Examples and statements of control problems
(Lecture 1)

Yuri L. Sachkov

yusachkov@gmail.com

«Geometric control theory, nonholonomic geometry, and their applications»

Lecture course in Dept. of Mathematics and Mechanics

Lomonosov Moscow State University

2 October 2024

Plan of the course

1. Examples and statements of control problems
2. Controllability of linear systems, local controllability of nonlinear systems.
3. Orbit theorem, Rashevsky-Chow, Frobenius, Krener theorems.
4. Pontryagin maximum principle on manifolds and Lie groups.
5. Sub-Riemannian geometry on Lie groups.
6. Applications in mechanics, robotics, vision, probability theory.
7. Measurable sets and functions, Carathéodory differential equations
8. Filippov's sufficient conditions for the existence of optimal control
9. Elements of chronological calculus by R.V.Gamkrelidze—A.A.Agrachev
10. Differential forms, elements of symplectic geometry
11. Proof of Pontryagin maximum principle on manifolds: geometric form, optimal control problems with different boundary conditions
12. (Sub)Lorentzian problems on Lie groups.
13. Almost Riemannian problems.

Literature 1

Primary sources:

1. A.A. Agrachev, Yu.L. Sachkov, *Control theory from the geometric viewpoint*, Berlin Heidelberg New York Tokyo. Springer-Verlag. 2004.
   Russian translation: А.А. Аграчев, Ю. Л. Сачков, *Геометрическая теория управления*, М.: Физматлит, 2005.
2. Сачков Ю.Л. *Введение в геометрическую теорию управления*, М.: URSS, 2021.
   English translation: Yu.L. Sachkov, *Introduction to geometric control*, Springer, 2022.

Literature 2

Secondary sources:

1. V. Jurdjevic, *Geometric Control Theory*, Cambridge University Press, 1997.
2. R. Montgomery, *A tour of subriemannnian geometries, their geodesics and applications*, Amer. Math. Soc., 2002
3. А. А. Аграчев, Некоторые вопросы субримановой геометрии, *УМН*, 71:6(432) (2016), 3–36
   English translation: A. A. Agrachev, Topics in sub-Riemannian geometry, *Russian Math. Surveys*, 71:6 (2016), 989–1019
4. A. Agrachev, D. Barilari, U. Boscain, *A Comprehensive Introduction to sub-Riemannian Geometry from Hamiltonian viewpoint*, Cambridge Studies in Advanced Mathematics, Cambridge Univ. Press, 2019

Web page of the course:

`http://control.botik.ru/?page_id=3574`

Any questions:

`yusachkov@gmail.com`

Plan of lecture

1. Examples of optimal control problems
2. Statements of the main problems of this course:
   2.1 controllability problem,
   2.2 optimal control problem.
3. Smooth manifolds and vector fields.

Examples of optimal control problems: 1. Stopping a train

Given:

- material point of mass $m > 0$ with coordinate $x \in \mathbb{R}$
- force F bounded by the absolute value by $F_{\max} > 0$
- initial position x_0 and initial velocity $\dot{x}_0$ of the material point

Find:

- force F that steers the point to the origin with zero velocity, for a minimal time.

$$
\begin{aligned}
&\dot{x}_1 = x_2, \qquad (x_1, x_2) \in \mathbb{R}^2, \\
&\dot{x}_2 = u, \qquad |u| \leq 1, \\
&(x_1, x_2)(0) = (x_0, \dot{x}_0), \qquad (x_1, x_2)(t_1) = (0, 0), \\
&t_1 \to \min .
\end{aligned}
$$

2. Control of linear oscillator

Given:

- pendulum that performs small oscillations under the action of a force bounded by the absolute value

Find:

- force that steers the pendulum from an arbitrary position and velocity to the stable equilibrium for a minimum time.

$$
\begin{aligned}
&\dot{x}_1 = x_2, \qquad x = (x_1, x_2) \in \mathbb{R}^2,\\
&\dot{x}_2 = -x_1 + u, \qquad |u| \le 1,\\
&x(0) = x^0, \qquad x(t_1) = 0,\\
&t_1 \to \min .
\end{aligned}
$$

3. The Markov–Dubins car

Given:

- model of a car given by a unit vector attached at a point $(x, y) \in \mathbb{R}^2$, with orientation $\theta \in S^1$
- The car moves forward with the unit velocity and can simultaneously rotate with an angular velocity $|\dot{\theta}| \leq 1$
- an initial and a terminal state of the car

Find:

- angular velocity in such a way that the time of motion is as minimum as possible.

$$
\begin{aligned}
&\dot{x} = \cos\theta, \qquad && q = (x, y, \theta) \in \mathbb{R}^2_{x,y} \times S^1_\theta = M, \\
&\dot{y} = \sin\theta, && |u| \leq 1, \\
&\dot{\theta} = u, && \\
&q(0) = q_0, && q(t_1) = q_1, \\
&t_1 \to \min .
\end{aligned}
$$

4. The sub-Riemannian problem on the group of motions of the plane

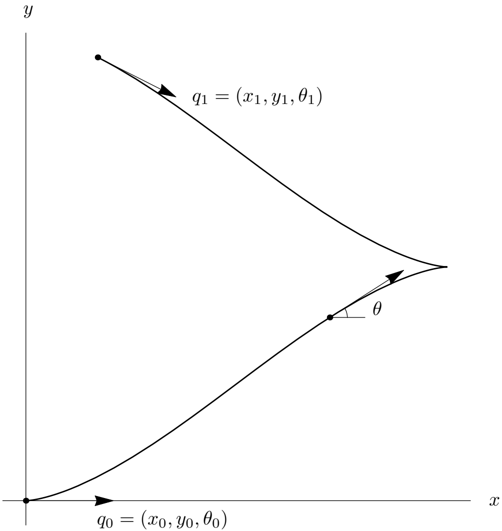
Given:

- model of a car in the plane that can move forward or backward with an arbitrary linear velocity and simultaneously rotate with an arbitrary angular velocity
- state of the car is given by its position in the plane and orientation angle
- an initial and a terminal state of the car

Find:

- motion of the car from a given initial state to a given terminal state, so that the length of the path in the space of positions and orientations is as minimum as possible.

4. The sub-Riemannian problem on the group of motions of the plane



$$
\begin{aligned}
&\dot{x} = u \cos\theta, && q = (x, y, \theta) \in \mathbb{R}^2 \times S^1,\\
&\dot{y} = u \sin\theta, && (u, v) \in \mathbb{R}^2,\\
&\dot{\theta} = v, &&\\
&q(0) = q_0, && q(t_1) = q_1,\\
&l = \int_0^{t_1} \sqrt{u^2 + v^2}\, dt \to \min .
\end{aligned}
$$

5. Euler elasticae

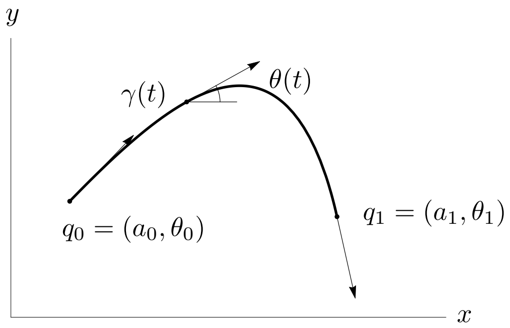
Given:

- uniform elastic rod of length l in the plane
- the rod has fixed endpoints and tangents at endpoints

Find:

- the profile of the rod.

5. Euler elasticae



$$\dot{x} = \cos\theta, \qquad q = (x, y, \theta) \in \mathbb{R}^2 \times S^1,$$

$$\dot{y} = \sin\theta, \qquad u \in \mathbb{R},$$

$$\dot{\theta} = u,$$

$$q(0) = q_0, \qquad q(t_1) = q_1,$$

$t_1 = l$ is the length of the rod,

$$J = \frac{1}{2}\int_0^{t_1} u^2\, dt \to \min .$$

6. The plate-ball problem

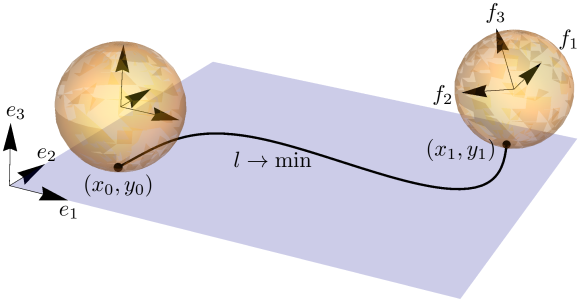
Given:

- uniform sphere roll without slipping or twisting on a horizontal plane
- imagine: the sphere rolls between two horizontal planes, a fixed lower one and a moving upper one
- absence of slipping: the contact point of the sphere with the plane has zero instantaneous velocity
- absence of twisting means that the angular velocity vector of the sphere is horizontal
- admissible motions are obtained by horizontal motions of the upper plane
- initial and terminal states of the sphere.

Find:

- roll the sphere so that the length of the curve in the plane traced by the contact point was the minimum possible.

6. The plate-ball problem



$$\dot{x} = u, \quad \dot{y} = v, \qquad (u, v) \in \mathbb{R}^2,$$

$$\dot{R} = R \begin{pmatrix} 0 & 0 & -u \\ 0 & 0 & -v \\ u & v & 0 \end{pmatrix},$$

$$q = (x, y, R) \in \mathbb{R}^2 \times \mathrm{SO}(3),$$

$$q(0) = q_0, \qquad q(t_1) = q_1,$$

$$l = \int_0^{t_1} \sqrt{u^2 + v^2}\, dt \to \min.$$

7. Anthropomorphic curve reconstruction

Given:

- greyscale image as a set of isophotes (level lines of brightness)
- image corrupted in some domain.

Find:

- anthropomorphic reconstruction of the image.

7. Anthropomorphic curve reconstruction

- D. Hubel and T. Wiesel (1981 Nobel Prize): a human brain stores curves not as sequences of planar points (x_i, y_i), but as sequences of positions and orientations (x_i, y_i, θ_i)
- model of the primary visual cortex $V1$ of the human brain by J. Petitot, G. Citti and A. Sarti: corrupted curves of images are reconstructed according to a variational principle
- human brain lifts images $(x(t), y(t))$ from the plane to the space of positions and orientations $(x(t), y(t), \theta(t))$.

$$\dot{x} = u \cos\theta, \quad \dot{y} = u \sin\theta, \quad \dot{\theta} = v,$$

$$J = \int_0^{t_1} (u^2 + v^2)\, dt \to \min \quad \Leftrightarrow \quad l = \int_0^{t_1} \sqrt{u^2 + v^2}\, dt \to \min,$$

- Optimal trajectories for the sub-Riemannian problem on the group of motions of the plane solve the problem of anthropomorphic curve reconstruction!

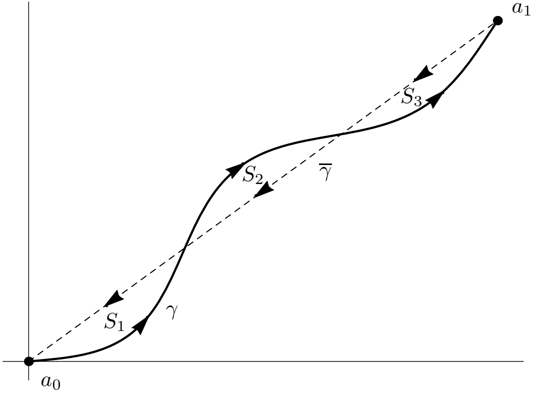
8. Dido's problem

Given:

- points $a_0, a_1 \in \mathbb{R}^2$
- Lipschitzian curve $\overline{\gamma} \subset \mathbb{R}^2$ connecting a_1 with a_0
- number $S \in \mathbb{R}$.

Find:

- the shortest Lipschitzian curve $\gamma \subset \mathbb{R}^2$ connecting a_0 with a_1 for which the closed curve $\gamma \cup \overline{\gamma}$ bounds a domain in $\mathbb{R}^2$ of the algebraic area S.



8. Dido's problem

- coordinates x, y in the plane $\mathbb{R}^2$ with the origin a_0. Then $a_0 = (0, 0)$, $a_1 = (x_1, y_1)$, $\gamma(t) = (x(t), y(t))$, $t \in [0, t_1]$, $\overline{\gamma}(t) = (\overline{x}(t), \overline{y}(t))$, $t \in [0, \overline{t}_1]$.
- closed curve $\widehat{\gamma} = \gamma \cup \overline{\gamma}$ and a domain bounded by it: $D \subset \mathbb{R}^2$, $\partial D = \widehat{\gamma}$

$$S(D) = \frac{1}{2} \oint_{\widehat{\gamma}} x\,dy - y\,dx = \frac{1}{2} \int_0^{t_1} (x\dot{y} - y\dot{x})\,dt - \overline{I},$$

$$\dot{x}(t) =: u_1(t), \qquad \dot{y}(t) =: u_2(t), \qquad \dot{z}(t) = \frac{1}{2}(xu_2 - yu_1).$$

$$\dot{q} = u_1 X_1(q) + u_2 X_2(q), \qquad u = (u_1, u_2) \in \mathbb{R}^2, \quad q = (x, y, z) \in \mathbb{R}^3,$$

$$X_1 = \frac{\partial}{\partial x} - \frac{y}{2}\frac{\partial}{\partial z}, \qquad X_2 = \frac{\partial}{\partial y} + \frac{x}{2}\frac{\partial}{\partial z},$$

$$q(0) = q_0 = (0, 0, 0), \qquad q(t_1) = q_1 = (x_1, y_1, z_1),$$

$$l(\gamma) = \int_0^{t_1} \sqrt{\dot{x}^2 + \dot{y}^2}\,dt = \int_0^{t_1} \sqrt{u_1^2 + u_2^2}\,dt \to \min.$$

Dynamical systems and control systems

- *smooth dynamical system*, or an *ordinary differential equation*:

$$\dot{q} = f(q), \qquad q \in M$$

- deterministic: given an initial condition $q(0) = q_0$ and a time $t > 0$, there exists a unique solution $q(t)$
- *control system*:

$$\dot{q} = f(q, u), \qquad q \in M, \quad u \in U. \tag{1}$$

- *control* function $u = u(t) \in U \quad \Rightarrow \quad$ a nonautonomous ODE

$$\dot{q} = f(q, u(t)). \tag{2}$$

- Together with an initial condition $q(0) = q_0$, ODE (2) determines a unique solution — a *trajectory* $q_u(t)$.
- Regularity assumptions for controls $u(\cdot)$: piecewise constant, piecewise continuous, L^∞, L^1, ...

Attainable sets

- *Attainable set* of control system (1) from a point q_0 for arbitrary times:

$$\mathcal{A}_{q_0} = \{q_u(t) \mid q_u(0) = q_0, \quad u \in L^\infty([0,t],U), \quad t \geq 0\}.$$

- attainable set from the point q_0 for a time $t_1 \geq 0$:

$$\mathcal{A}_{q_0}(t_1) = \{q_u(t_1) \mid q_u(0) = q_0, \quad u \in L^\infty([0,t_1],U)\},$$

- attainable set from the point q_0 for times not greater than $t_1 \geq 0$:

$$\mathcal{A}_{q_0}(\leq t_1) = \bigcup_{t=0}^{t_1} \mathcal{A}_{q_0}(t).$$

Controllability problem

A control system (1) is called:

- *globally (completely) controllable* if $\mathcal{A}_{q_0} = M$ for any $q_0 \in M$
- *globally controllable from a point* $q_0 \in M$ if $\mathcal{A}_{q_0} = M$
- *locally controllable at* q_0 if $q_0 \in \operatorname{int} \mathcal{A}_{q_0}$
- *small time locally controllable (STLC) at* q_0 if $q_0 \in \operatorname{int} \mathcal{A}_{q_0}(\leq t_1)$ for any $t_1 > 0$

- Local controllability problem: necessary conditions and sufficient conditions of STLC for arbitrary dimension of the state space M, but local controllability tests are available only for the case $\dim M = 2$.
- Global controllability problem: global controllability conditions only for very symmetric systems (linear systems, left-invariant systems on Lie groups).

Optimal control problem

- Let $q_1 \in \mathcal{A}_{q_0}(t_1)$. Then typically q_0, q_1 are connected by continuum of trajectories
- *Cost functional* to be minimized:

$$J = \int_0^{t_1} \varphi(q, u)\, dt.$$

- *Optimal control problem*:

$$\begin{aligned} &\dot{q} = f(q, u), \qquad q \in M, \quad u \in U, \\ &q(0) = q_0, \qquad q(t_1) = q_1, \\ &J = \int_0^{t_1} \varphi(q, u)\, dt \to \min. \end{aligned}$$

- Other important mathematical control problems: equivalence, stabilization, observability, etc., which we do not touch upon.

Smooth manifolds

A smooth k-dimensional *submanifold* $M \subset \mathbb{R}^n$ is usually defined by one of the following equivalences:

(a) implicitly by a system of regular equations:

$$f_1(x) = \cdots = f_{n-k}(x) = 0, \qquad x \in \mathbb{R}^n,$$
$$\operatorname{rank}\left(\frac{\partial f_1}{\partial x}, \ldots, \frac{\partial f_{n-k}}{\partial x}\right) = n - k,$$

(b) or by a regular parametrization:

$$x = \Phi(y), \qquad y \in \mathbb{R}^k, \quad x \in \mathbb{R}^n,$$
$$\operatorname{rank} \frac{\partial \Phi}{\partial y} = k.$$

An abstract smooth *manifold* M (not embedded into $\mathbb{R}^n$) is defined via a system of charts (local coordinates) that mutually agree.

Tangent vectors and spaces

A mapping between smooth manifolds is called *smooth* if it is smooth (of class C^∞) in local coordinates.
The *tangent space* to a smooth submanifold $M \subset \mathbb{R}^n$ at a point $x \in M$ is defined as follows for the two definitions above of a submanifold:

(a) $T_x M = \operatorname{Ker} \frac{\partial f}{\partial x}(x)$,

(b) $T_x M = \operatorname{Im} \frac{\partial \Phi}{\partial y}(y)$, $x = \Phi(y)$.

Given a smooth mapping $F : M \to N$ between smooth manifolds, for any $q \in M$ the *differential* $F_{*q} : T_q M \to T_{F(q)} N$ is defined as follows:

$$F_{*q} v = \left.\frac{d}{dt}\right|_{t=0} F(\gamma(t)),$$

where $\gamma : (-\varepsilon, \varepsilon) \to M$ is a smooth curve such that $\gamma(0) = q$, $\dot{\gamma}(0) = v$.

Smooth vector fields and their commutativity

- A smooth *vector field* on a manifold M is a smooth mapping $M \ni q \mapsto V(q) \in T_qM$. Notation: $V \in \mathrm{Vec}(M)$.
- A *trajectory of a vector field* V through a point $q_0 \in M$ is a solution to the Cauchy problem $\dot{q}(t) = V(q(t)), \quad q(0) = q_0$.
- Suppose that a trajectory $q(t)$ exists for all times $t \in \mathbb{R}$, then we denote $e^{tV}(q_0) := q(t)$. The one-parameter group of diffeomorphisms $e^{tV} : M \to M$ is the *flow* of the vector field V.
- We say that vector fields V and W *commute* if their flows commute:

$$e^{tV} \circ e^{sW} = e^{sW} \circ e^{tV}, \qquad t, s \in \mathbb{R}.$$

- In the general case vector fields V and W do not commute and the curve

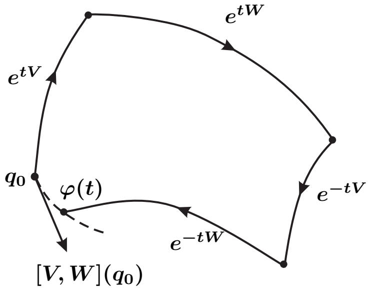
$$\varphi(t) = e^{-tW} \circ e^{-tV} \circ e^{tW} \circ e^{tV}(q_0)$$

  satisfies the inequality $\varphi(t) \neq q_0$, $t \in \mathbb{R}$.
- The leading nontrivial term of the Taylor expansion of $\varphi(t)$, $t \to 0$, is taken as the measure of noncommutativity of vector fields V and W.

Lie brackets of vector fields

- The *commutator* (*Lie bracket*) of the vector fields V, W at the point q_0 is defined as $[V, W](q_0) := \frac{1}{2}\ddot{\varphi}(0)$, so that

$$\varphi(t) = q_0 + t^2[V, W](q_0) + o(t^2), \qquad t \to 0.$$



- In local coordinates $[V, W] = \frac{\partial W}{\partial x} V - \frac{\partial V}{\partial x} W$.

Exercises 1

1. Describe the attainable sets $\mathcal{A}_{q_0}$ for examples 1–5, 8. Which of these systems are controllable?
2. Describe in example 6:

$$\begin{aligned}&\mathrm{Lie}_q(X_1, X_2)\\ &\quad = \mathrm{span}(X_1(q), X_2(q), [X_1, X_2](q), [X_1, [X_1, X_2]](q), [X_2, [X_1, X_2]](q), \dots),\end{aligned}$$

   where X_1 and X_2 are the vector fields in the right-hand side of system in slide 13:

$$\dot{q} = u_1 X_1 + u_2 X_2, \qquad q \in \mathbb{R}^2 \times \mathrm{SO}(3).$$

3. When a second-order curve $\{(x_1, x_2) \in \mathbb{R}^2 \mid \sum_{0 \le i_1 + i_2 \le 2} c_{i_1 i_2} x_1^{i_1} x_2^{i_2} = 0\}$ is a smooth manifold? Compute its dimension.

Exercises 2

4. Show that the two-dimensional sphere S^2 and the group SO(3) of rotations of the 3-space are smooth manifolds. Compute their tangent spaces.
5. Prove in example 7:

$$I = \int_0^{t_1} \sqrt{u_1^2 + u_2^2}\, dt \to \min \quad \Leftrightarrow \quad J = \int_0^{t_1} (u_1^2 + u_2^2)\, dt \to \min$$

   for a fixed terminal time t_1.
6. Prove the formula $[V, W] = \frac{\partial W}{\partial x} V - \frac{\partial V}{\partial x} W$.