Krener's theorem and Optimal control problem
(Lecture 4)

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3. Seeing the Ox:

On a yonder branch perches a nightingale cheerfully singing;
The sun is warm, and a soothing breeze blows, on the bank the willows are green;
The ox is there all by himself, nowhere is he to hide himself;
The splendid head decorated with stately horns what painter can reproduce him?

Pu-ming, "The Ten Oxherding Pictures"

Reminder: Plan of the previous lecture

1. The Orbit theorem.
2. Corollaries of the Orbit theorem:
   - Rashevskii–Chow theorem,
   - Lie algebra rank controllability condition,
   - Frobenius theorem.

Plan of this lecture

1. Krener's theorem
2. Statement of optimal control problem
3. Existence of optimal controls
4. Elements of symplectic geometry
5. Statement of Pontryagin maximum principle

Comparison of topologies of M and $M^{\mathcal{F}}$

Proposition

The "strong" topology of $M^{\mathcal{F}}$ is not weaker than the manifold topology of M.

Proof.

Take any open subset $S \subset M$. We have to show that S is open in $M^{\mathcal{F}}$, i.e., that S is a union of elements of the "strong" topology base $G_q(W_0)$. Take any $q \in S$, let $m = \dim \mathcal{O}_q$. Consider the mapping $G_q(t_1, \dots, t_m) = e^{t_m V_m} \circ \dots \circ e^{t_1 V_1}(q)$, $\mathbb{R}^m \to M$. Since the mappings $t_i \mapsto e^{t_i V_i}(q)$, $\mathbb{R} \to M$, are continuous, then

$$\exists \varepsilon > 0 \; \forall t \in \mathbb{R}^m, \; |t| < \varepsilon \qquad G_q(t) \in S.$$

Let $W_0 = \{t \in \mathbb{R}^m \mid |t| < \varepsilon\}$, then $G_q(W_0) \subset S$. So $S = \bigcup_{q \in S} G_q(W_0)$ is open in $M^{\mathcal{F}}$. □

Exercises: 1) When the topology of $M^{\mathcal{F}}$ is stronger than the topology of M? 2) When the topology of $\mathcal{O}_q$ induced by $M^{\mathcal{F}}$ is stronger than the topology of $\mathcal{O}_q$ induced by M?

Attainable sets of full-rank systems

- Let $\mathcal{F} \subset \text{Vec}(M)$ be a full-rank system:

$$\forall q \in M \qquad \text{Lie}_q(\mathcal{F}) = T_q M.$$

  The assumption of full rank is not very strong in the analytic case: if it is violated, we can consider the restriction of $\mathcal{F}$ to its orbit, and this restriction is full-rank.
- What is the possible structure of *attainable sets* of $\mathcal{F}$?
- It is easy to construct systems in the two-dimensional plane that have the following attainable sets:
  - a smooth full-dimensional manifold without boundary;
  - a full-dimensional manifold with smooth boundary;
  - a full-dimensional manifold with non-smooth boundary, with corner or cusp singularity.

Possible attainable sets of full-rank systems

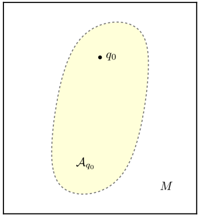


Figure: Smooth manifold without boundary

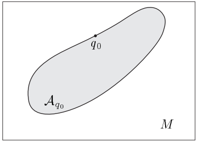


Figure: Manifold with smooth boundary

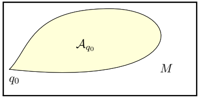


Figure: Manifold with a corner singularity of the boundary

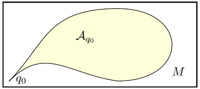


Figure: Manifold with a cusp singularity of the boundary

Impossible attainable sets of full-rank systems

- But it is impossible to construct an attainable set that is:
  - a lower-dimensional submanifold;
  - a set whose boundary points are isolated from its interior points.

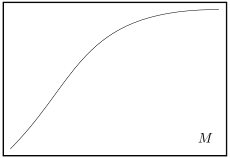


Figure: Forbidden attainable set: subset of lower dimension

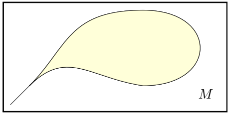


Figure: Forbidden attainable set: subset with isolated boundary points

- These possibilities are forbidden respectively by the following theorem.

Krener's theorem

Theorem (Krener)

Let $\mathcal{F} \subset \mathrm{Vec}(M)$, *and let* $\mathrm{Lie}_q \mathcal{F} = T_q M$ *for any* $q \in M$. *Then:*

(1) $\mathrm{int}\, \mathcal{A}_q \neq \varnothing$ *for any* $q \in M$,

(2) $\mathrm{cl}(\mathrm{int}\, \mathcal{A}_q) \supset \mathcal{A}_q$ *for any* $q \in M$.

Proof of Krener's theorem: 1/2

- Since item (2) implies item (1), we prove item (2): $\mathrm{cl}(\mathrm{int}\,\mathcal{A}_q) \supset \mathcal{A}_q$.
- We argue by induction on dimension of M. If $\dim M = 0$, then $\mathcal{A}_q = \{q\} = M$, and the statement is obvious. Let $\dim M > 0$.
- Take any $q_1 \in \mathcal{A}_q$, and fix any neighbourhood $q_1 \in W(q_1) \subset M$. We show that $\mathrm{int}\,\mathcal{A}_q \cap W(q_1) \neq \varnothing$.
- There exists $f_1 \in \mathcal{F}$ such that $f_1(q_1) \neq 0$, otherwise $\mathcal{F}(q_1) = \{0\} = \mathrm{Lie}_{q_1}(\mathcal{F}) = T_{q_1}M$, a contradiction. Consider the following set for a small $\varepsilon_1 > 0$:

$$N_1 = \{e^{t_1 f_1}(q_1) \mid 0 < t_1 < \varepsilon_1\} \subset W(q_1) \cap \mathcal{A}_q.$$

- N_1 is a smooth 1-dimensional manifold. If $\dim M = 1$, then N_1 is open, thus $N_1 \subset \mathrm{int}\,\mathcal{A}_q$, so $\mathrm{int}\,\mathcal{A}_q \cap W(q_1) \neq \varnothing$. Since the neighbourhood $W(q_1)$ is arbitrary, $q_1 \in \mathrm{cl}(\mathrm{int}\,\mathcal{A}_q)$.

Proof of Krener's theorem: 2/2

- Let $\dim M > 1$. There exist $q_2 = e^{t_1^1 f_1}(q_1) \in N_1 \cap W(q_1)$ and $f_2 \in \mathcal{F}$ such that $f_2(q_2) \notin T_{q_2} N_1$. Otherwise $\dim \mathcal{F}(q_2) = \dim \mathrm{Lie}_{q_2}(\mathcal{F}) = \dim T_{q_2} M = 1$ for any $q_2 \in N_2 \cap W$, and $\dim M = 1$.
- Consider the following set for a small $\varepsilon_2 > 0$:

$$N_2 = \{e^{t_2 f_2} \circ e^{t_1 f_1}(q_1) \mid t_1^1 < t_1 < t_1^1 + \varepsilon_2,\ 0 < t_2 < \varepsilon_2\} \subset W(q_1) \cap \mathcal{A}_q.$$

- N_2 is a smooth 2-dimensional manifold.
- If $\dim M = 2$, then N_2 is open, thus $N_2 \subset \mathrm{int}\,\mathcal{A}_q \cap W(q_1) \neq \varnothing$ and $q_1 \in \mathrm{cl}(\mathrm{int}\,\mathcal{A}_q)$.
- If $\dim M > 2$, we proceed by induction. □

A control system $\mathcal{F} \subset \mathrm{Vec}(M)$ is called *accessible* at a point $q \in M$ if $\mathrm{int}\,\mathcal{A}_q \neq \emptyset$. In the analytic case the accessibility property is equivalent to the full-rank condition (exercise).

Example: Stopping a train (1/2)

- The control system has the form

$$\dot{x} = f_1(x) + u f_2(x), \qquad x = (x_1, x_2) \in \mathbb{R}^2, \quad |u| \leq 1,$$
$$f_1 = x_2 \frac{\partial}{\partial x_1}, \qquad f_2 = \frac{\partial}{\partial x_2}.$$

- We have $[f_1, f_2] = -\frac{\partial}{\partial x_1}$, whence the system $\mathcal{F} = \{f_1 + u f_2 \mid u \in [-1, 1]\}$ is full-rank: $\mathrm{Lie}_x(\mathcal{F}) = \mathrm{span}\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right)(x) = T_x\mathbb{R}^2 \qquad \forall x \in \mathbb{R}^2$.
- Thus

$$\mathcal{O}_x = \mathbb{R}^2 \qquad \forall x \in \mathbb{R}^2.$$

- In order to find the attainable sets, we compute trajectories of the system with a constant control $u \neq 0$: they are the parabolas

$$\frac{x_2^2}{2} = u x_1 + C.$$

Example: Stopping a train (1/2)

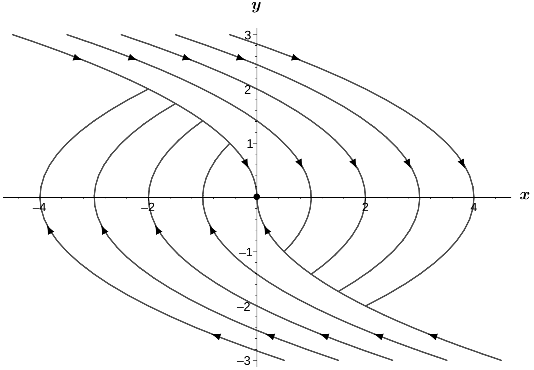


Figure: Reaching the origin from an arbitrary initial point

- Now it is visually obvious that the system is controllable.

Example: Markov–Dubins car (1/2)

- The control system has the form

$$\dot{q} = f_1(q) + u f_2(q), \qquad q = (x, y, \theta) \in M = \mathbb{R}^2 \times S^1, \quad |u| \le 1,$$

$$f_1 = \cos\theta \frac{\partial}{\partial x} + \sin\theta \frac{\partial}{\partial y}, \qquad f_2 = \frac{\partial}{\partial \theta}.$$

- We have

$$[f_1, f_2] = \sin\theta \frac{\partial}{\partial x} - \cos\theta \frac{\partial}{\partial y} =: f_3.$$

- Thus the system $\mathcal{F} = \{f_1 + u f_2 \mid u \in [-1, 1]\}$ is full-rank:

$$\mathrm{Lie}_q(\mathcal{F}) = \mathrm{span}(f_1(q), f_2(q), f_3(q)) = T_q M \qquad \forall q \in M,$$

consequently,

$$\mathcal{O}_q = M \qquad \forall q \in M.$$

- In order to describe the attainable sets, we replace the initial system $\mathcal{F}$ by a restricted system $\mathcal{F}_1 = \{f_1 \pm f_2\} \subset \mathcal{F}$ and prove that $\mathcal{F}_1$ is controllable (then $\mathcal{F}$ is controllable as well).

Example: Markov–Dubins car (2/2)

- Trajectories of the restricted system $\dot{x} = \cos\theta$, $\dot{y} = \sin\theta$, $\dot{\theta} = \pm 1$, have the form

$$\theta = \theta_0 \pm t, \qquad x = x_0 \pm (\sin(\theta_0 \pm t) - \sin\theta_0), \qquad y = y_0 \pm (\cos\theta_0 - \cos(\theta_0 \pm t)).$$

- These trajectories are periodic: $e^{(t+2\pi n)(f_1 \pm f_2)} = e^{t(f_1 \pm f_2)}, \qquad t \in \mathbb{R}, \quad n \in \mathbb{Z}$. So a shift along the fields $f_1 \pm f_2$ in the negative time can be obtained as a shift in the positive time.
- Consequently, if we introduce the system $\mathcal{F}_2 = \{f_1 \pm f_2, \ -f_1 \pm f_2\}$, then we get

$$\mathcal{A}_q(\mathcal{F}_2) = \mathcal{A}_q(\mathcal{F}_1), \qquad q \in M.$$

- But the system $\mathcal{F}_2$ is symmetric and full-rank, thus $\mathcal{A}_q(\mathcal{F}_2) = \mathcal{O}_q(\mathcal{F}_2) = M$, whence

$$\mathcal{A}_q(\mathcal{F}) = \mathcal{A}_q(\mathcal{F}_1) = M \text{ for all } q \in M.$$

  That is, the Markov–Dubins car is completely controllable in the space $\mathbb{R}^2 \times S^1$.

Statement of optimal control problem

- We consider the following *optimal control problem*:

$$\dot{q} = f(q, u), \qquad q \in M, \quad u \in U \subset \mathbb{R}^m, \tag{1}$$

$$q(0) = q_0, \qquad q(t_1) = q_1, \tag{2}$$

$$J[u] = \int_0^{t_1} \varphi(q, u)\, dt \to \min, \tag{3}$$

$$t_1 \text{ fixed or free.}$$

- A solution $q(t)$, $t \in [0, t_1]$, to this problem is said to be *(globally) optimal*.
- The following assumptions are made for the dynamics $f(q, u)$:
  - the mapping $q \mapsto f(q, u)$ is smooth for any $u \in U$,
  - the mapping $(q, u) \mapsto f(q, u)$ is continuous for any $q \in M$, $u \in \mathrm{cl}(U)$,
  - the mapping $(q, u) \mapsto \frac{\partial f}{\partial q}(q, u)$ is continuous for any $q \in M$, $u \in \mathrm{cl}(U)$.
- The same assumptions are made for the function $\varphi(q, u)$ that determines the cost functional J.
- Admissible control is $u \in L^\infty([0, t_1], U)$.

Reduction to the study of attainable sets

- In order to include the functional J into dynamics of the system, introduce a new variable equal to the running value of the cost functional along a trajectory $q_u(t)$:

$$y(t) = \int_0^t \varphi(q, u)\, dt.$$

- Respectively, we introduce an extended state $\widehat{q} = \begin{pmatrix} y \\ q \end{pmatrix} \in \mathbb{R} \times M$ that satisfies an *extended control system*

$$\frac{d\widehat{q}}{dt} = \begin{pmatrix} \dot{y} \\ \dot{q} \end{pmatrix} = \begin{pmatrix} \varphi(q, u) \\ f(q, u) \end{pmatrix} =: \widehat{f}(\widehat{q}, u).$$

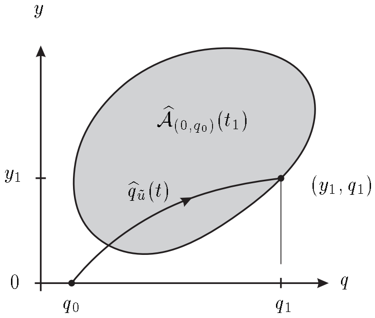
- The boundary conditions for this system are

$$\widehat{q}(0) = \begin{pmatrix} 0 \\ q_0 \end{pmatrix}, \qquad \widehat{q}(t_1) = \begin{pmatrix} J \\ q_1 \end{pmatrix}.$$

Reduction to the study of attainable sets

- A trajectory $q_{\tilde{u}}(t)$ is optimal for the optimal control problem with fixed time t_1 if and only if the corresponding trajectory $\widehat{q}_{\tilde{u}}(t)$ of the extended system comes to a point (y_1, q_1) of the attainable set $\widehat{\mathcal{A}}_{(0,q_0)}(t_1)$ such that

$$\widehat{\mathcal{A}}_{(0,q_0)}(t_1) \cap \{(y, q_1) \mid y < y_1\} = \emptyset.$$



- For the problem with free terminal time an analogous condition is written for the attainable set $\widehat{\mathcal{A}}_{(0,q_0)}$.

Corollary

If the attainable set $\widehat{\mathcal{A}}_{(0,q_0)}(t_1)$ *is compact and* $q_1 \in \mathcal{A}_{q_0}(t_1)$*, then the optimal control problem* (1)–(3) *with fixed time* t_1 *has a solution.*

Theorem (Filippov)

Suppose that control system (1) *satisfies the hypotheses:*

(1) *the set* U *is compact,*

(2) *the set* $f(q, U)$ *is convex for all* $q \in M$*,*

(3) *there exists a compact set* $K \subset M$ *such that for all* $q \in M \backslash K$*,* $u \in U$ *there holds the equality* $f(q, u) = 0$.

Then the attainable sets $\mathcal{A}_{q_0}(t)$*,* $\mathcal{A}_{q_0}(\leq t)$ *are compact for any* $q_0 \in M$*,* $t > 0$.

Proof.

See A.A. Agrachev, Yu.L. Sachkov, *Control theory from the geometric viewpoint*,
А.А. Аграчев, Ю. Л. Сачков, *Геометрическая теория управления*. □

Corollary

Let the optimal control problem (1)–(3) *satisfy the hypotheses:*

(1) *the set* U *is compact,*

(2) *the set* $\left\{ \begin{pmatrix} \varphi(q,u) \\ f(q,u) \end{pmatrix} \mid u \in U \right\}$ *is convex for all* $q \in M$*,*

(3) *there exists a compact set* $K \subset \mathbb{R} \times M$ *such that* $\widehat{\mathcal{A}}_{(0,q_0)}(t_1) \subset K$*,*

(4) $q_1 \in \mathcal{A}_{q_0}(t_1)$.

Then the problem (1)–(3) *with fixed time* t_1 *has a solution.*

Proof of the existence conditions for optimal control problem

- *Proof*. There exists a compact set $K' \subset \mathbb{R} \times M$ such that $K \subset \operatorname{int} K'$. Take a function $a \in C^\infty(\mathbb{R} \times M)$ such that

$$a\big|_K \equiv 1, \qquad a\big|_{(\mathbb{R}\times M)\setminus K'} \equiv 0.$$

- Consider a new extended control system:

$$\frac{d\widehat{q}}{dt} = a(\widehat{q})\widehat{f}(\widehat{q}, u), \qquad \widehat{q} \in \mathbb{R} \times M, \quad u \in U.$$

- This system has compact attainable sets for time t_1 , which coincide with the corresponding attainable sets of the extended system.
- Then optimal control problem (1)–(3) has a solution (by Filippov's theorem). □

Existence of solutions to time-optimal problem

Now consider a *time-optimal problem*

$$\dot{q} = f(q,u), \qquad q \in M, \quad u \in U \subset \mathbb{R}^m, \tag{4}$$

$$q(0) = q_0, \qquad q(t_1) = q_1, \tag{5}$$

$$t_1 \to \min. \tag{6}$$

Corollary

Let the following conditions hold:

(1) *the set U is compact,*

(2) *the set $f(q,U)$ is convex for all $q \in M$,*

(3) *there exist $t_1 > 0$ and a compact set $K \subset M$ such that*

$$q_1 \in \mathcal{A}_{q_0}(\leq t_1) \subset K.$$

Then time-optimal problem (4)–(6) *has a solution.*

Elements of symplectic geometry

- Let M be an n-dimensional smooth manifold. Then the disjoint union of its tangent spaces $TM = \bigsqcup_{q\in M} T_qM = \{(q,v) \mid q \in M,\ v \in T_qM\}$ is called its *tangent bundle*.
- If $(q_1,\ldots,q_n)$ are local coordinates on M, then any tangent vector $v \in T_qM$ has a decomposition $v = \sum_{i=1}^n v_i \frac{\partial}{\partial q_i}$. So $(q_1,\ldots,q_n;\ v_1,\ldots,v_n)$ are local coordinates on TM, which is thus a $2n$-dimensional smooth manifold.
- For any point $q \in M$, the dual space $(T_qM)^* = T_q^*M$ is called the *cotangent space* to M at q. Thus T_q^*M consists of linear forms on T_qM. The disjoint union $T^*M = \bigsqcup_{q\in M} T_q^*M = \{(q,p) \mid q \in M,\ p \in T_q^*M\}$ is called the *cotangent bundle*.
- If $(q_1,\ldots,q_n)$ are local coordinates on M, then any covector $\lambda \in T^*M$ has a decomposition $\lambda = \sum_{i=1}^n p_i\, dq_i$. Thus $(q_1,\ldots,q_n;\ p_1,\ldots,p_n)$ are local coordinates on T^*M called the *canonical coordinates*. So T^*M is a smooth $2n$-dimensional manifold.
- The *canonical projection* is the mapping $\pi\colon\ T^*M \to M, \quad T_q^*M \ni \lambda \mapsto q \in M.$

Elements of symplectic geometry

- The *Liouville (tautological) differential 1-form* $s \in \Lambda^1(T^*M)$ is defined as follows:

$$\langle s_\lambda, w \rangle = \langle \lambda, \pi_* w \rangle, \qquad \lambda \in T^*M, \quad w \in T_\lambda(T^*M).$$

  In the canonical coordinates on T^*M, $s = p\,dq$.
- The canonical *symplectic structure* on T^*M is the differential 2-form $\sigma = ds \in \Lambda^2(T^*M)$. In the canonical coordinates $\sigma = dp \wedge dq = \sum_{i=1}^n dp_i \wedge dq_i$.
- A *Hamiltonian* (*Hamiltonian function*) is an arbitrary function $h \in C^\infty(T^*M)$.
- The *Hamiltonian vector field* $\vec{h} \in \mathrm{Vec}(T^*M)$ with the Hamiltonian function h is defined by the equality $dh = \sigma(\,\cdot\,, \vec{h})$. In the canonical coordinates:

$$h = h(q, p),$$

$$\vec{h} = \frac{\partial h}{\partial p}\frac{\partial}{\partial q} - \frac{\partial h}{\partial q}\frac{\partial}{\partial p} = \sum_{i=1}^n \left(\frac{\partial h}{\partial p_i}\frac{\partial}{\partial q_i} - \frac{\partial h}{\partial q_i}\frac{\partial}{\partial p_i} \right).$$

Elements of symplectic geometry

- The corresponding *Hamiltonian system of ODEs* is
$$\dot{\lambda} = \vec{h}(\lambda), \qquad \lambda \in T^*M.$$
- In the canonical coordinates:
$$\begin{cases} \dot{q} = \dfrac{\partial h}{\partial p}, \\ \dot{p} = -\dfrac{\partial h}{\partial q}, \end{cases} \qquad \text{or} \qquad \begin{cases} \dot{q}_i = \dfrac{\partial h}{\partial p_i}, \\ \dot{p}_i = -\dfrac{\partial h}{\partial q_i}, \end{cases} \qquad i = 1, \dots, n.$$
- The *Poisson bracket* of Hamiltonians $h, g \in C^\infty(T^*M)$ is the Hamiltonian $\{h, g\} \in C^\infty(T^*M)$ defined by the equalities
$$\{h, g\} = \vec{h}g = \sigma(\vec{h}, \vec{g}).$$
- In the canonical coordinates:
$$\{h, g\} = \frac{\partial h}{\partial p}\frac{\partial g}{\partial q} - \frac{\partial h}{\partial q}\frac{\partial g}{\partial p} = \sum_{i=1}^{n}\left(\frac{\partial h}{\partial p_i}\frac{\partial g}{\partial q_i} - \frac{\partial h}{\partial q_i}\frac{\partial g}{\partial p_i}\right).$$

Lemma

Let $a, b, c \in C^\infty(T^*M)$ *and* $\alpha, \beta \in \mathbb{R}$. *Then:*

(1) $\{a, b\} = -\{b, a\}$,

(2) $\{a, a\} = 0$,

(3) $\{\{a, b\}, c\} + \{\{b, c\}, a\} + \{\{c, a\}, b\} = 0$,

(4) $\{\alpha a + \beta b, c\} = \alpha\{a, c\} + \beta\{b, c\}$,

(5) $\{ab, c\} = \{a, c\}b + a\{b, c\}$,

(6) $[\vec{a}, \vec{b}] = \vec{d},\ d = \{a, b\}$.

Theorem (Noether)

Let $a, h \in C^\infty(T^*M)$. *Then*

$$a(e^{t\vec{h}}(\lambda)) \equiv \text{const} \quad \Leftrightarrow \quad \{h, a\} = 0.$$

Elements of symplectic geometry

Now we describe the last construction of symplectic geometry necessary for us — *linear on fibers of T^*M Hamiltonians*. Let $X \in \mathrm{Vec}(M)$. The corresponding linear on fibers of T^*M Hamiltonian is defined as follows: $h_X(\lambda) = \langle \lambda, X(q) \rangle, \qquad q = \pi(\lambda)$.
In the canonical coordinates:

$$X = \sum_{i=1}^{n} X_i \frac{\partial}{\partial q_i}, \qquad h_X(q,p) = \sum_{i=1}^{n} p_i X_i.$$

Lemma

Let $X, Y \in \mathrm{Vec}(M)$. *Then:*

(1) $\{h_X, h_Y\} = h_{[X,Y]}$,

(2) $[\vec{h}_X, \vec{h}_Y] = \vec{h}_{[X,Y]}$,

(3) $\pi_* \vec{h}_X = X$.

The vector field $\vec{h}_X \in \mathrm{Vec}(T^*M)$ is called the *Hamiltonian lift* of the vector field $X \in \mathrm{Vec}(M)$.

Hamiltonians of Pontryagin maximum principle

- Return to the optimal control problem

$$
\begin{aligned}
&\dot q = f(q,u), \qquad q \in M, \quad u \in U \subset \mathbb{R}^m,\\
&q(0) = q_0, \qquad q(t_1) = q_1,\\
&J = \int_0^{t_1} \varphi(q,u)\,dt \to \min,\\
&t_1 \text{ fixed.}
\end{aligned}
$$

- Define a family of *Hamiltonians of PMP*

$$
h_u^\nu(\lambda) = \langle \lambda, f(q,u) \rangle + \nu \varphi(q,u), \qquad \nu \in \mathbb{R}, \quad u \in U, \quad \lambda \in T^*M, \quad q = \pi(\lambda).
$$

Statement of Pontryagin maximum principle

Theorem (PMP)

If a control $u(t)$ *and the corresponding trajectory* $q(t), t \in [0, t_1]$*, are optimal, then there exist a curve* $\lambda_t \in \mathrm{Lip}([0, t_1], T^*M)$*,* $\lambda_t \in T^*_{q(t)}M$*, and a number* $\nu \leq 0$ *such that the following conditions hold for almost all* $t \in [0, t_1]$*:*

(1) $\dot{\lambda}_t = \vec{h}^{\nu}_{u(t)}(\lambda_t)$,

(2) $h^{\nu}_{u(t)}(\lambda_t) = \max_{w \in U} h^{\nu}_{w}(\lambda_t)$,

(3) $(\lambda_t, \nu) \neq (0, 0)$.

If the terminal time t_1 is free, then the following condition is added to (1)–(3):

(4) $h^{\nu}_{u(t)}(\lambda_t) \equiv 0$.

A curve λ_t that satisfies PMP is called an *extremal*, a curve $q(t)$ — an *extremal trajectory*, a control $u(t)$ — an *extremal control*.

Exercises

1. When the topology of $M^{\mathcal{F}}$ is stronger than the topology of M?
2. When the topology of $\mathcal{O}_q$ induced by $M^{\mathcal{F}}$ is stronger than the topology of $\mathcal{O}_q$ induced by M
3. Construct examples of control systems having an attainable set of the following structure:
   - a smooth manifold without boundary,
   - a manifold with a smooth boundary,
   - a manifold with boundary having an angle singularity,
   - a manifold with boundary having a cusp singularity.
4. Prove in detail the induction step in Krener's theorem.
5. Prove that in the analytic case the accessibility property is equivalent to the full-rank condition.
6. Infer existence of time-optimal trajectories from Filippov's theorem.