Lie groups and Lie algebras. Controllability of linear and nonlinear systems *(Lecture 2)*

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«Geometric control theory, nonholonomic geometry, and their applications»

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1. Searching for the Ox:

Alone in the wilderness, lost in the jungle, the boy is searching, searching!
The swelling waters, the far-away mountains, and the unending path;
Exhausted and in despair, he knows not where to go,
He only hears the evening cicadas singing in the maple-woods.

Pu-ming, "The Ten Oxherding Pictures"

Reminder: Plan of the previous lecture

1. Examples of optimal control problems
2. Statements of the main problems of this course:
   2.1 controllability problem,
   2.2 optimal control problem.
3. Smooth manifolds and vector fields.

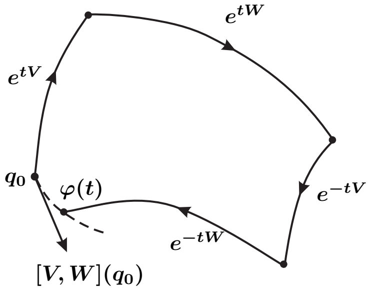
Plan of this lecture

1. Lie bracket of vector fields
2. Lie groups, Lie algebras, and left-invariant optimal control problems
3. Controllability of linear systems
4. Local controllability of nonlinear systems

Lie bracket of vector fields

- The *commutator* (*Lie bracket*) of the vector fields V, W at the point q_0 is defined as $[V, W](q_0) := \frac{1}{2}\ddot{\varphi}(0)$, so that

$$\varphi(t) = q_0 + t^2[V, W](q_0) + o(t^2), \qquad t \to 0.$$



- In local coordinates $[V, W] = \frac{\partial W}{\partial x} V - \frac{\partial V}{\partial x} W$.

Example: Car in the plane

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{pmatrix} = u \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix} + v \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \qquad V = \cos\theta \frac{\partial}{\partial x} + \sin\theta \frac{\partial}{\partial y}, \quad W = \frac{\partial}{\partial \theta}.$$

$$[V, W] = \frac{\partial W}{\partial q} V - \frac{\partial V}{\partial q} W = 0 \cdot V - \begin{pmatrix} 0 & 0 & -\sin\theta \\ 0 & 0 & \cos\theta \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \sin\theta \\ -\cos\theta \\ 0 \end{pmatrix}.$$

Another way of computing Lie brackets, via commutator of differential operators:

$$[V, W] = V \circ W - W \circ V = \left(\cos\theta \frac{\partial}{\partial x} + \sin\theta \frac{\partial}{\partial y} \right) \frac{\partial}{\partial \theta} - \frac{\partial}{\partial \theta} \left(\cos\theta \frac{\partial}{\partial x} + \sin\theta \frac{\partial}{\partial y} \right)$$

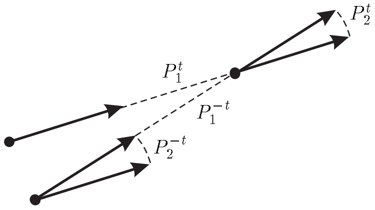
$$= \sin\theta \frac{\partial}{\partial x} - \cos\theta \frac{\partial}{\partial y}.$$

Example: Car in the plane

- Notice the visual meaning of the vector fields $V, W, [V, W]$ for the car:
  - V generates the motion forward
  - W generates rotations of the car
  - $[V, W]$ generates motion of the car in the direction perpendicular to its orientation.
- Choosing alternating motions of the car:

  $$\text{forward} \to \text{rotation counter-clockwise} \to \text{backward} \to \text{rotation clockwise},$$

  we can move the car infinitesimally in the forbidden direction. So the Lie bracket $[V, W]$ is generated by a car during parking manoeuvres in a limited space.



Lie groups

- A set G is called a *Lie group* if it is a smooth manifold endowed with a group structure such that the following mappings are smooth:

$$(g, h) \mapsto gh, \qquad g \mapsto g^{-1}.$$

  Let $\mathrm{Id} \in G$ denote the identity element of the group G.
- Denote by $\mathbb{R}^{n \times n}$ the set of al real $n \times n$ matrices. The set

$$\mathrm{GL}(n, \mathbb{R}) = \{g \in \mathbb{R}^{n \times n} \mid \det g \neq 0\}$$

  is a Lie group w.r.t. the matrix product, it is called the *general linear group*.
- A *linear Lie group* is a closed subgroup of $\mathrm{GL}(n, \mathbb{R})$.

Theorem

A closed subgroup of a Lie group is a Lie subgroup.

Lie algebras

- A set $\mathfrak{g}$ is called a *Lie algebra* if it is a vector space endowed with a binary operation $[\cdot\,,\,\cdot]$ called *Lie bracket* that satisfies the following properties:
  1. bilinearity: $[ax+by,z]=a[x,z]+b[y,z], \qquad x,\ y,\ z\in\mathfrak{g}, \quad a,\ b\in\mathbb{R}$,
  2. skew symmetry: $[x,y]=-[y,x], \qquad x,\ y\in\mathfrak{g}$,
  3. Jacobi identity: $[x,[y,z]]+[y,[z,x]]+[z,[x,y]]=0, \qquad x,\ y,\ z\in\mathfrak{g}$.
- For any element g of a Lie group G, the mapping $L_g\ :\ h\mapsto gh, \qquad G\to G$, is called the *left translation* by g. A vector field $X\in\mathrm{Vec}(G)$ is called *left-invariant* if it is preserved by left translations: $(L_g)_*(X(h))=X(gh), \qquad g,\ h\in G$.
- Lie bracket of left-invariant vector fields is left-invariant. Thus left-invariant vector fields on a Lie group G form a Lie algebra $\mathfrak{g}$ called the *Lie algebra of the Lie group* G.
- There is a linear isomorphism $\mathfrak{g}\cong T_{\mathrm{Id}}G$, which defines the structure of a Lie algebra on $T_{\mathrm{Id}}G$. Thus the tangent space $T_{\mathrm{Id}}G$ is also called the Lie algebra of the Lie group G.

Examples of Lie groups G and their Lie algebras $\mathfrak{g}$

- Denote the vector space $\mathbb{R}^{n\times n} = \{A = (a_{ij}) \mid a_{ij} \in \mathbb{R},\ i, j = 1, \ldots, n\}$.
- The *general linear group*: $\mathrm{GL}(n, \mathbb{R}) = \{A \in \mathbb{R}^{n\times n} \mid \det A \neq 0\}$, its Lie algebra $\mathfrak{gl}(n, \mathbb{R}) = \mathbb{R}^{n\times n}$ with Lie bracket $[A, B] = AB - BA$.
- The *special linear group*: $\mathrm{SL}(n, \mathbb{R}) = \{A \in \mathbb{R}^{n\times n} \mid \det A = 1\}$, $\mathfrak{sl}(n, \mathbb{R}) = \{A \in \mathbb{R}^{n\times n} \mid \operatorname{tr} A = 0\}$.
- The *special orthogonal group*: $\mathrm{SO}(n) = \{A \in \mathbb{R}^{n\times n} \mid AA^{\mathsf{T}} = \mathrm{Id},\ \det A = 1\}$, $\mathfrak{so}(n) = \{A \in \mathbb{R}^{n\times n} \mid A + A^{\mathsf{T}} = 0\}$.
- The *special Euclidean group*:
$$\mathrm{SE}(n) = \left\{ \begin{pmatrix} Y & b \\ 0 & 1 \end{pmatrix} \in \mathbb{R}^{(n+1)\times(n+1)} \mid Y \in \mathrm{SO}(n),\ b \in \mathbb{R}^n \right\} \subset \mathrm{GL}(n+1),$$
$$\mathfrak{se}(n) = \left\{ \begin{pmatrix} A & b \\ 0 & 0 \end{pmatrix} \mid A \in \mathfrak{so}(n),\ b \in \mathbb{R}^n \right\}.$$

Left-invariant vector fields and optimal control problems

- For a Lie group G, the tangent space is $T_g G = (L_g)_* T_{\mathrm{Id}} G, \qquad g \in G$.
- In the case of a linear Lie group $G \subset \mathrm{GL}(n, \mathbb{R})$, $(L_g)_* A = gA$, $g \in G$, $A \in T_{\mathrm{Id}} G$.
- Thus *left-invariant* vector fields on a linear Lie group G have the form

$$V(g) = gA, \qquad g \in G, \quad A \in T_{\mathrm{Id}} G.$$

- A control system on a Lie group G

$$\dot{g} = f(g, u), \qquad g \in G, \quad u \in U,$$

  is called *left-invariant* if its dynamics is preserved by left translations:

$$(L_h)_* f(g, u) = f(hg, u), \qquad g,\ h \in G, \quad u \in U.$$

- An optimal control problem on G is called *left-invariant* if both its dynamics and the cost functional are preserved by left translations.
- If an optimal control problem is left-invariant on a Lie group, we can set $g(0) = \mathrm{Id}$.

Controllability of linear systems: Cauchy's formula

Linear control systems:

$$\dot{x} = Ax + \sum_{i=1}^{k} u_i b_i = Ax + Bu, \qquad x \in \mathbb{R}^n, \quad u = (u_1, \ldots, u_k) \in \mathbb{R}^k$$

Find solutions by the variation of constants method:

$$x(t) = e^{At} C(t), \qquad e^{At} = \sum\nolimits_{k=0}^{\infty} (At)^k / k!,$$

$$\dot{x} = Ae^{At} C + e^{At} \dot{C} = Ae^{At} C + Bu,$$

$$\dot{C}(t) = e^{-At} Bu(t) \quad \Rightarrow \quad C(t) = \int_0^t e^{-As} Bu(s)\, ds + C_0,$$

$$x(t) = e^{At} \left(\int_0^t e^{-As} Bu(s)\, ds + C_0 \right), \qquad x(0) = C_0 = x_0,$$

$$x(t) = e^{At} \left(x_0 + \int_0^t e^{-As} Bu(s)\, ds \right)$$ — *Cauchy's formula* for linear systems.

Kalman controllability test

A control system in $\mathbb{R}^n$ is called *globally controllable* from a point $x_0 \in \mathbb{R}^n$ for time $t_1 > 0$ (for time not greater than t_1) if $\mathcal{A}_{x_0}(t_1) = \mathbb{R}^n$ (resp. $\mathcal{A}_{x_0}(\leq t_1) = \mathbb{R}^n$).

Theorem (R. Kalman)

Let $t_1 > 0$ and $x_0 \in \mathbb{R}^n$. A linear system $\dot{x} = Ax + Bu$ is globally controllable from x_0 for time t_1 iff $\operatorname{span}(B, AB, \ldots, A^{n-1}B) = \mathbb{R}^n$.

Proof of the Kalman controllability test

- The mapping $L^1 \ni u(\cdot) \mapsto x(t_1) \in \mathbb{R}^n$ is affine, thus its image $\mathcal{A}_{x_0}(t_1)$ is an affine subspace of $\mathbb{R}^n$.
- Rewrite the definition of controllability taking into account Cauchy's formula:

$$\mathcal{A}_{x_0}(t_1) = \mathbb{R}^n \Leftrightarrow \operatorname{Im}\ e^{At_1}\left(x_0 + \int_0^{t_1} e^{-At} Bu(t)\,dt\right) = \mathbb{R}^n$$

$$\Leftrightarrow \operatorname{Im} \int_0^{t_1} e^{-At} Bu(t)\,dt = \mathbb{R}^n.$$

- *Necessity*. Let $\mathcal{A}_{x_0}(t_1) = \mathbb{R}^n$, but $\operatorname{span}(B, AB, \ldots, A^{n-1}B) \neq \mathbb{R}^n$.
- Then $\exists\ 0 \neq p \in \mathbb{R}^{n*}$ s.t. $pA^iB = 0, \quad i = 0, \ldots, n-1$.
- By the Cayley–Hamilton theorem, $A^n = \sum_{i=0}^{n-1} \alpha_i A^i$ for some $\alpha_i \in \mathbb{R}$. Thus

$$A^m = \sum_{i=0}^{n-1} \beta_i^m A^i, \quad \beta_i^m \in \mathbb{R}, \quad m = 0, 1, 2, \ldots.$$

Proof of the Kalman controllability test

- Consequently,

$$pA^mB = \sum_{i=0}^{n-1} \beta_i^m pA^iB = 0, \qquad m = 0, 1, 2, \dots,$$

$$pe^{-At}B = p\sum_{m=0}^{\infty} \frac{(-At)^m}{m!}B = 0,$$

  and $\operatorname{Im} \int_0^{t_1} e^{-At}Bu(t)\,dt \neq \mathbb{R}^n$, a contradiction.
- Necessity proved.

Proof of the Kalman controllability test

- *Sufficiency*. Let $\mathrm{span}(B, AB, \dots, A^{n-1}B) = \mathbb{R}^n$, but $\mathrm{Im} \int_0^{t_1} e^{-At} Bu(t)\, dt \neq \mathbb{R}^n$.
- Then $\exists\ 0 \neq p \in \mathbb{R}^{n*}$ s.t.

$$p \int_0^{t_1} e^{-At} Bu(t)\, dt = 0 \qquad \forall u \in L^1([0, t_1], \mathbb{R}^k).$$

- Let $e_1, \dots, e_k$ be the standard frame in $\mathbb{R}^k$. For any $\tau \in [0, t_1]$ and any $i = 1, \dots, k$, define the following controls:

$$u(t) = \begin{cases} e_i, & t \in [0, \tau], \\ 0, & t \in (\tau, t_1]. \end{cases}$$

- We have $\int_0^{t_1} e^{-At} Bu(t)\, dt = \int_0^{\tau} e^{-At} b_i\, dt = \frac{\mathrm{Id} - e^{-A\tau}}{A} b_i$, thus $p \frac{\mathrm{Id} - e^{-A\tau}}{A} B = 0$.
- We differentiate successively previous identity at $\tau = 0$ and obtain $pB = pAB = \cdots = pA^{n-1}B = 0$, a contradiction. □

Final remarks on controllability of linear systems

- The control used in the proof of Kalman's controllability test is piecewise constant. Thus if Kalman's condition holds, then linear system is controllable for any time $t_1 > 0$ with piecewise-constant controls.
- For linear systems, controllability for the class of admissible controls $u(\cdot) \in L^1$ is equivalent to controllability for any class of admissible controls $u(\cdot) \in L$ where L is a linear subspace of L^1 containing piecewise constant functions.
- The following conditions are equivalent for a linear system:
  - the Kalman controllability condition
  - $\forall\, t_1 > 0\ \forall\, x_0 \in \mathbb{R}^n$ the system is globally controllable from x_0 for time t_1
  - $\forall\, t_1 > 0\ \forall\, x_0 \in \mathbb{R}^n$ the system is globally controllable from x_0 for time not greater than t_1
  - $\exists\, t_1 > 0\ \exists\, x_0 \in \mathbb{R}^n$ such the linear system is globally controllable from x_0 for time t_1
  - $\exists\, t_1 > 0\ \exists\, x_0 \in \mathbb{R}^n$ such the linear system is globally controllable from x_0 for time not greater than t_1.
- In these cases a linear system is called *controllable*.

Local controllability of nonlinear systems

- Nonlinear system

$$\dot{x} = f(x, u), \qquad x \in \mathbb{R}^n, \quad u \in U \subset \mathbb{R}^m. \tag{1}$$

- A point $(x_0, u_0) \in \mathbb{R}^n \times U$ is called an *equilibrium point* of system (1) if $f(x_0, u_0) = 0$. Let $u_0 \in \operatorname{int} \ U$.
- *Linearization* of system (1) at the equilibrium point (x_0, u_0):

$$\dot{y} = Ay + Bv, \qquad y \in \mathbb{R}^n, \quad v \in \mathbb{R}^m, \tag{2}$$

$$A = \left.\frac{\partial f}{\partial x}\right|_{(x_0,u_0)}, \quad B = \left.\frac{\partial f}{\partial u}\right|_{(x_0,u_0)}.$$

Theorem (linearization principle for controllability)

If linearization (2) at an equilibrium point (x_0, u_0) is controllable, then for any $t_1 > 0$ nonlinear system (1) is locally controllable at the point x_0 for time t_1:

$$\forall \, t_1 > 0 \quad x_0 \in \operatorname{int} \mathcal{A}_{x_0}(t_1).$$

Proof of linearization principle for controllability

- Fix any $t_1 > 0$.
- Let $e_1, \dots, e_n$ be the standard frame in $\mathbb{R}^n$. Since linearization is controllable, then

$$\forall i = 1, \dots, n \quad \exists v_i \in L^\infty([0, t_1], \mathbb{R}^m) : \quad y_{v_i}(0) = 0, \quad y_{v_i}(t_1) = e_i. \tag{3}$$

- Construct the following family of controls:

$$u(z, t) = u_0 + z_1 v_1(t) + \cdots + z_n v_n(t), \quad z = (z_1, \dots, z_n) \in \mathbb{R}^n.$$

- Since $u_0 \in \operatorname{int} U$, for sufficiently small $|z|$ and any $t \in [0, t_1]$, the control $u(z, t) \in U$, thus it is admissible for the nonlinear system.
- Consider the corresponding family of trajectories of the nonlinear system:

$$x(z, t) = x_{u(z,\cdot)}(t), \quad x(z, 0) = x_0, \quad z \in B,$$

where B is a small open ball in $\mathbb{R}^n$ centred at the origin.

Proof of linearization principle for controllability

- Since

$$x(z, t_1) \in \mathcal{A}_{x_0}(t_1), \quad z \in B,$$

  then the mapping

$$F\colon z \mapsto x(z, t_1), \quad B \to \mathbb{R}^n$$

  satisfies the inclusion

$$F(B) \subset \mathcal{A}_{x_0}(t_1).$$

- It remains to show that $x_0 \in \operatorname{int} F(B)$. Define the matrix function

$$W(t) = \left.\frac{\partial x(z, t)}{\partial z}\right|_{z=0}.$$

- We show that $\det W(t_1) = \left.\frac{\partial F}{\partial z}\right|_{z=0} \neq 0$. This would imply that

$$x_0 = F(0) \in \operatorname{int} F(B) \subset \mathcal{A}_{x_0}(t_1).$$

Proof of linearization principle for controllability

- Differentiating the identity $\frac{\partial x}{\partial t} = f(x, u(z,t))$ w.r.t. z, we get

$$\frac{\partial}{\partial t}\left.\frac{\partial x}{\partial z}\right|_{z=0} = \left.\frac{\partial f}{\partial x}\right|_{(x_0,u_0)}\left.\frac{\partial x}{\partial z}\right|_{z=0} + \left.\frac{\partial f}{\partial u}\right|_{(x_0,u_0)}\left.\frac{\partial u}{\partial z}\right|_{z=0}$$

  since $u(0,t) \equiv u_0$ and $x(0,t) \equiv x_0$.
- Thus we get a matrix ODE $\dot{W}(t) = AW(t) + B(v_1(t), \dots, v_n(t))$ with the initial condition $W(0) = \left.\frac{\partial x(z,0)}{\partial z}\right|_{z=0} = \left.\frac{\partial x_0}{\partial z}\right|_{z=0} = 0$.
- This matrix ODE means that columns of the matrix $W(t)$ are solutions to the linearised system with the control $v_i(t)$. Since $y_{v_i}(t_1) = e_i$, we have $W(t_1) = (e_1, \dots, e_n)$, so $\det W(t_1) = 1 \neq 0$.
- By the implicit function theorem, we have $x_0 \in \operatorname{int} F(B)$, thus $x_0 \in \operatorname{int} \mathcal{A}_{x_0}(t_1)$. $\square$

Example: Application of the linearization principle for controllability

$$\dot{x} = uf_1(x) + (1-u)f_2(x), \qquad x = (x_1, x_2) \in \mathbb{R}^2, \quad u \in [0,1], \tag{4}$$

$$f_1(x) = \frac{\partial}{\partial x_1}, \qquad f_2(x) = -\frac{\partial}{\partial x_1} + x_1 \frac{\partial}{\partial x_2}. \tag{5}$$

- $(x^0, u^0) = (0, \frac{1}{2})$ is an equilibrium point and $u^0 \in \mathrm{int}([0,1])$.
- The linearization of system (4) at the equilibrium point (x^0, u^0) has the form

$$\dot{y} = Ay + Bv, \qquad y \in \mathbb{R}^2, \quad v \in \mathbb{R}, \tag{6}$$

$$A = \begin{pmatrix} 0 & 0 \\ \frac{1}{2} & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 2 \\ 0 \end{pmatrix}.$$

- Check Kalman's condition: $\mathrm{rank}(B, AB) = \mathrm{rank} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} = 2$, thus linear system (6) is controllable.
- So nonlinear system (4) is locally controllable at the point x^0 for any time $t_1 > 0$.

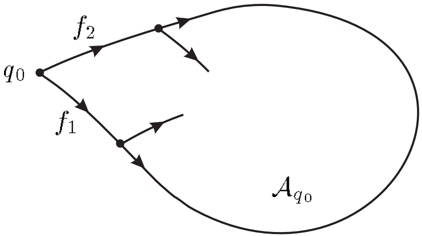
Orbit of a control system

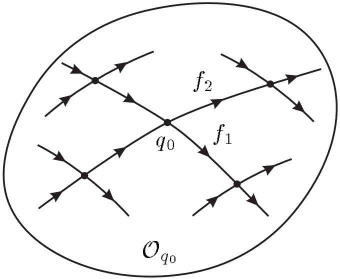
- A *control system* on a smooth manifold M is an arbitrary set of vector fields $\mathcal{F} \subset \mathrm{Vec}(M)$.
- The *attainable set* of the system $\mathcal{F}$ from a point $q_0 \in M$:

$$\mathcal{A}_{q_0} = \{e^{t_N f_N} \circ \cdots \circ e^{t_1 f_1}(q_0) \mid t_i \geq 0, \quad f_i \in \mathcal{F}, \quad N \in \mathbb{N}\}.$$

- The *orbit* of the system $\mathcal{F}$ through the point q_0:

$$\mathcal{O}_{q_0} = \{e^{t_N f_N} \circ \cdots \circ e^{t_1 f_1}(q_0) \mid t_i \in \mathbb{R}, \quad f_i \in \mathcal{F}, \quad N \in \mathbb{N}\}.$$





Basic properties of attainable sets and orbits

1. $\mathcal{A}_{q_0} \subset \mathcal{O}_{q_0}$, obvious
2. $\mathcal{O}_{q_0}$ has a "simpler" structure than $\mathcal{A}_{q_0}$
3. $\mathcal{A}_{q_0}$ has a "reasonable" structure inside $\mathcal{O}_{q_0}$.

- A system $\mathcal{F}$ is called *symmetric* if $\mathcal{F} = -\mathcal{F}$.

4. $\mathcal{F} = -\mathcal{F} \quad \Rightarrow \quad \mathcal{A}_{q_0} = \mathcal{O}_{q_0}$.

Exercises 1

1. Show that the following sets are linear Lie groups:
   - the *special linear group*

$$\mathrm{SL}(n,\mathbb{R}) = \{g \in \mathrm{GL}(n,\mathbb{R}) \mid \det g = 1\},$$

   - the *special orthogonal group*

$$\mathrm{SO}(n) = \left\{g \in \mathrm{GL}(n,\mathbb{R}) \mid \det g = 1,\ g^{-1} = g^T\right\},$$

   - the *special Euclidean group*

$$\mathrm{SE}(n) = \left\{\begin{pmatrix} A & v \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}(n+1,\mathbb{R}) \mid A \in \mathrm{SO}(n),\ v \in \mathbb{R}^n\right\},$$

   - the *special unitary group*

$$\mathrm{SU}(n) = \left\{\begin{pmatrix} A & B \\ -B & A \end{pmatrix} \in \mathbb{R}^{2n\times 2n} \mid A, B \in \mathbb{R}^{n\times n},\ AA^T + BB^T = \mathrm{Id},\right.$$

$$\left. BA^T - AB^T = 0,\ \det(A + iB) = 1\right\},$$

   compute their dimensions.

Exercises 2

2. Prove that the 2D sphere S^2 is not a Lie group. Hint: there is no smooth nowhere vanishing vector field on S^2.
3. Prove that the product

$$(x_1, y_1, z_1) \cdot (x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2 + (x_1 y_2 - x_2 y_1)/2),$$
$$(x_i, y_i, z_i) \in \mathbb{R}^3, \qquad i = 1, 2,$$

   turns $\mathbb{R}^3$ into a Lie group called the *Heisenberg group*. Show that Dido's problem is left-invariant on this Lie group.
4. For the sub-Riemannian problem on the group of motions of the plane, find equilibrium points and study controllability of linearization at these points.
5. For Euler's elastic problem, find equilibrium points and study controllability of linearization at these points.
6. Prove local and global controllability of system (4), (5) geometrically, with the help of the phase portraits of the vector fields f_1, f_2.