Pontryagin maximum principle - 1

(Lecture 7)

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Plan of previous lecture

1. Exterior differential
2. Lie derivative of differential forms
3. Liouville form and symplectic form
4. Hamiltonian vector fields
5. Linear on fibers Hamiltonians

Plan of this lecture

1. Geometric statement of PMP and discussion
2. Proof of the geometric statement of PMP with fixed terminal time
3. Geometric statement of PMP for free time
4. PMP for optimal control problems

Pontryagin Maximum Principle

Geometric statement of PMP and discussion

- Consider an optimal control problem for a control system

$$\dot{q} = f_u(q), \qquad q \in M, \quad u \in U \subset \mathbb{R}^m, \tag{1}$$

with the initial condition

$$q(0) = q_0. \tag{2}$$

- Define the following family of Hamiltonians:

$$h_u(\lambda) = \langle \lambda, f_u(q) \rangle, \qquad \lambda \in T_q^* M, \; q \in M, \; u \in U.$$

- In terms of the previous lecture,

$$h_u(\lambda) = f_u^*(\lambda).$$

- Fix an arbitrary instant $t_1 > 0$.
- In Lecture 1 we reduced the optimal control problem to the study of boundary of attainable sets.

Reduction to Study of Attainable Sets

Theorem 1

Let $q_{\tilde{u}}(t)$, $t \in [0, t_1]$, be an optimal trajectory in the optimal control problem with the fixed terminal time t_1. Then $\widehat{q}_{\tilde{u}}(t_1) \in \partial \widehat{\mathcal{A}}_{(0,q_0)}(t_1)$.

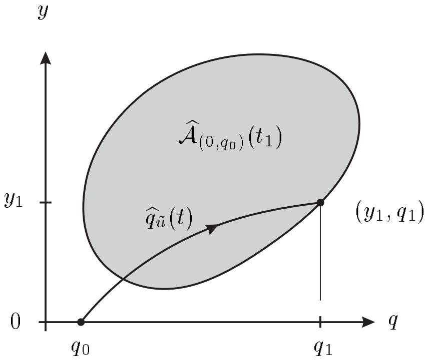

Figure: $q_{\tilde{u}}(t)$ optimal

- Now we give a *necessary optimality condition* in this geometric setting.

Theorem 2 (PMP)

Let $\widetilde{u}(t)$, $t \in [0, t_1]$, be an admissible control and $\widetilde{q}(t) = q_{\widetilde{u}}(t)$ the corresponding solution of Cauchy problem (1), (2). If $\widetilde{q}(t_1) \in \partial \mathcal{A}_{q_0}(t_1)$, then there exists a Lipschitzian curve in the cotangent bundle

$$\lambda_t \in T^*_{\widetilde{q}(t)} M, \qquad 0 \leq t \leq t_1,$$

such that

$$\begin{aligned} &\lambda_t \neq 0, && (3)\\ &\dot{\lambda}_t = \vec{h}_{\widetilde{u}(t)}(\lambda_t), && (4)\\ &h_{\widetilde{u}(t)}(\lambda_t) = \max_{u \in U} h_u(\lambda_t) && (5) \end{aligned}$$

for almost all $t \in [0, t_1]$.

- If $u(t)$ is an admissible control and λ_t a Lipschitzian curve in T^*M such that conditions (11)–(13) hold, then the pair $(u(t), \lambda_t)$ is said to satisfy PMP
- In this case the curve λ_t is called an *extremal*, and its projection $\widetilde{q}(t) = \pi(\lambda_t)$ is called an *extremal trajectory*.

Remark 1

If a pair $(\widetilde{u}(t), \lambda_t)$ satisfies PMP, then

$$h_{\widetilde{u}(t)}(\lambda_t) = \text{const}, \qquad t \in [0, t_1]. \tag{6}$$

Indeed, since the admissible control $\widetilde{u}(t)$ is bounded, we can take maximum in (13) over the compact $\overline{\{\widetilde{u}(t) \mid t \in [0, t_1]\}} = \widetilde{U}$.
Further, the function $\varphi(\lambda) = \max_{u \in \widetilde{U}} h_u(\lambda)$ is Lipschitzian w.r.t. $\lambda \in T^*M$. We show that this function has zero derivative.

For optimal control $\widetilde{u}(t)$,

$$\varphi(\lambda_t) \geq h_{\widetilde{u}(\tau)}(\lambda_t), \qquad \varphi(\lambda_\tau) = h_{\widetilde{u}(\tau)}(\lambda_\tau),$$

thus

$$\frac{\varphi(\lambda_t) - \varphi(\lambda_\tau)}{t - \tau} \geq \frac{h_{\widetilde{u}(\tau)}(\lambda_t) - h_{\widetilde{u}(\tau)}(\lambda_\tau)}{t - \tau}, \qquad t > \tau.$$

Consequently,

$$\left.\frac{d}{d\,t}\right|_{t=\tau} \varphi(\lambda_t) \geq \{h_{\widetilde{u}(\tau)}, h_{\widetilde{u}(\tau)}\} = 0$$

if τ is a differentiability point of $\varphi(\lambda_t)$. Similarly,

$$\frac{\varphi(\lambda_t) - \varphi(\lambda_\tau)}{t - \tau} \leq \frac{h_{\widetilde{u}(\tau)}(\lambda_t) - h_{\widetilde{u}(\tau)}(\lambda_\tau)}{t - \tau}, \qquad t < \tau,$$

thus $\left.\frac{d}{d\,t}\right|_{t=\tau} \varphi(\lambda_t) \leq 0$. So

$$\frac{d}{d\,t}\varphi(\lambda_t) = 0,$$

and identity (6) follows.

- The Hamiltonian system of PMP

$$\dot{\lambda}_t = \vec{h}_{u(t)}(\lambda_t) \tag{7}$$

  is an extension of the initial control system (1) to the cotangent bundle.
- Indeed, in canonical coordinates $\lambda = (\xi, x) \in T^*M$, the Hamiltonian system yields

$$\dot{x} = \frac{\partial h_{u(t)}}{\partial \xi} = f_{u(t)}(x).$$

- That is, solutions λ_t to (7) are Hamiltonian lifts of solutions $q(t)$ to (1):

$$\pi(\lambda_t) = q_u(t).$$

- Before proving Pontryagin Maximum Principle, we discuss its statement.

- First we give a heuristic explanation of the way the covector curve λ_t appears naturally in the study of trajectories coming to boundary of the attainable set.
- Let

$$q_1 = \widetilde{q}(t_1) \in \partial \mathcal{A}_{q_0}(t_1). \tag{8}$$

- The idea is to take a normal covector to the attainable set $\mathcal{A}_{q_0}(t_1)$ near q_1, more precisely — a normal covector to a kind of a convex tangent cone to $\mathcal{A}_{q_0}(t_1)$ at q_1.
- By virtue of inclusion (8), this convex cone is proper.
- Thus it has a hyperplane of support, i.e., a linear hyperplane in $T_{q_1}M$ bounding a half-space that contains the cone.

- Further, the hyperplane of support is a kernel of a normal covector $\lambda_{t_1} \in T^*_{q_1} M$, $\lambda_{t_1} \neq 0$, see fig. below:

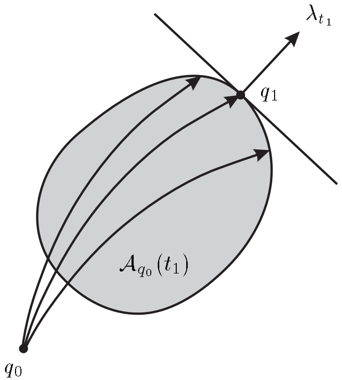

Figure: Hyperplane of support and normal covector to attainable set $\mathcal{A}_{q_0}(t_1)$ at the point q_1

- The covector λ_{t_1} is an analog of Lagrange multipliers.

- In order to construct the whole curve λ_t, $t \in [0, t_1]$, consider the flow generated by the control $\widetilde{u}(\cdot)$:

$$P_{t,t_1} = \overrightarrow{\exp} \int_t^{t_1} f_{\widetilde{u}(\tau)}\, d\tau, \qquad t \in [0, t_1].$$

- It is easy to see that

$$P_{t,t_1}(\mathcal{A}_{q_0}(t)) \subset \mathcal{A}_{q_0}(t_1), \qquad t \in [0, t_1].$$

- Indeed, if a point $q \in \mathcal{A}_{q_0}(t)$ is reachable from q_0 by a control $u(\tau)$, $\tau \in [0, t]$, then the point $P_{t,t_1}(q)$ is reachable from q_0 by the control

$$v(\tau) = \begin{cases} u(\tau), & \tau \in [0, t], \\ \widetilde{u}(\tau), & \tau \in [t, t_1]. \end{cases}$$

- Further, the diffeomorphism $P_{t,t_1} : M \to M$ satisfies the condition

$$P_{t,t_1}(\widetilde{q}(t)) = \widetilde{q}(t_1) = q_1, \qquad t \in [0, t_1].$$

- Thus if $\widetilde{q}(t) \in \operatorname{int} \mathcal{A}_{q_0}(t)$, then $q_1 \in \operatorname{int} \mathcal{A}_{q_0}(t_1)$.
- By contradiction, inclusion (8) implies that

$$\widetilde{q}(t) \in \partial \mathcal{A}_{q_0}(t), \qquad t \in [0, t_1].$$

- The tangent cone to $\mathcal{A}_{q_0}(t)$ at the point $\widetilde{q}(t) = P_{t_1,t}(q_1)$ has the normal covector $\lambda_t = P^*_{t,t_1}(\lambda_{t_1})$.
- By the previous lecture, the curve λ_t, $t \in [0, t_1]$, is a trajectory of the Hamiltonian vector field $\vec{h}_{\widetilde{u}(t)}$, i.e., of the Hamiltonian system of PMP.

- One can easily get the maximality condition of PMP as well.
- The tangent cone to $\mathcal{A}_{q_0}(t_1)$ at q_1 should contain the infinitesimal attainable set from the point q_1:

$$f_U(q_1) - f_{\widetilde{u}(t_1)}(q_1),$$

i.e., the set of vectors obtained by variations of the control $\widetilde{u}$ near t_1.
- Thus the covector λ_{t_1} should determine a hyperplane of support to this set:

$$\langle \lambda_{t_1}, f_u - f_{\widetilde{u}(t_1)} \rangle \leq 0, \qquad u \in U.$$

- In other words,

$$h_u(\lambda_{t_1}) = \langle \lambda_{t_1}, f_u \rangle \leq \langle \lambda_{t_1}, f_{\widetilde{u}(t_1)} \rangle = h_{\widetilde{u}(t_1)}(\lambda_{t_1}), \qquad u \in U.$$

- Translating the covector λ_{t_1} by the flow P^*_{t,t_1}, we arrive at the maximality condition of PMP:

$$h_u(\lambda_t) \leq h_{\widetilde{u}(t)}(\lambda_t), \qquad u \in U, \quad t \in [0, t_1].$$

- The following statement shows the power of PMP.

Proposition 1

Assume that the maximized Hamiltonian of PMP

$$H(\lambda) = \max_{u \in U} h_u(\lambda), \qquad \lambda \in T^*M,$$

is defined and C^2-smooth on $T^*M \setminus \{\lambda = 0\}$.
If a pair $(\widetilde{u}(t), \lambda_t)$, $t \in [0, t_1]$, satisfies PMP, then

$$\dot{\lambda}_t = \vec{H}(\lambda_t), \qquad t \in [0, t_1]. \tag{9}$$

Conversely, if a Lipschitzian curve $\lambda_t \neq 0$ is a solution to the Hamiltonian system (9), then one can choose an admissible control $\widetilde{u}(t)$, $t \in [0, t_1]$, such that the pair $(\widetilde{u}(t), \lambda_t)$ satisfies PMP.

- That is, in the favorable case when the maximized Hamiltonian H is C^2-smooth, PMP reduces the problem to the study of solutions to just one Hamiltonian system (9).

- From the point of view of dimension, this reduction is the best one we can expect.
- Indeed, for a full-dimensional attainable set ($\dim \mathcal{A}_{q_0}(t_1) = n$) we have $\dim \partial\mathcal{A}_{q_0}(t_1) = n - 1$, i.e., we need an $(n-1)$-parameter family of curves to describe the boundary $\partial\mathcal{A}_{q_0}(t_1)$.
- On the other hand, the family of solutions to Hamiltonian system (9) with the initial condition $\pi(\lambda_0) = q_0$ is n-dimensional.
- Taking into account that the Hamiltonian H is homogeneous:

$$H(c\lambda) = cH(\lambda), \qquad c > 0,$$

thus

$$e^{t\vec{H}}(c\lambda_0) = ce^{t\vec{H}}(\lambda_0), \qquad \pi \circ e^{t\vec{H}}(c\lambda_0) = \pi \circ e^{t\vec{H}}(\lambda_0),$$

we obtain the required $(n-1)$-dimensional family of curves.
- Now we prove Proposition 1.

Proof.

- We show that if an admissible control $\widetilde{u}(t)$ satisfies the maximality condition (13), then
$$\vec{h}_{\widetilde{u}(t)}(\lambda_t) = \vec{H}(\lambda_t), \qquad t \in [0, t_1]. \tag{10}$$
- By definition of the maximized Hamiltonian H,
$$H(\lambda) - h_{\widetilde{u}(t)}(\lambda) \geq 0 \qquad \lambda \in T^*M, \quad t \in [0, t_1].$$
- On the other hand, by the maximality condition of PMP (13), along the extremal λ_t this inequality turns into equality:
$$H(\lambda_t) - h_{\widetilde{u}(t)}(\lambda_t) = 0, \qquad t \in [0, t_1].$$
- That is why
$$d_{\lambda_t} H = d_{\lambda_t} h_{\widetilde{u}(t)}, \qquad t \in [0, t_1].$$
- But a Hamiltonian vector field is obtained from differential of the Hamiltonian by a standard linear transformation, thus equality (10) follows.

- Conversely, let $\lambda_t \neq 0$ be a trajectory of the Hamiltonian system $\dot{\lambda}_t = \vec{H}(\lambda_t)$.
- In the same way as in the proof of Filippov's theorem, one can choose an admissible control $\widetilde{u}(t)$ that realizes maximum along λ_t:

$$H(\lambda_t) = h_{\widetilde{u}(t)}(\lambda_t) = \max_{u \in U} h_u(\lambda_t).$$

- As we have shown above, then there holds equality (10). So the pair $(\widetilde{u}(t), \lambda_t)$ satisfies PMP.

□

The geometric statement of PMP with fixed terminal time

Theorem 1 (PMP)

Let $\widetilde{u}(t)$, $t \in [0, t_1]$, be an admissible control and $\widetilde{q}(t) = q_{\widetilde{u}}(t)$ the corresponding solution of Cauchy problem (1), (2). If $\widetilde{q}(t_1) \in \partial \mathcal{A}_{q_0}(t_1)$, then there exists a Lipschitzian curve in the cotangent bundle

$$\lambda_t \in T^*_{\widetilde{q}(t)} M, \qquad 0 \leq t \leq t_1,$$

such that

$$\begin{aligned} &\lambda_t \neq 0, && (11)\\ &\dot{\lambda}_t = \vec{h}_{\widetilde{u}(t)}(\lambda_t), && (12)\\ &h_{\widetilde{u}(t)}(\lambda_t) = \max_{u \in U} h_u(\lambda_t) && (13) \end{aligned}$$

for almost all $t \in [0, t_1]$.

Proof of the geometric statement of PMP with fixed terminal time

- We start from two auxiliary lemmas.
- Denote the positive orthant in $\mathbb{R}^m$ as

$$\mathbb{R}^m_+ = \{(x_1, \dots, x_m) \in \mathbb{R}^m \mid x_i \geq 0, \;\; i = 1, \dots, m\}.$$

Lemma 2

Let a vector-function $F : \mathbb{R}^m \to \mathbb{R}^n$ *be Lipschitzian,* $F(0) = 0$*, and differentiable at* 0*:*

$$\exists \; F_0' = \left. \frac{d\,F}{d\,x} \right|_0 .$$

Assume that

$$F_0'(\mathbb{R}^m_+) = \mathbb{R}^n.$$

Then for any neighborhood of the origin $O_0 \subset \mathbb{R}^m$

$$0 \in \operatorname{int} F(O_0 \cap \mathbb{R}^m_+).$$

Remark 2

The statement of this lemma holds if the orthant $\mathbb{R}^m_+$ is replaced by an arbitrary convex cone $C \subset \mathbb{R}^m$. In this case the proof given below works without any changes.

Proof of Lemma 2.

- Choose points $y_0, \dots, y_n \in \mathbb{R}^n$ that generate an n-dimensional simplex centered at the origin: $\frac{1}{n+1}\sum_{i=0}^{n} y_i = 0.$
- Since the mapping $F_0' : \mathbb{R}^m_+ \to \mathbb{R}^n$ is surjective and the positive orthant $\mathbb{R}^m_+$ is a convex cone, it is easy to show that restriction to the interior $F_0'|_{\operatorname{int}\mathbb{R}^m_+}$ is also surjective:

$$\exists\ v_i \in \operatorname{int}\mathbb{R}^m_+ \quad \text{such that} \quad F_0' v_i = y_i, \quad i = 0, \dots, n.$$

- The points $y_0, \dots, y_n$ are affinely independent in $\mathbb{R}^n$, thus their preimages $v_0, \dots, v_n$ are also affinely independent in $\mathbb{R}^m$.

- The mean

$$v = \frac{1}{n+1} \sum_{i=0}^{n} v_i$$

belongs to $\operatorname{int} \mathbb{R}^m_+$ and satisfies the equality

$$F_0' v = 0.$$

- Further, the subspace

$$W = \operatorname{span}\{v_i - v \mid i = 0, \ldots, n\} \subset \mathbb{R}^m$$

is n-dimensional.

- Since $v \in \operatorname{int} \mathbb{R}^m_+$, we can find an n-dimensional ball $B_\delta \subset W$ of a sufficiently small radius δ centered at the origin such that

$$v + B_\delta \subset \operatorname{int} \mathbb{R}^m_+.$$

- Since $F_0'(v_i - v) = F_0' v_i$, then $F_0' W = \mathbb{R}^n$, i.e., the linear mapping $F_0' : W \to \mathbb{R}^n$ is invertible.

- Consider the following family of mappings:

$$G_\alpha : B_\delta \to \mathbb{R}^n, \qquad \alpha \in [0, \alpha_0),$$
$$G_\alpha(w) = \frac{1}{\alpha} F(\alpha(v + w)), \qquad \alpha > 0,$$
$$G_0(w) = F_0' w.$$

- By the hypotheses of this lemma,

$$F(x) = F_0' x + o(x), \qquad x \in \mathbb{R}^m, \; x \to 0,$$

thus

$$G_\alpha(w) = \frac{1}{\alpha}(F_0'(\alpha(v + w)) + o(\alpha(v + w))) = F_0' w + o(1), \; \alpha \to 0, \; w \in B_\delta. \tag{14}$$

- Since the mapping F is Lipschitzian, all mappings G_α are Lipschitzian with a common constant.
- Thus the family G_α is equicontinuous. Equality (14) means that uniformly in $w \in B_\delta$ we have $G_\alpha \to G_0, \qquad \alpha \to 0.$

- So the continuous mapping $G_\alpha \circ G_0^{-1} : G_0(B_\delta) \to \mathbb{R}^n$ is uniformly close to the identity mapping, hence the difference $\mathrm{Id} - G_\alpha \circ G_0^{-1}$ is uniformly close to the zero mapping.
- For any $\tilde{x} \in \mathbb{R}^n$ sufficiently close to the origin, the continuous mapping

  $$\mathrm{Id} - G_\alpha \circ G_0^{-1} + \tilde{x}$$

  transforms the set $G_0(B_\delta)$ into itself.
- By Brower's fixed point theorem, this mapping has a fixed point $x \in G_0(B_\delta)$:

  $$x - G_\alpha \circ G_0^{-1}(x) + \tilde{x} = x,$$

  i.e.,

  $$G_\alpha \circ G_0^{-1}(x) = \tilde{x}.$$
- It follows that $\operatorname{int} G_\alpha(B_\delta) \ni 0$, consequently, $\operatorname{int} F(\alpha(v + B_\delta)) \ni 0$ for small $\alpha > 0$. Thus $\operatorname{int} F(O_0 \cap \mathbb{R}^m_+) \ni 0$ for a small neighborhood $O_0 \in \mathbb{R}^m$. $\square$

- Now we start to compute a convex approximation of the attainable set $\mathcal{A}_{q_0}(t_1)$ at the point $q_1 = \widetilde{q}(t_1)$ corresponding to a reference control $\widetilde{u}(\cdot)$.
- Take any admissible control $u(t)$ and express the endpoint of a trajectory via Variations Formula:

$$\begin{aligned} q_u(t_1) &= q_0 \circ \overrightarrow{\exp} \int_0^{t_1} f_{u(\tau)}\, d\tau = q_0 \circ \overrightarrow{\exp} \int_0^{t_1} f_{\widetilde{u}(\tau)} + \left(f_{u(\tau)} - f_{\widetilde{u}(\tau)}\right) d\tau \\ &= q_0 \circ \overrightarrow{\exp} \int_0^{t_1} f_{\widetilde{u}(\tau)}\, d\tau \circ \overrightarrow{\exp} \int_0^{t_1} \left(P_\tau^{t_1}\right)_* \left(f_{u(\tau)} - f_{\widetilde{u}(\tau)}\right) d\tau \\ &= q_1 \circ \overrightarrow{\exp} \int_0^{t_1} \left(P_\tau^{t_1}\right)_* \left(f_{u(\tau)} - f_{\widetilde{u}(\tau)}\right) d\tau. \end{aligned}$$

- Introduce the following vector field depending on two parameters:

$$g_{\tau,u} = \left(P_\tau^{t_1}\right)_* \left(f_u - f_{\widetilde{u}(\tau)}\right), \qquad \tau \in [0, t_1], \quad u \in U. \tag{15}$$

- We showed that

$$q_u(t_1) = q_1 \circ \overrightarrow{\exp} \int_0^{t_1} g_{\tau,u(\tau)}\, d\tau. \tag{16}$$

- Notice that $g_{\tau,\widetilde{u}(\tau)} \equiv 0, \qquad \tau \in [0, t_1].$

Lemma 3

Let $\mathcal{T} \subset [0, t_1]$ *be the set of Lebesgue points of the control* $\widetilde{u}(\cdot)$*. If*

$$\operatorname{cone}\{g_{\tau,u}(q_1) \mid \tau \in \mathcal{T},\ u \in U\} = T_{q_1}M,$$

then $q_1 \in \operatorname{int} \mathcal{A}_{q_0}(t_1)$.

Remark 3

The set $\operatorname{cone}\{g_{\tau,u}(q_1) \mid \tau \in \mathcal{T},\ u \in U\} \subset T_{q_1}M$ is a local convex approximation of the attainable set $\mathcal{A}_{q_0}(t_1)$ at the point q_1 corresponding to a reference control $\widetilde{u}(\cdot)$.

- Recall that a point $\tau \in [0, t_1]$ is called a *Lebesgue point* of a function $u \in L^1[0, t_1]$ if $\lim\limits_{t \to \tau} \dfrac{1}{|t - \tau|} \displaystyle\int_\tau^t |u(\theta) - u(\tau)|\, d\theta = 0$.
- At Lebesgue points of u, the integral $\displaystyle\int_0^t u(\theta)\, d\theta$ is differentiable and $\dfrac{d}{d\,t}\left(\displaystyle\int_0^t u(\theta)\, d\theta\right) = u(t)$.

- The set of Lebesgue points has the full measure in the domain $[0, t_1]$.

Proof of Lemma 3.

- We can choose vectors

$$g_{\tau_i,u_i}(q_1) \in T_{q_1}M, \qquad \tau_i \in \mathcal{T}, \quad u_i \in U, \quad i = 1, \ldots, k,$$

  that generate the whole tangent space as a positive convex cone:

$$\text{cone}\,\{g_{\tau_i,u_i}(q_1) \mid i = 1, \ldots, k\} = T_{q_1}M,$$

  moreover, we can choose points τ_i distinct: $\tau_i \neq \tau_j$, $i \neq j$.
- Indeed, if $\tau_i = \tau_j$ for some $i \neq j$, we can find a sufficiently close Lebesgue point $\tau_j' \neq \tau_j$ such that the difference $g_{\tau_j',u_j}(q_1) - g_{\tau_j,u_j}(q_1)$ is as small as we wish.
- This is possible since for any $\tau \in \mathcal{T}$ and any $\varepsilon > 0$

$$\frac{1}{|t-\tau|}\,\text{meas}\{t' \in [\tau, t] \mid |u(t') - u(\tau)| \leq \varepsilon\} \to 1 \text{ as } t \to \tau.$$

- We suppose that $\tau_1 < \tau_2 < \cdots < \tau_k$.

- We define a family of variations of controls that follow the reference control $\widetilde{u}(\cdot)$ everywhere except neighborhoods of τ_i, and follow u_i near τ_i (such variations are called *needle-like*).
- More precisely, for any $s = (s_1, \ldots, s_k) \in \mathbb{R}^k_+$ consider a control of the form

$$u_s(t) = \begin{cases} u_i, & t \in [\tau_i, \tau_i + s_i], \\ \widetilde{u}(t), & t \notin \cup_{i=1}^k [\tau_i, \tau_i + s_i]. \end{cases} \qquad (17)$$

- For small s, the segments $[\tau_i, \tau_i + s_i]$ do not overlap since $\tau_i \neq \tau_j$, $i \neq j$.
- In view of formula (16), the endpoint of the trajectory corresponding to the control constructed is expressed as follows:

$$\begin{aligned} q_{u_s}(t_1) &= q_0 \circ \overrightarrow{\exp} \int_0^{t_1} f_{u_s(t)}\, dt \\ &= q_1 \circ \overrightarrow{\exp} \int_{\tau_1}^{\tau_1 + s_1} g_{t,u_1}\, dt \circ \overrightarrow{\exp} \int_{\tau_2}^{\tau_2 + s_2} g_{t,u_2}\, dt \circ \cdots \\ &\quad \circ \overrightarrow{\exp} \int_{\tau_k}^{\tau_k + s_k} g_{t,u_k}\, dt. \end{aligned}$$

- The mapping

$$F \; : \; s = (s_1, \ldots, s_k) \mapsto q_{u_s}(t_1)$$

is Lipschitzian, differentiable at $s = 0$, and

$$\left.\frac{\partial F}{\partial s_i}\right|_{s=0} = g_{\tau_i, u_i}(q_1).$$

- By Lemma 2,

$$F(0) = q_1 \in \operatorname{int} F(O_0 \cap \mathbb{R}^k_+)$$

for any neighborhood $O_0 \subset \mathbb{R}^k$.

- But the curve $q_{u_s}(t)$, $t \in [0, t_1]$, is an admissible trajectory for small $s \in \mathbb{R}^k_+$, thus $F(O_0 \cap \mathbb{R}^k_+) \subset \mathcal{A}_{q_0}(t_1)$ and $q_1 \in \operatorname{int} \mathcal{A}_{q_0}(t_1)$.

□