Differential Forms and Symplectic Geometry-2

(Lecture 6)

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Plan of previous lecture

1. Derivative of flow with respect to parameter
2. Differential 1-forms
3. Differential k-forms
4. Exterior differential

Plan of this lecture

1. Exterior differential
2. Lie derivative of differential forms
3. Liouville form and symplectic form
4. Hamiltonian vector fields
5. Linear on fibers Hamiltonians

Exterior differential

- First of all, it is obvious from the Stokes formula that $d : \Lambda^k M \to \Lambda^{k+1} M$ is a linear operator.
- Further, if $F : M \to N$ is a diffeomorphism, then

$$d\widehat{F}\omega = \widehat{F}d\omega, \qquad \omega \in \Lambda^k N. \tag{1}$$

- Indeed, if $W \subset M$, then

$$\int_{F(W)} \omega = \int_W \widehat{F}\omega, \qquad \omega \in \Lambda^k N,$$

thus

$$\begin{aligned} \int_W d\widehat{F}\omega &= \int_{\partial W} \widehat{F}\omega = \int_{F(\partial W)} \omega = \int_{\partial F(W)} \omega = \int_{F(W)} d\omega \\ &= \int_W \widehat{F}d\omega, \end{aligned}$$

and equality (1) follows.

- Another basic property of exterior differential is given by the equality

$$d \circ d = 0,$$

which follows since $\partial(\partial N) = \emptyset$ for any submanifold with boundary $N \subset M$.
- Exterior differential is an antiderivation:

$$d(\omega_1 \wedge \omega_2) = (d\omega_1) \wedge \omega_2 + (-1)^{k_1} \omega_1 \wedge d\omega_2, \qquad \omega_i \in \Lambda^{k_i} M,$$

this equality is dual to the formula of boundary $\partial(N_1 \times N_2)$.
- In local coordinates exterior differential is computed as follows: if

$$\omega = \sum_{i_1 < \cdots < i_k} a_{i_1 \dots i_k} dx_{i_1} \wedge \dots \wedge dx_{i_k}, \qquad a_{i_1 \dots i_k} \in C^\infty,$$

then

$$d\omega = \sum_{i_1 < \cdots < i_k} (da_{i_1 \dots i_k}) \wedge dx_{i_1} \wedge \dots \wedge dx_{i_k},$$

this formula is forced by above properties of differential forms.

Lie derivative of differential forms

- The "infinitesimal version" of the pull-back $\widehat{P}$ of a differential form by a flow P is given by the following operation.
- *Lie derivative* of a differential form $\omega \in \Lambda^k M$ along a vector field $f \in \operatorname{Vec} M$ is the differential form $L_f\omega \in \Lambda^k M$ defined as follows:

$$L_f \omega \stackrel{\text{def}}{=} \left.\frac{d}{d\,\varepsilon}\right|_{\varepsilon=0} \widehat{e^{\varepsilon f}}\,\omega. \tag{2}$$

- Since

$$\widehat{e^{tf}}(\omega_1 \wedge \omega_2) = \widehat{e^{tf}}\omega_1 \wedge \widehat{e^{tf}}\omega_2,$$

Lie derivative L_f is a derivation of the algebra of differential forms:

$$L_f(\omega_1 \wedge \omega_2) = (L_f\omega_1) \wedge \omega_2 + \omega_1 \wedge L_f\omega_2.$$

- Further, we have

$$\widehat{e^{tf}} \circ d = d \circ \widehat{e^{tf}},$$

thus

$$L_f \circ d = d \circ L_f.$$

- For 0-forms, Lie derivative is just the directional derivative:

$$L_f a = fa, \qquad a \in C^\infty(M),$$

since $\widehat{e^{tf}} a = a \circ e^{tf}$ is a substitution of variables.

- Now we obtain a useful formula for the action of Lie derivative on differential forms of an arbitrary order.
- Consider, along with exterior differential

$$d : \Lambda^k M \to \Lambda^{k+1} M$$

the *interior product* of a differential form ω with a vector field $f \in \operatorname{Vec} M$:

$$i_f : \Lambda^k M \to \Lambda^{k-1} M,$$

$$(i_f\omega)(v_1, \dots, v_{k-1}) \stackrel{\text{def}}{=} \omega(f, v_1, \dots, v_{k-1}), \qquad \omega \in \Lambda^k M, \ v_i \in T_q M,$$

which acts as substitution of f for the first argument of ω. By definition, for 0-order forms

$$i_f a = 0, \qquad a \in \Lambda^0 M.$$

- Interior product is an antiderivation, as well as the exterior differential:

$$i_f(\omega_1 \wedge \omega_2) = (i_f\omega_1) \wedge \omega_2 + (-1)^{k_1}\omega_1 \wedge i_f\omega_2, \qquad \omega_i \in \Lambda^{k_i} M.$$

- Now we prove that Lie derivative of a differential form of an arbitrary order can be computed by the following formula:

$$L_f = d \circ i_f + i_f \circ d \tag{3}$$

  called *Cartan's formula*, for short "$L = di + id$".

- Notice first of all that the right-hand side in (3) has the required order:

$$d \circ i_f + i_f \circ d \; : \; \Lambda^k M \to \Lambda^k M.$$

- Further, $d \circ i_f + i_f \circ d$ is a derivation as it is obtained from two antiderivations.

- Moreover, this derivation commutes with differential:

$$d \circ (d \circ i_f + i_f \circ d) = d \circ i_f \circ d,$$
$$(d \circ i_f + i_f \circ d) \circ d = d \circ i_f \circ d.$$

- Now we check the formula $L = di + id$ on 0-forms: if $a \in \Lambda^0 M$, then

$$(d \circ i_f)a = 0,$$
$$(i_f \circ d)a = \langle da, f \rangle = fa = L_f a.$$

  So the formula $L = di + id$ holds for 0-forms.

- The properties of the mappings L_f and $d \circ i_f + i_f \circ d$ established and the coordinate representation of differential forms reduce the general case of k-forms to the case of 0-forms.
- Cartan's formula $L = di + id$ is proved for k-forms.

- The differential definition (2) of Lie derivative can be integrated, i.e., there holds the following equality on $\Lambda^k M$:

$$\widehat{\left(\overrightarrow{\exp}\int_0^t f_\tau \, d\tau\right)} = \overrightarrow{\exp}\int_0^t L_{f_\tau}\, d\tau, \tag{4}$$

in the following sense.

- Denote the flow $P_{t_0}^{t_1} = \overrightarrow{\exp}\displaystyle\int_{t_0}^{t_1} f_\tau \, d\tau$ of a nonautonomous vector field f_τ on M.
- The family of operators on differential forms $\widehat{P_0^t} : \Lambda^k M \to \Lambda^k M$ is a unique solution of the Cauchy problem

$$\frac{d}{d\,t}\widehat{P_0^t} = \widehat{P_0^t} \circ L_{f_t}, \qquad \widehat{P_0^t}\Big|_{t=0} = \mathrm{Id}, \tag{5}$$

compare with Cauchy problems for the flow P_0^t and for the family of operators $\mathrm{Ad}\, P_0^t$, and this solution is denoted as

$$\overrightarrow{\exp}\int_0^t L_{f_\tau}\, d\tau \stackrel{\mathrm{def}}{=} \widehat{P_0^t} = \widehat{\left(\overrightarrow{\exp}\int_0^t f_\tau \, d\tau\right)}.$$

- In order to verify the ODE in (5), we prove first the following equality for operators on forms:

$$\frac{d}{d\,\varepsilon}\bigg|_{\varepsilon=0} \widehat{P_t^{t+\varepsilon}}\omega = L_{f_t}\omega, \qquad \omega \in \Lambda^k M. \tag{6}$$

- This equality is straightforward for 0-order forms:

$$\frac{d}{d\,\varepsilon}\bigg|_{\varepsilon=0} \widehat{P_t^{t+\varepsilon}} a = \frac{d}{d\,\varepsilon}\bigg|_{\varepsilon=0} a \circ P_t^{t+\varepsilon} = f_t a = L_{f_t} a, \qquad a \in C^\infty(M).$$

- Further, the both operators $\frac{d}{d\,\varepsilon}\big|_{\varepsilon=0} \widehat{P_t^{t+\varepsilon}}$ and L_{f_t} commute with d and satisfy the Leibniz rule w.r.t. product of a function with a differential form.
- Then equality (6) follows for forms of arbitrary order, as in the proof of Cartan's formula.

- Now we easily verify the ODE in (5):

$$\frac{d}{d\,t}\widehat{P_0^t} = \frac{d}{d\,\varepsilon}\bigg|_{\varepsilon=0}\widehat{P_0^{t+\varepsilon}} = \frac{d}{d\,\varepsilon}\bigg|_{\varepsilon=0}\left(P_0^t \circ P_t^{t+\varepsilon}\right)\widehat{\ }$$

by the composition rule for pull-back of differential forms

$$= \frac{d}{d\,\varepsilon}\bigg|_{\varepsilon=0}\widehat{P_0^t}\circ\widehat{P_t^{t+\varepsilon}} = \widehat{P_0^t}\circ\frac{d}{d\,\varepsilon}\bigg|_{\varepsilon=0}\widehat{P_t^{t+\varepsilon}}$$

$$= \widehat{P_0^t}\circ L_{f_t}.$$

Exercise 1

Prove uniqueness for Cauchy problem (5).

- For an autonomous vector field $f \in \operatorname{Vec} M$, equality (4) takes the form

$$\widehat{e^{tf}} = e^{tL_f}.$$

- Notice that the Lie derivatives of differential forms L_f and vector fields $(-\operatorname{ad} f)$ are in a certain sense dual one to another, see equality (7) below.
- That is, the function

$$\langle \omega, X \rangle \; : \; q \mapsto \langle \omega_q, X(q) \rangle, \qquad q \in M,$$

defines a pairing of $\Lambda^1 M$ and $\operatorname{Vec} M$ over $C^\infty(M)$. Then the equality

$$\langle \widehat{P}\omega, X \rangle = P\langle \omega, \operatorname{Ad} P^{-1} X \rangle, \qquad P \in \operatorname{Diff} M, \; X \in \operatorname{Vec} M, \; \omega \in \Lambda^1 M,$$

has an infinitesimal version of the form

$$\langle L_Y \omega, X \rangle = Y\langle \omega, X \rangle - \langle \omega, (\operatorname{ad} Y)X \rangle, \qquad X, \; Y \in \operatorname{Vec} M, \; \omega \in \Lambda^1 M. \qquad (7)$$

- Taking into account Cartan's formula $L = di + id$, we immediately obtain the following important equality:

$$d\omega(Y, X) = Y\langle \omega, X \rangle - X\langle \omega, Y \rangle - \langle \omega, [Y, X] \rangle, \quad X, \; Y \in \operatorname{Vec} M, \; \omega \in \Lambda^1 M. \; (8)$$

Elements of Symplectic Geometry

Liouville form and symplectic form

- We have already seen that the cotangent bundle $T^*M = \cup_{q\in M} T_q^*M$ of an n-dimensional manifold M is a $2n$-dimensional manifold. Any local coordinates $x = (x_1, \ldots, x_n)$ on M determine canonical local coordinates on T^*M of the form $(\xi, x) = (\xi_1, \ldots, \xi_n; x_1, \ldots, x_n)$ in which any covector $\lambda \in T_{q_0}^*M$ has the decomposition $\lambda = \sum_{i=1}^{n} \xi_i \, dx_i|_{q_0}$.
- The *"tautological" 1-form* (or *Liouville 1-form*) on the cotangent bundle

$$s \in \Lambda^1(T^*M)$$

is defined as follows.

- Let $\lambda \in T^*M$ be a point in the cotangent bundle and $w \in T_\lambda(T^*M)$ a tangent vector to T^*M at λ.
- Denote by π the canonical projection from T^*M to M:

$$\pi \, : \, T^*M \to M,$$
$$\pi \, : \, \lambda \mapsto q, \qquad \lambda \in T_q^*M.$$

- Differential of π is a linear mapping

$$\pi_* : T_\lambda(T^*M) \to T_q M, \qquad q = \pi(\lambda).$$

- The tautological 1-form s at the point λ acts on the tangent vector w in the following way:

$$\langle s_\lambda, w \rangle \stackrel{\text{def}}{=} \langle \lambda, \pi_* w \rangle.$$

- That is, we project the vector $w \in T_\lambda(T^*M)$ to the vector $\pi_* w \in T_q M$, and then act by the covector $\lambda \in T_q^* M$.
- So

$$s_\lambda \stackrel{\text{def}}{=} \lambda \circ \pi_*.$$

- The title "tautological" is explained by the coordinate representation of the form s.
- In canonical coordinates (ξ, x) on T^*M, we have:

$$\lambda = \sum_{i=1}^{n} \xi_i dx_i, \tag{9}$$

$$w = \sum_{i=1}^{n} \alpha_i \frac{\partial}{\partial \xi_i} + \beta_i \frac{\partial}{\partial x_i}.$$

- The projection written in canonical coordinates

$$\pi \ : \ (\xi, x) \mapsto x$$

is a linear mapping, its differential acts as follows:

$$\pi_* \left(\frac{\partial}{\partial \xi_i} \right) = 0, \qquad i = 1, \dots, n,$$

$$\pi_* \left(\frac{\partial}{\partial x_i} \right) = \frac{\partial}{\partial x_i}, \qquad i = 1, \dots, n.$$

- Thus
$$\pi_* w = \sum_{i=1}^{n} \beta_i \frac{\partial}{\partial x_i},$$
consequently,
$$\langle s_\lambda, w\rangle = \langle \lambda, \pi_* w\rangle = \sum_{i=1}^{n} \xi_i \beta_i.$$
- But $\beta_i = \langle dx_i, w\rangle$, so the form s has in coordinates (ξ, x) exactly the same expression
$$s_\lambda = \sum_{i=1}^{n} \xi_i dx_i \tag{10}$$
as the covector λ, see (9).
- Although, definition of the form s does not depend on any coordinates.

Remark 1

In mechanics, the tautological form s is denoted as $p\,dq$.

- Consider the exterior differential of the 1-form s:

$$\sigma \stackrel{\text{def}}{=} ds.$$

- The differential 2-form $\sigma \in \Lambda^2(T^*M)$ is called the *canonical symplectic structure* on T^*M.
- In canonical coordinates, we obtain from (10):

$$\sigma = \sum_{i=1}^{n} d\xi_i \wedge dx_i. \tag{11}$$

- This expression shows that the form σ is nondegenerate, i.e., the bilinear skew-symmetric form

$$\sigma_\lambda : T_\lambda(T^*M) \times T_\lambda(T^*M) \to \mathbb{R}$$

has no kernel:

$$\sigma(w, \cdot) = 0 \quad \Rightarrow \quad w = 0, \qquad w \in T_\lambda(T^*M).$$

- In the following basis in the tangent space $T_\lambda(T^*M)$

$$\frac{\partial}{\partial x_1}, \frac{\partial}{\partial \xi_1}, \ \dots, \ \frac{\partial}{\partial x_n}, \frac{\partial}{\partial \xi_n},$$

the form σ_λ has the block matrix
$$\begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & 1 \\ & & & -1 & 0 \end{pmatrix}.$$

- The form σ is closed: $d\sigma = 0$ since it is exact: $\sigma = ds$, and $d \circ d = 0$.

Remarks

(1) A closed nondegenerate exterior differential 2-form on a $2n$-dimensional manifold is called a *symplectic structure*. A manifold with a symplectic structure is called a *symplectic manifold*. The cotangent bundle T^*M with the canonical symplectic structure σ is the most important example of a symplectic manifold.
(2) In mechanics, the 2-form σ is known as the form $dp \wedge dq$.

Hamiltonian vector fields

- Due to the symplectic structure $\sigma \in \Lambda^2(T^*M)$, we can develop the Hamiltonian formalism on T^*M.
- A *Hamiltonian* is an arbitrary smooth function on the cotangent bundle:

$$h \in C^\infty(T^*M).$$

- To any Hamiltonian h, we associate the *Hamiltonian vector field*

$$\vec{h} \in \mathrm{Vec}(T^*M)$$

  by the rule:

$$\sigma_\lambda(\cdot, \vec{h}) = d_\lambda h, \qquad \lambda \in T^*M. \tag{12}$$

- In terms of the interior product $i_v\omega(\cdot,\cdot) = \omega(v,\cdot)$, the Hamiltonian vector field is a vector field $\vec{h}$ that satisfies

$$i_{\vec{h}}\sigma = -dh.$$

- Since the symplectic form σ is nondegenerate, the mapping

$$w \mapsto \sigma_\lambda(\cdot, w)$$

is a linear isomorphism

$$T_\lambda(T^*M) \to T^*_\lambda(T^*M),$$

thus the Hamiltonian vector field $\vec{h}$ in (12) exists and is uniquely determined by the Hamiltonian function h.

- In canonical coordinates (ξ, x) on T^*M we have

$$dh = \sum_{i=1}^{n} \left(\frac{\partial h}{\partial \xi_i} d\xi_i + \frac{\partial h}{\partial x_i} dx_i \right),$$

then in view of (11)

$$\vec{h} = \sum_{i=1}^{n} \left(\frac{\partial h}{\partial \xi_i} \frac{\partial}{\partial x_i} - \frac{\partial h}{\partial x_i} \frac{\partial}{\partial \xi_i} \right). \qquad (13)$$

- So the *Hamiltonian system* of ODEs corresponding to h

$$\dot{\lambda} = \vec{h}(\lambda), \qquad \lambda \in T^*M,$$

reads in canonical coordinates as follows:

$$\begin{cases} \dot{x}_i = \dfrac{\partial h}{\partial \xi_i}, & i = 1, \dots, n, \\ \dot{\xi}_i = -\dfrac{\partial h}{\partial x_i}, & i = 1, \dots, n. \end{cases}$$

- The Hamiltonian function can depend on a parameter: h_t, $t \in \mathbb{R}$. Then the nonautonomous Hamiltonian vector field $\vec{h}_t$, $t \in \mathbb{R}$ is defined in the same way as in the autonomous case.
- The flow of a Hamiltonian system preserves the symplectic form σ.

Proposition 1

Let $\vec{h}_t$ be a nonautonomous Hamiltonian vector field on T^*M. Then

$$\widehat{\left(\overrightarrow{\exp} \int_0^t \vec{h}_\tau \, d\tau \right)} \sigma = \sigma.$$

Proof:

- In view of equality (4), we have

$$\widehat{\left(\overrightarrow{\exp} \int_0^t \vec{h}_\tau \, d\tau \right)} = \overrightarrow{\exp} \int_0^t L_{\vec{h}_\tau} \, d\tau,$$

  thus the statement of this proposition can be rewritten as $L_{\vec{h}_t} \sigma = 0.$
- But this Lie derivative is easily computed by Cartan's formula:

$$L_{\vec{h}_t} \sigma = i_{\vec{h}_t} \circ \underbrace{d\sigma}_{=0} + d \circ \underbrace{i_{\vec{h}_t} \sigma}_{=-dh_t} = -d \circ dh_t = 0.$$

- Moreover, there holds a local converse statement: if a flow preserves σ, then it is locally Hamiltonian.
- Indeed,

$$\widehat{\left(\overrightarrow{\exp} \int_0^t f_\tau \, d\tau\right)} \sigma = \sigma \quad \Leftrightarrow \quad L_{f_t}\sigma = 0,$$

further

$$L_{f_t}\sigma = i_{f_t} \circ \underbrace{d\sigma}_{=0} + d \circ i_{f_t}\sigma,$$

thus

$$L_{f_t}\sigma = 0 \quad \Leftrightarrow \quad d \circ i_{f_t}\sigma = 0.$$

- If the form $i_{f_t}\sigma$ is closed, then it is locally exact (Poincaré's Lemma), i.e., there exists a Hamiltonian h_t such that locally $f_t = \vec{h}_t$.
- Essentially, only Hamiltonian flows preserve σ (globally, "multi-valued Hamiltonians" can appear).
- If a manifold M is simply connected, then there holds a global statement: a flow on T^*M is Hamiltonian if and only if it preserves the symplectic structure.

- The *Poisson bracket* of Hamiltonians $a, b \in C^\infty(T^*M)$ is a Hamiltonian
$$\{a, b\} \in C^\infty(T^*M)$$
defined in one of the following equivalent ways:
$$\{a, b\} = \vec{a}b = \langle db, \vec{a}\rangle = \sigma(\vec{a}, \vec{b}) = -\sigma(\vec{b}, \vec{a}) = -\vec{b}a.$$
- It is obvious that Poisson bracket is bilinear and skew-symmetric:
$$\{a, b\} = -\{b, a\}.$$
- In canonical coordinates (ξ, x) on T^*M,
$$\{a, b\} = \sum_{i=1}^{n} \left(\frac{\partial a}{\partial \xi_i} \frac{\partial b}{\partial x_i} - \frac{\partial a}{\partial x_i} \frac{\partial b}{\partial \xi_i} \right). \tag{14}$$
- Leibniz rule for Poisson bracket easily follows from definition:
$$\{a, bc\} = \{a, b\}c + b\{a, c\}$$
(here bc is the usual pointwise product of functions b and c).

- Symplectomorphisms of cotangent bundle preserve Hamiltonian vector fields; the action of a symplectomorphism $P \in \mathrm{Diff}(T^*M)$, $\widehat{P}\sigma = \sigma$, on a Hamiltonian vector field $\vec{h}$ reduces to the action of P on the Hamiltonian function as substitution of variables:

$$\mathrm{Ad}\, P\, \vec{h} = \overrightarrow{Ph}\ .$$

- This follows from the chain

$$\sigma\left(X, \mathrm{Ad}\, P\, \vec{h}\right) = \widehat{P}\sigma\left(X, \mathrm{Ad}\, P\, \vec{h}\right) = P\sigma\left(\mathrm{Ad}\, P^{-1} X, \vec{h}\right)$$
$$= P\langle dh, \mathrm{Ad}\, P^{-1} X\rangle = X(Ph) = \sigma\left(X, \overrightarrow{Ph}\right), \qquad X \in \mathrm{Vec}(T^*M).$$

- In particular, a Hamiltonian flow transforms a Hamiltonian vector field into a Hamiltonian vector field:

$$\mathrm{Ad}\, P^t\, \vec{b}_t = \overrightarrow{P^t b_t}, \qquad P^t = \overrightarrow{\exp} \int_0^t \vec{a}_\tau\, d\tau. \tag{15}$$

- Infinitesimally, this equality implies Jacobi identity for Poisson bracket.

Proposition 2

$$\{a, \{b, c\}\} + \{b, \{c, a\}\} + \{c, \{a, b\}\} = 0, \qquad a, b, c \in C^\infty(T^*M). \tag{16}$$

Proof:

- Any symplectomorphism $P \in \mathrm{Diff}(T^*M)$, $\widehat{P}\sigma = \sigma$, preserves Poisson brackets:

$$P\{b, c\} = P\sigma\left(\vec{b}, \vec{c}\right) = \widehat{P}\sigma\left(\mathrm{Ad}\, P\, \vec{b}, \mathrm{Ad}\, P\, \vec{c}\right) = \sigma\left(\overrightarrow{Pb}, \overrightarrow{Pc}\right) = \{Pb, Pc\}.$$

- Taking $P = e^{t\vec{a}}$ and differentiating at $t = 0$, we come to Jacobi identity:

$$\{a, \{b, c\}\} = \{\{a, b\}, c\} + \{b, \{a, c\}\}.$$

- So the space of all Hamiltonians $C^\infty(T^*M)$ forms a Lie algebra with Poisson bracket as a product.
- The correspondence

$$a \mapsto \vec{a}, \qquad a \in C^\infty(T^*M), \tag{17}$$

is a homomorphism from the Lie algebra of Hamiltonians to the Lie algebra of Hamiltonian vector fields on M. This follows from the next statement.

Corollary 1

$\overrightarrow{\{a, b\}} = [\vec{a}, \vec{b}]$ *for any Hamiltonians* $a, b \in C^\infty(T^*M)$.

Proof:

- Jacobi identity can be rewritten as

$$\{\{a, b\}, c\} = \{a, \{b, c\}\} - \{b, \{a, c\}\},$$

i.e.,

$$\overrightarrow{\{a, b\}}\ c = \vec{a} \circ \vec{b}\, c - \vec{b} \circ \vec{a}\, c = [\vec{a}, \vec{b}]\, c, \qquad c \in C^\infty(T^*M).$$

- It is easy to see from the coordinate representation (13) that the kernel of the mapping $a \mapsto \vec{a}$ consists of constant functions, i.e., this is isomorphism up to constants.
- On the other hand, this homomorphism is far from being onto all vector fields on T^*M.
- Indeed, a general vector field on T^*M is locally defined by arbitrary $2n$ smooth real functions of $2n$ variables, while a Hamiltonian vector field is determined by just one real function of $2n$ variables, a Hamiltonian.

Theorem 2 (Nöther)

A function $a \in C^\infty(T^*M)$ is an integral of a Hamiltonian system of ODEs

$$\dot{\lambda} = \vec{h}(\lambda), \qquad \lambda \in T^*M, \tag{18}$$

i.e.,

$$e^{t\vec{h}} a = a \qquad t \in \mathbb{R},$$

if and only if it Poisson-commutes with the Hamiltonian:

$$\{a, h\} = 0.$$

Proof:

- $e^{t\vec{h}} a \equiv a \Leftrightarrow 0 = \vec{h} a = \{h, a\}$.

Corollary 3

$e^{t\vec{h}} h = h$, *i.e., any Hamiltonian $h \in C^\infty(T^*M)$ is an integral of the corresponding Hamiltonian system* (18).

- Further, Jacobi identity for Poisson brackets implies that the set of integrals of the Hamiltonian system (18) forms a Lie algebra with respect to Poisson brackets.

Corollary 4

$\{h, a\} = \{h, b\} = 0 \Rightarrow \{h, \{a, b\}\} = 0.$

Remark 2

The Hamiltonian formalism developed generalizes for arbitrary symplectic manifolds.

Linear on fibers Hamiltonians

- We introduce a construction that works only on T^*M. Given a vector field $X \in \operatorname{Vec} M$, we define a Hamiltonian function

$$X^* \in C^\infty(T^*M),$$

which is linear on fibers T_q^*M, as follows:

$$X^*(\lambda) = \langle \lambda, X(q) \rangle, \qquad \lambda \in T^*M, \quad q = \pi(\lambda).$$

- In canonical coordinates (ξ, x) on T^*M we have:

$$X = \sum_{i=1}^{n} a_i(x) \frac{\partial}{\partial x_i}, \qquad X^* = \sum_{i=1}^{n} \xi_i a_i(x). \tag{19}$$

- This coordinate representation implies that

$$\{X^*, Y^*\} = [X, Y]^*, \qquad X, Y \in \operatorname{Vec} M,$$

i.e., Poisson brackets of Hamiltonians linear on fibers in T^*M contain usual Lie brackets of vector fields on M.

- The Hamiltonian vector field $\overrightarrow{X^*} \in \mathrm{Vec}(T^*M)$ corresponding to the Hamiltonian function X^* is called the *Hamiltonian lift* of the vector field $X \in \mathrm{Vec}\, M$.
- It is easy to see from the coordinate representation (19) that

$$\pi_* \overrightarrow{X^*} = X.$$

- Now we pass to nonautonomous vector fields. Let X_t be a nonautonomous vector field and

$$P_{\tau,t} = \overrightarrow{\exp} \int_{\tau}^{t} X_\theta \, d\theta$$

the corresponding flow on M.

- The flow $P = P_{\tau,t}$ acts on M:

$$P : M \to M, \qquad P : q_0 \mapsto q_1,$$

its differential pushes tangent vectors forward:

$$P_* : T_{q_0} M \to T_{q_1} M,$$

and the dual mapping P^* pulls covectors back:

$$P^* : T^*_{q_1} M \to T^*_{q_0} M.$$

- Thus we have a flow on covectors (i.e., on points of the cotangent bundle):

$$P^*_{\tau,t} : T^* M \to T^* M.$$

- Let V_t be the nonautonomous vector field on T^*M that generates the flow $P^*_{\tau,t}$:

$$V_t = \left.\frac{d}{d\,\varepsilon}\right|_{\varepsilon=0} P^*_{t,t+\varepsilon}.$$

- Then

$$\frac{d}{d\,t} P^*_{\tau,t} = \left.\frac{d}{d\,\varepsilon}\right|_{\varepsilon=0} P^*_{\tau,t+\varepsilon} = \left.\frac{d}{d\,\varepsilon}\right|_{\varepsilon=0} P^*_{t,t+\varepsilon} \circ P^*_{\tau,t} = V_t \circ P^*_{\tau,t},$$

so the flow $P^*_{\tau,t}$ is a solution to the Cauchy problem

$$\frac{d}{d\,t} P^*_{\tau,t} = V_t \circ P^*_{\tau,t}, \qquad P^*_{\tau,\tau} = \mathrm{Id},$$

i.e., it is the left chronological exponential:

$$P^*_{\tau,t} = \overleftarrow{\exp} \int_{\tau}^{t} V_\theta \, d\theta.$$

- It turns out that the nonautonomous field V_t is simply related with the Hamiltonian vector field corresponding to the Hamiltonian X_t^*:

$$V_t = -\overrightarrow{X_t^*}\ . \tag{20}$$

- Indeed, the flow $P_{\tau,t}^*$ preserves the tautological form s, thus

$$L_{V_t} s = 0.$$

- By Cartan's formula,

$$i_{V_t}\sigma = -d\langle s, V_t\rangle,$$

i.e., the field V_t is Hamiltonian:

$$V_t = \overrightarrow{\langle s, V_t\rangle}\ .$$

- But $\pi_* V_t = -X_t$, consequently,

$$\langle s, V_t\rangle = -X_t^*,$$

and equality (20) follows.

- Taking into account the relation between the left and right chronological exponentials, we obtain

$$P^*_{\tau,t} = \overleftarrow{\exp} \int_\tau^t - \overrightarrow{X^*_\theta} \, d\theta = \overrightarrow{\exp} \int_t^\tau \overrightarrow{X^*_\theta} \, d\theta.$$

- We proved the following statement.

Proposition 3

Let X_t be a complete nonautonomous vector field on M. Then

$$\left(\overrightarrow{\exp} \int_\tau^t X_\theta \, d\theta \right)^* = \overrightarrow{\exp} \int_t^\tau \overrightarrow{X^*_\theta} \, d\theta.$$

- In particular, for autonomous vector fields $X \in \operatorname{Vec} M$,

$$\left(e^{tX} \right)^* = e^{-t\overrightarrow{X^*}}.$$