Pontryagin maximum principle and its applications
(Lecture 5)

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«Geometric control theory, nonholonomic geometry, and their applications»

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4. Catching the Ox:

With the energy of his whole being, the boy has at last taken hold of the ox:
But how wild his will, how ungovernable his power!
At times he struts up a plateau,
When lo! he is lost again in a misty unpenetrable mountain-pass.

Pu-ming, "The Ten Oxherding Pictures"

Reminder: Plan of the previous lecture

1. Krener's theorem
2. Statement of optimal control problem
3. Existence of optimal controls
4. Elements of symplectic geometry
5. Statement of Pontryagin maximum principle

Plan of this lecture

1. Statement of Pontryagin maximum principle
2. Solution to examples of optimal control problems
3. Sub-Riemannian problems
4. Optimality conditions

Hamiltonians of Pontryagin maximum principle

- Optimal control problem

$$
\begin{aligned}
&\dot{q} = f(q,u), \qquad q \in M, \quad u \in U \subset \mathbb{R}^m,\\
&q(0) = q_0, \qquad q(t_1) = q_1,\\
&J = \int_0^{t_1} \varphi(q,u)\, dt \to \min,\\
&t_1 \text{ fixed or free.}
\end{aligned}
$$

- Define a family of *Hamiltonians of PMP*

$$
h_u^\nu(\lambda) = \langle \lambda, f(q,u) \rangle + \nu\varphi(q,u), \qquad \nu \in \mathbb{R}, \quad u \in U, \quad \lambda \in T^*M, \quad q = \pi(\lambda).
$$

Statement of Pontryagin maximum principle

Theorem (PMP)

If a control $u(t)$ and the corresponding trajectory $q(t), t \in [0, t_1]$, are optimal in the problem with fixed t_1, then there exist a curve $\lambda_t \in \mathrm{Lip}([0, t_1], T^*M)$, $\lambda_t \in T^*_{q(t)}M$, and a number $\nu \leq 0$ such that the following conditions hold for almost all $t \in [0, t_1]$:

(1) $\dot{\lambda}_t = \vec{h}^{\nu}_{u(t)}(\lambda_t)$,

(2) $h^{\nu}_{u(t)}(\lambda_t) = \max_{w \in U} h^{\nu}_w(\lambda_t)$,

(3) $(\lambda_t, \nu) \neq (0, 0)$.

If the terminal time t_1 is free, then the following condition is added to (1)–(3):

(4) $h^{\nu}_{u(t)}(\lambda_t) \equiv 0$.

A curve λ_t that satisfies PMP is called an *extremal*, a curve $q(t)$ — an *extremal trajectory*, a control $u(t)$ — an *extremal control*.

Time-optimal problem

- Let us apply PMP to the *time-optimal problem*

$$\dot q = f(q,u), \qquad q \in M, \quad u \in U,$$
$$q(0) = q_0, \qquad q(t_1) = q_1,$$
$$t_1 = \int_0^{t_1} 1\, dt \to \min .$$

- The Hamiltonian of PMP has the form $h_u^\nu(\lambda) = \langle \lambda, f(q,u)\rangle + \nu$. Introduce the *shortened Hamiltonian* $g_u(\lambda) = \langle \lambda, f(q,u)\rangle$.
- Then the statement of PMP for the time-optimal problem takes the form:

(1) $\dot\lambda_t = \vec h^\nu_{u(t)}(\lambda_t) = \vec g_{u(t)}(\lambda_t)$,

(2) $h^\nu_{u(t)}(\lambda_t) = \max_{w\in U} h^\nu_w(\lambda_t) \quad \Leftrightarrow \quad g_{u(t)}(\lambda_t) = \max_{w \in U} g_w(\lambda_t)$,

(3) $\lambda_t \neq 0$,

(4) $h^\nu_{u(t)}(\lambda_t) \equiv 0 \quad \Leftrightarrow \quad g_{u(t)}(\lambda_t) \equiv \text{const} \geq 0$.

The case of smooth maximized Hamiltonian

Denote the *maximized normal Hamiltonian of PMP*

$$H(\lambda) = \max_{u \in U} h_u^{-1}(\lambda), \qquad \lambda \in T^*M.$$

Theorem

Let $H \in C^2(T^*M)$. Then a curve λ_t is a normal extremal iff it is a trajectory of the Hamiltonian system $\dot{\lambda}_t = \vec{H}(\lambda_t)$.

Proof.

See A.A. Agrachev, Yu.L. Sachkov, *Control theory from the geometric viewpoint*,
А.А. Аграчев, Ю. Л. Сачков, *Геометрическая теория управления*. □

Example: Stopping a train (1/4)

- We have the time-optimal problem

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = u, \qquad x = (x_1, x_2) \in \mathbb{R}^2, \quad |u| \le 1,$$
$$x(0) = x^0, \quad x(t_1) = x^1 = (0,0), \qquad t_1 \to \min .$$

- The right-hand side of the control system $f(x,u) = (x_2, u)$ satisfies the bound

$$|f(x,u)| = \sqrt{x_2^2 + u^2} \le \sqrt{x_2^2 + 1} \le |x| + 1,$$

  thus $r = x^2$ satisfies the differential inequality
  $\dot{r} = 2\langle x, \dot{x} \rangle = 2\langle x, f(x,u) \rangle \le 2(r+1)$. By Gronwall's lemma
  $r(t) + 1 \le e^{2t}(r_0 + 1)$, thus attainable sets satisfy the a priori bound

$$\mathcal{A}_{x^0}(\le t) \subset \left\{ x \in \mathbb{R}^2 \mid |x| \le e^t \sqrt{(x^0)^2 + 1} \right\}.$$

- Therefore we can assume that there exists a compact set $K \subset \mathbb{R}^2$ such that the right-hand side of the control system vanishes outside of K (one of conditions of the Filippov theorem).

Example: Stopping a train (2/4)

- As we showed, $x^1 = (0,0) \in \mathcal{A}_{x^0}$ for any $x^0 \in \mathbb{R}^2$.
- The set of control parameters U is compact, and the set of admissible velocity vectors $f(x, U)$ is convex for any $x \in \mathbb{R}^2$. All hypotheses of the Filippov theorem are satisfied, thus optimal control exists.
- We apply PMP using the canonical coordinates (p_1, p_2, x_1, x_2) on $T^*\mathbb{R}^2$. We decompose a covector $\lambda = p_1\, dx_1 + p_2\, dx_2 \in T^*\mathbb{R}^2$, then the shortened Hamiltonian of PMP reads $h_u(\lambda) = p_1 x_2 + p_2 u$, and the Hamiltonian system $\dot{\lambda} = \vec{h}_u(\lambda)$ reads

$$\begin{aligned} \dot{x}_1 &= x_2, & \qquad \dot{p}_1 &= 0, \\ \dot{x}_2 &= u, & \qquad \dot{p}_2 &= -p_1. \end{aligned}$$

- The maximality condition of PMP has the form

$$h_u(\lambda) = p_1 x_2 + p_2 u \to \max_{|u| \leq 1},$$

and the nontriviality condition is $(p_1(t), p_2(t)) \neq (0,0)$.

Example: Stopping a train (3/4)

- The maximality condition yields:

$$p_2(t) > 0 \quad \Rightarrow \quad u(t) = 1, \qquad\qquad p_2(t) < 0 \quad \Rightarrow \quad u(t) = -1.$$

- Thus extremal trajectories are the parabolas

$$x_1 = \pm\frac{x_2^2}{2} + C,$$

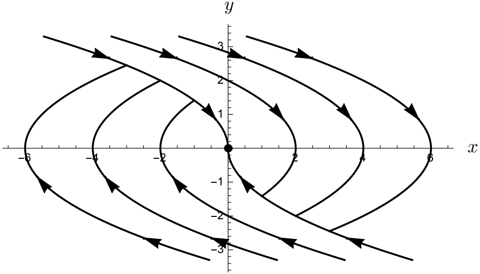
  and the number of switchings (discontinuities) of control is not greater than 1.
- Let us construct such trajectories backward in time, starting from $x^1 = (0,0)$:
  - the controls $u = 1$ and $u = -1$ generate two half-parabolas terminating at x^1:

$$x_1 = \frac{x_2^2}{2}, \quad x_2 \le 0 \quad \text{and} \quad x_1 = -\frac{x_2^2}{2}, \quad x_2 \ge 0,$$

  - denote the union of these half-parabolas as Γ,
  - after one switching, parabolic arcs with $u = 1$ terminating at the half-parabola $x_1 = -\frac{x_2^2}{2}, \quad x_2 \ge 0$, fill the part of the plane $\mathbb{R}^2$ below the curve Γ,
  - similarly, after one switching, parabolic arcs with $u = -1$ fill the part of the plane over the curve Γ.

Example: Stopping a train (4/4)

- So through each point of the plane $\mathbb{R}^2$ passes a unique extremal trajectory. In view of existence of optimal controls, the extremal trajectories are optimal.
- The optimal control found has explicit dependence on the current point of the plane: if $x_1 = \frac{x_2^2}{2}, \quad x_2 \leq 0$, or if the point (x_1, x_2) lies below the curve Γ, then $u(x_1, x_2) = 1$, otherwise, $u(x_1, x_2) = -1$.



- Such a dependence $u(x)$ of optimal control on the current point x of the state space is called an *optimal synthesis*, it is the best possible form of solution to an optimal control problem.

Example: The Markov-Dubins car (1/4)

- We have a time-optimal problem

$$\begin{aligned}
&\dot{x} = \cos\theta, \qquad q = (x, y, \theta) \in \mathbb{R}^2_{x,y} \times S^1_\theta = M,\\
&\dot{y} = \sin\theta, \qquad |u| \leq 1,\\
&\dot{\theta} = u,\\
&q(0) = q_0 = (0, 0, 0), \qquad q(t_1) = q_1,\\
&t_1 \to \min.
\end{aligned}$$

- The system is completely controllable.
- All conditions of the Filippov theorem are satisfied: U is compact, $f(q, U)$ are convex, the bound $|f(q, u)| \leq 2$ implies a priori bound of the attainable set. Thus optimal control exists.
- We apply PMP.

Example: The Markov-Dubins car (2/4)

- The vector fields

$$\begin{aligned} f_0 &= \cos\theta \frac{\partial}{\partial x} + \sin\theta \frac{\partial}{\partial y}, \\ f_1 &= \frac{\partial}{\partial \theta}, \\ f_2 &= [f_0, f_1] = \sin\theta \frac{\partial}{\partial x} - \cos\theta \frac{\partial}{\partial y} \end{aligned}$$

  form a frame in $T_q M$.

- Define the corresponding linear on fibers of T^*M Hamiltonians:

$$h_i(\lambda) = \langle \lambda, f_i \rangle, \qquad i = 0, 1, 2.$$

- The shortened Hamiltonian of PMP is

$$h_u(\lambda) = \langle \lambda, f_0 + u f_1 \rangle = h_0 + u h_1 .$$

Example: The Markov-Dubins car (3/4)

- The functions h_0, h_1, h_2 form a coordinate system on $T_q^* M$, and we write the Hamiltonian system of PMP in the non-canonical parametrization (h_0, h_1, h_2, q) of $T^* M$:

$$\dot{h}_0 = \vec{h}_u h_0 = \{h_0 + uh_1, h_0\} = u\langle \lambda, [f_1, f_0] \rangle = u\langle \lambda, -f_2 \rangle = -uh_2, \quad (1)$$

$$\dot{h}_1 = \{h_0 + uh_1, h_1\} = \langle \lambda, [f_0, f_1] \rangle = \langle \lambda, f_2 \rangle = h_2, \quad (2)$$

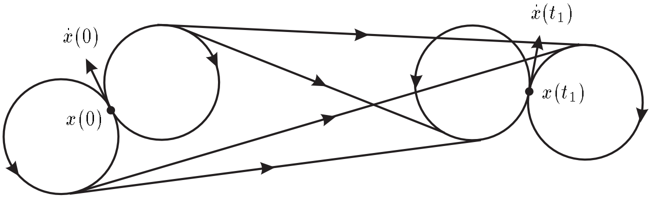
$$\dot{h}_2 = \{h_0 + uh_1, h_2\} = u\langle \lambda, [f_1, f_2] \rangle = u\langle \lambda, f_0 \rangle = uh_0, \quad (3)$$

$$\dot{q} = f_0 + uf_1.$$

- The maximality condition $h_u(\lambda) = h_0 + uh_1 \to \max\limits_{|u| \le 1}$ implies that if $h_1(\lambda_t) \neq 0$, then $u(t) = \operatorname{sgn} h_1(\lambda_t)$.
- Consider the case where the control is not determined by PMP: $h_1(\lambda_t) \equiv 0$ (this case is called *singular*). Then (2) gives $h_2(\lambda_t) \equiv 0$, thus $h_0(\lambda_t) \neq 0$ by the nontriviality condition of PMP, so $u(t) \equiv 0$ by (3). The corresponding extremal trajectory $(x(t), y(t))$ is a straight line.

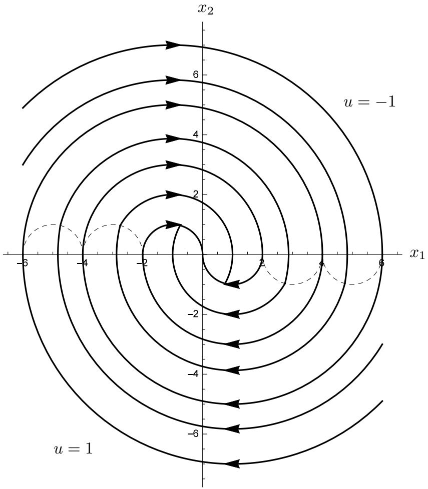
Example: The Markov-Dubins car (4/4)

- If $u(t) = \pm 1$, then the extremal trajectory $(x(t), y(t))$ is an arc of a unit circle.
- One can show that optimal trajectories have one of the following two types:
  1. arc of unit circle + line segment + arc of unit circle
  2. concatenation of three arcs of unit circles; in this case, if a, b, c are the times along the first, second, and third arc respectively, then $\pi < b < 2\pi$, $\min\{a, c\} < b$, and $\max\{a, c\} < b$.
- If boundary conditions are far one from another, then the optimal trajectory has type 1 and can explicitly be constructed as shown below.
- The optimal synthesis for the Markov-Dubins car is known, but it is rather complicated.



Example: Control of linear oscillator

- Optimal trajectories are concatenations of circular arcs.
- The optimal synthesis (exercise):



Sub-Riemannian structures and minimizers

- A *sub-Riemannian structure* on a smooth manifold M is a pair (Δ, g), where
$$\Delta = \{\Delta_q \subset T_q M \mid q \in M\},$$
is a distribution on M and
$$g = \{g_q \text{ inner product in } \Delta_q \mid q \in M\}$$
is an *inner product* (nondegenerate positive definite quadratic form) on Δ.
- The vector subspaces Δ_q and inner products g_q depend smoothly on $q \in M$, and $\dim \Delta_q \equiv \text{const}$.
- A curve $q \in \mathrm{Lip}([0, t_1], M)$ is called *horizontal* (*admissible*) if
$$\dot{q}(t) \in \Delta_{q(t)} \text{ for almost all } t \in [0, t_1].$$
- The *sub-Riemannian length* of a horizontal curve $q(\cdot)$ is defined as
$$l(q(\cdot)) = \int_0^{t_1} \sqrt{g(\dot{q}, \dot{q})}\, dt.$$

Sub-Riemannian structures and minimizers

- The *sub-Riemannian* (*Carnot–Carathéodory*) *distance* between points $q_0, q_1 \in M$ is
$$d(q_0, q_1) = \inf\{l(q(\cdot)) \mid q(\cdot) \text{ horizontal}, \; q(0) = q_0, \; q(t_1) = q_1\}.$$
- A horizontal curve $q(\cdot)$ is called a *sub-Riemannian length minimizer* if
$$l(q(\cdot)) = d(q(0), q(t_1)).$$
- Thus length minimizers are solutions to a *sub-Riemannian optimal control problem*:
$$\begin{aligned} &\dot{q}(t) \in \Delta_{q(t)}, \\ &q(0) = q_0, \qquad q(t_1) = q_1, \\ &l(q(\cdot)) \to \min . \end{aligned}$$
- Suppose that a sub-Riemannian structure (Δ, g) has a *global orthonormal frame* $f_1, \ldots, f_k \in \mathrm{Vec}(M)$:
$$\Delta_q = \mathrm{span}(f_1(q), \ldots, f_k(q)), \quad q \in M, \quad g(f_i, f_j) = \delta_{ij}, \quad i, j = 1, \ldots, k.$$

Sub-Riemannian structures and minimizers

- Then the optimal control problem for sub-Riemannian minimizers takes the standard form:

$$\dot{q} = \sum_{i=1}^{k} u_i f_i(q), \qquad q \in M, \quad u = (u_1, \dots, u_k) \in \mathbb{R}^k, \tag{4}$$

$$q(0) = q_0, \qquad q(t_1) = q_1, \tag{5}$$

$$l = \int_0^{t_1} \left(\sum_{i=1}^{k} u_i^2 \right)^{1/2} dt \to \min . \tag{6}$$

- The sub-Riemannian length does not depend on parametrization of a horizontal curve $q(t)$. Namely, if

$$\widetilde{q}(s) = q(t(s)), \qquad t(\,\cdot\,) \in \mathrm{Lip}([0, s_1], [0, t_1]), \qquad t'(s) > 0,$$

is a reparametrization of a curve $q(t)$, then $l(\widetilde{q}(\,\cdot\,)) = l(q(\,\cdot\,))$ (exercise).

Sub-Riemannian structures and minimizers

- Along with the length functional, it is convenient to consider the *energy* functional

$$J(q(\cdot)) = \frac{1}{2}\int_0^{t_1} g(\dot{q},\dot{q})\,dt.$$

- Denote $\|\dot{q}\| = \sqrt{g(\dot{q},\dot{q})}$.

Sub-Riemannian structures and minimizers

Lemma

Let the terminal time t_1 be fixed. Then minimizers of energy are exactly length minimizers of constant velocity:

$$J(q(\cdot)) \to \min \quad \Leftrightarrow \quad I(q(\cdot)) \to \min, \qquad \|\dot{q}\| = \text{const}\,.$$

Proof.

By the Cauchy–Schwarz inequality,

$$(I(q(\cdot)))^2 = \left(\int_0^{t_1} \|\dot{q}\| \cdot 1\, dt\right)^2 \leq \int_0^{t_1} \|\dot{q}\|^2\, dt \cdot \int_0^{t_1} 1^2\, dt = 2J(q(\cdot))\, t_1,$$

moreover, equality is attained here only for $\|\dot{q}\| \equiv \text{const}$.
It is obvious that on constant velocity curves the problems $I \to \min$ and $J \to \min$ are equivalent. And for $\|\dot{q}\| \not\equiv \text{const}$ we have $I < 2t_1 J$, i.e., J does not attain minimum. □

Sub-Riemannian optimal control problem

$$\dot{q} = \sum_{i=1}^{k} u_i f_i(q), \qquad q \in M, \quad u = (u_1, \dots, u_k) \in \mathbb{R}^k,$$

$$q(0) = q_0, \qquad q(t_1) = q_1,$$

$$l = \int_0^{t_1} \left(\sum_{i=1}^{k} u_i^2 \right)^{1/2} dt \to \min,$$

or, which is equivalent,

$$J = \frac{1}{2} \int_0^{t_1} \sum_{i=1}^{k} u_i^2 \, dt \to \min .$$

The Lie algebra rank condition for SR problems

- The system $\mathcal{F} = \left\{ \sum_{i=1}^{k} u_i f_i \mid u_i \in \mathbb{R} \right\}$ is symmetric, thus $\mathcal{A}_q = \mathcal{O}_q$ for any $q \in M$.
- Assume that M and $\mathcal{F}$ are real-analytic, and M is connected.
- Then for any point $q_0 \in M$, by Lie algebra rank condition,

$$\begin{aligned} \mathcal{A}_{q_0} = M \Leftrightarrow \mathcal{O}_{q_0} = M \\ \Leftrightarrow \mathrm{Lie}_q(\mathcal{F}) = \mathrm{Lie}_q(f_1, \dots, f_k) = T_q M \qquad \forall q \in M. \end{aligned}$$

The Filippov theorem for SR problems

- We can equivalently rewrite the optimal control problem for SR minimizers as the following time-optimal problem:

$$\dot{q} = \sum_{i=1}^{k} u_i f_i(q), \qquad \sum_{i=1}^{k} u_i^2 \leq 1, \quad q \in M,$$
$$q(0) = q_0, \qquad q(t_1) = q_1,$$
$$t_1 \to \min.$$

- Let us check hypotheses of the Filippov theorem for this problem.
- The set of control parameters $U = \{u \in \mathbb{R}^k \mid \sum_{i=1}^k u_i^2 \leq 1\}$ is compact, and the sets of admissible velocities $\left\{\sum_{i=1}^k u_i f_i(q) \mid u \in U\right\} \subset T_q M$ are convex.
- If we prove an a priori estimate for the attainable sets $\mathcal{A}_{q_0}(\leq t_1)$, then the Filippov theorem guarantees existence of length minimizers.

The Pontryagin maximum principle for SR problems

- Introduce the linear on fibers of T^*M Hamiltonians $h_i(\lambda) = \langle \lambda, f_i \rangle, \quad i = 1, \ldots, k$. Then the Hamiltonian of PMP for SR problem takes the form

$$h_u^\nu(\lambda) = \sum_{i=1}^{k} u_i h_i(\lambda) + \frac{\nu}{2} \sum_{i=1}^{k} u_i^2.$$

- *The normal case: Let $\nu = -1$.*
- The maximality condition $\sum_{i=1}^k u_i h_i - \frac{1}{2} \sum_{i=1}^k u_i^2 \to \max_{u_i \in \mathbb{R}}$ yields $u_i = h_i$, then the Hamiltonian takes the form

$$h_u^{-1}(\lambda) = \frac{1}{2} \sum_{i=1}^{k} h_i^2(\lambda) =: H(\lambda).$$

- The function $H(\lambda)$ is called the *normal maximized Hamiltonian*. Since it is smooth, in the normal case extremals satisfy the Hamiltonian system $\dot{\lambda} = \vec{H}(\lambda)$.

The abnormal case

- *Let $\nu = 0$.*
- The maximality condition

$$\sum_{i=1}^{k} u_i h_i \to \max_{u_i \in \mathbb{R}}$$

implies that $h_i(\lambda_t) \equiv 0, \quad i = 1, \ldots, k.$

- Thus abnormal extremals satisfy the conditions:

$$\dot{\lambda}_t = \sum_{i=1}^{k} u_i(t) \vec{h}_i(\lambda_t),$$

$$h_1(\lambda_t) = \cdots = h_k(\lambda_t) \equiv 0.$$

- Normal length minimizers are projections of solutions to the smooth Hamiltonian system $\dot{\lambda} = \vec{H}(\lambda)$, thus they are smooth. An important *open question* of sub-Riemannian geometry is whether abnormal length minimizers are smooth.

Exercises

1. Infer PMP for time-optimal problem (slide 7) from the general statement of PMP.
2. Construct the optimal synthesis for the linear oscillator.
3. Prove that the sub-Riemannian length does not depend on parametrization of a horizontal curve.