* if
$$h_Y(\lambda) = \langle \lambda, Y(q) \rangle, \qquad q = \pi(\lambda), \quad \lambda \in T^*M,$$
for some vector field $Y \in \mathrm{Vec}(M)$. Notation: $h_Y \in C^\infty_{\mathrm{lin}}(T^*M)$.
- An *affine on fibers of T^*M function* is a sum of a constant on fibers and a linear on fibers functions:
$$C^\infty_{\mathrm{aff}}(T^*M) = C^\infty_{\mathrm{cst}}(T^*M) + C^\infty_{\mathrm{lin}}(T^*M).$$

Proof of Theorem 3: 3/7, Proof of Lemma 5

- Remark: Let $v, \omega \in T_\lambda(T^*M)$. The equality $v = \omega$ holds if and only if

$$vg = \omega g \qquad \forall g \in C^\infty_{\mathrm{aff}}(T^*M).$$

Indeed, the value $vg = \langle d_\lambda g, v\rangle$ depends only on the first order Taylor polynomial of the function g.

- So we check the required equality $\dot\varphi(t) = \vec{h}_X(\varphi(t))$ for affine on fibers of T^*M functions.
- Let $a = \alpha \circ \pi \in C^\infty_{\mathrm{cst}}(T^*M)$, we check the equality $\dot\varphi(t)a = \vec{h}_X a$. We have

$$\vec{h}_X a = \{h_X, a\} = \sum_{i=1}^n \frac{\partial h_X}{\partial p_i}\frac{\partial \alpha}{\partial q_i} = \sum_{i=1}^n X_i \frac{\partial \alpha}{\partial q_i} = X\alpha,$$

$$\dot\varphi(t)a = \frac{d}{dt} a(\varphi(t)) = \frac{d}{dt}\alpha \circ e^{tX}(q_0) = (X\alpha)(\varphi(t)),$$

and the required equality is proved for functions $a \in C^\infty_{\mathrm{cst}}(T^*M)$.

- Now let $h_Y \in C^\infty_{\mathrm{lin}}(T^*M)$, we check the equality $\dot{\varphi}(t)h_Y = \vec{h}_X h_Y$. We have

$$\vec{h}_X h_Y = \{h_X, h_Y\} = h_{[X,Y]}.$$

- On the other hand,

$$\begin{aligned}
\dot{\varphi}(t)h_Y &= \frac{d}{dt} h_Y \circ \varphi(t) = \frac{d}{d\tau}\bigg|_{\tau=0} h_Y \circ \varphi(t+\tau) \\
&= \frac{d}{d\tau}\bigg|_{\tau=0} h_Y \circ (e^{-\tau X})^* \circ (e^{-tX})^*(\lambda_0) \\
&= \frac{d}{d\tau}\bigg|_{\tau=0} \left\langle (e^{-\tau X})^* \circ (e^{-tX})^*(\lambda_0),\ Y(e^{(t+\tau)X}(q_0)) \right\rangle \\
&= \left\langle \varphi(t), \frac{d}{d\tau}\bigg|_{\tau=0} e_*^{-\tau X} Y(e^{\tau X} \circ e^{tX}(q_0)) \right\rangle \\
&= \left\langle \varphi(t), [X, Y](e^{tX}(q_0)) \right\rangle = h_{[X,Y]}(\varphi(t)).
\end{aligned}$$

Proof of Theorem 3: 5/7, Proof of Lemma 5

- In the penultimate transition we used the equality

$$\left.\frac{d}{d\tau}\right|_{\tau=0} e_*^{-\tau X} Y(e^{\tau X}(q)) = [X, Y](q), \qquad (9)$$

which we prove now.

- We have

$$\left.\frac{d}{d\tau}\right|_{\tau=0} e_*^{-\tau X} Y(e^{\tau X}(q)) = \left.\frac{\partial^2}{\partial\tau\partial s}\right|_{\tau=0,s=0} e^{-\tau X} \circ e^{sY} \circ e^{\tau X}(q).$$

- We compute Taylor expansions of the compositions in the right-hand side:

$$e^{\tau X}(q) = q + \tau X(q) + o(\tau),$$

$$\begin{aligned} e^{sY} \circ e^{\tau X} &= e^{sY}(q + \tau X(q) + o(\tau)) \\ &= q + \tau X(q) + o(\tau) + sY(q + \tau X(q) + o(\tau)) + o(s) \\ &= q + \tau X(q) + sY(q) + s\tau \frac{\partial Y}{\partial q} X(q) + \dots, \end{aligned}$$

Proof of Theorem 3: 6/7, Proof of Lemma 5

- Consequently,

$$
\begin{aligned}
e^{-\tau X}\circ e^{sY}\circ e^{\tau X}(q) &= q+\tau X(q)+sY(q)+s\tau\frac{\partial Y}{\partial q}X(q)\\
&\qquad\qquad -\tau X(q)-\tau s\frac{\partial X}{\partial q}Y(q)+\dots\\
&= q+sY(q)+s\tau[X,Y](q)+\dots,
\end{aligned}
$$

  thus

$$
\left.\frac{\partial^2}{\partial\tau\partial s}\right|_{\tau=0,s=0} e^{-\tau X}\circ e^{sY}\circ e^{\tau X}(q)=[X,Y](q),
$$

  and equality (9) follows.
- Lemma 5 is proved.

Proof of Theorem 3: 7/7

- Similarly to Lemma 5 for an autonomous vector field X, one proves the equality $\dot{\lambda}_t = \sum_{i=1}^{k} \overline{u}_i(t)\vec{h}_i(\lambda_t)$ for a curve $\lambda_t = (P_t^{-1})^* \lambda_0$ in the case of a nonautonomous vector field $f_{\overline{u}(t)}$ (Exercise.)
- This completes the proof of Theorem 3.
- As we noticed above, Theorem 1 follows from Theorems 2 and 3.
- The Pontryagin maximum principle for sub-Riemannian problems is proved.