Elements of Chronological Calculus-3. Differential Forms and Symplectic Geometry

(Lecture 5)

Yuri Sachkov

Program Systems Institute
Russian Academy of Sciences
Pereslavl-Zalessky, Russia
yusachkov@gmail.com

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Plan of previous lecture

1. Estimates and convergence of the series
2. Left chronological exponential
3. Uniqueness for functional and operator ODEs
4. Autonomous vector fields
5. Action of diffeomorphisms on vector fields
6. Commutation of flows
7. Variations formula

Plan of this lecture

1. Derivative of flow with respect to parameter
2. Differential 1-forms
3. Differential k-forms
4. Exterior differential

Definition of the right chronological exponential

- The Cauchy problem $\dot{q} = V_t(q)$, $q(0) = q_0$, rewritten as a linear equation for Lipschitzian w.r.t. t families of functionals on $C^\infty(M)$:
$$\dot{q}(t) = q(t) \circ V_t, \qquad q(0) = q_0, \tag{1}$$
is satisfied for the family of functionals
$$q(t, q_0) \,:\, C^\infty(M) \to \mathbb{R}, \qquad q_0 \in M, \quad t \in \mathbb{R}$$
constructed before.
- We proved that this Cauchy problem has no other solutions.
- Thus the flow defined as
$$P^t \;:\; q_0 \mapsto q(t, q_0) \tag{2}$$
is a unique solution of the operator Cauchy problem $\dot{P}^t = P^t \circ V_t$, $P^0 = \mathrm{Id}$ (where Id is the identity operator), in the class of Lipschitzian flows on M.
- The flow P^t determined in (2) is called the *right chronological exponential* of the field V_t and is denoted as $P^t = \overrightarrow{\exp} \int_0^t V_\tau \, d\tau$.

Formal series expansion

- Purely formally passing to the limit $n \to \infty$, we obtained a formal series for the solution $q(t)$ to problem $\dot{q}(t) = q(t) \circ V_t$, $q(0) = q_0$:

$$q_0 \circ \left(\mathrm{Id} + \sum_{n=1}^{\infty} \int_{\Delta_n(t)} \cdots \int V_{\tau_n} \circ \cdots \circ V_{\tau_1} \, d\tau_n \, \ldots \, d\tau_1 \right),$$

thus for the solution P^t to operator Cauchy problem $\dot{P}^t = P^t \circ V_t$, $P^0 = \mathrm{Id}$:

$$\mathrm{Id} + \sum_{n=1}^{\infty} \int_{\Delta_n(t)} \cdots \int V_{\tau_n} \circ \cdots \circ V_{\tau_1} \, d\tau_n \, \ldots \, d\tau_1 . \tag{3}$$

Convergence of the series

- Unfortunately, series (3) never converges on $C^\infty(M)$ in the weak sense (if $V_t \not\equiv 0$): there always exists a smooth function on M, on which it diverges.
- Although, one can show that series (3) gives an asymptotic expansion for the chronological exponential $P^t = \overrightarrow{\exp} \int_0^t V_\tau \, d\tau$:

$$\overrightarrow{\exp} \int_0^t V_\tau \, d\tau \approx \mathrm{Id} + \sum_{n=1}^{\infty} \int \cdots \int_{\Delta_n(t)} V_{\tau_n} \circ \cdots \circ V_{\tau_1} \, d\tau_n \ldots d\tau_1 . \tag{4}$$

- In the sequel we will use terms of the zeroth, first, and second orders of the series obtained:

$$\overrightarrow{\exp} \int_0^t V_\tau \, d\tau \approx \mathrm{Id} + \int_0^t V_\tau \, d\tau + \iint_{0 \le \tau_2 \le \tau_1 \le t} V_{\tau_2} \circ V_{\tau_1} \, d\tau_2 \, d\tau_1 + \cdots \, .$$

Variations formula

- Consider an ODE of the form
$$\dot{q} = V_t(q) + W_t(q). \tag{5}$$
We think of V_t as an initial vector field and W_t as its perturbation.
- Our aim is to find a formula for the flow Q^t of the new field $V_t + W_t$ as a perturbation of the flow $P^t = \overrightarrow{\exp} \int_0^t V_\tau \, d\tau$ of the initial field V_t.
- We obtained the required decomposition of the perturbed flow:
$$\overrightarrow{\exp} \int_0^t (V_\tau + W_\tau) \, d\tau = \overrightarrow{\exp} \int_0^t \left(\overrightarrow{\exp} \int_0^\tau \operatorname{ad} V_\theta \, d\theta \right) W_\tau \, d\tau \circ \overrightarrow{\exp} \int_0^t V_\tau \, d\tau. \tag{6}$$
- This equality is called the *variations formula*.
- It can be written as follows:
$$\overrightarrow{\exp} \int_0^t (V_\tau + W_\tau) \, d\tau = \overrightarrow{\exp} \int_0^t (\operatorname{Ad} P^\tau) \, W_\tau \, d\tau \circ P^t, \qquad P^t = \overrightarrow{\exp} \int_0^t V_\tau \, d\tau.$$
- So the perturbed flow is a composition of the initial flow P^t with the flow of the perturbation W_t twisted by P^t.

- Now we obtain another form of the variations formula, with the flow P^t to the left of the twisted flow.
- We have

$$
\begin{aligned}
\overrightarrow{\exp} \int_0^t (V_\tau + W_\tau)\, d\tau &= \overrightarrow{\exp} \int_0^t (\operatorname{Ad} P^\tau)\, W_\tau \, d\tau \circ P^t \\
&= P^t \circ \left(P^t\right)^{-1} \circ \overrightarrow{\exp} \int_0^t (\operatorname{Ad} P^\tau)\, W_\tau \, d\tau \circ P^t \\
&= P^t \circ \overrightarrow{\exp} \int_0^t \left(\operatorname{Ad} \left(P^t\right)^{-1} \circ \operatorname{Ad} P^\tau\right) W_\tau \, d\tau \\
&= P^t \circ \overrightarrow{\exp} \int_0^t \left(\operatorname{Ad} \left(\left(P^t\right)^{-1} \circ P^\tau\right)\right) W_\tau \, d\tau .
\end{aligned}
$$

- Notice that

$$
\left(P^t\right)^{-1} \circ P^\tau = \overrightarrow{\exp} \int_t^\tau V_\theta \, d\theta .
$$

- Thus

$$\overrightarrow{\exp} \int_0^t (V_\tau + W_\tau)\, d\tau = P^t \circ \overrightarrow{\exp} \int_0^t \left(\overrightarrow{\exp} \int_t^\tau \operatorname{ad} V_\theta \, d\theta \right) W_\tau \, d\tau$$
$$= \overrightarrow{\exp} \int_0^t V_\tau \, d\tau \circ \overrightarrow{\exp} \int_0^t \left(\overrightarrow{\exp} \int_t^\tau \operatorname{ad} V_\theta \, d\theta \right) W_\tau \, d\tau. \quad (7)$$

- For autonomous vector fields $V, W \in \operatorname{Vec} M$, the variations formulas (6), (7) take the form:

$$e^{t(V+W)} = \overrightarrow{\exp} \int_0^t e^{\tau \operatorname{ad} V} W \, d\tau \circ e^{tV} = e^{tV} \circ \overrightarrow{\exp} \int_0^t e^{(\tau - t) \operatorname{ad} V} W \, d\tau. \quad (8)$$

- In particular, for $t = 1$ we have

$$e^{V+W} = \overrightarrow{\exp} \int_0^1 e^{\tau \operatorname{ad} V} W \, d\tau \circ e^V .$$

Derivative of flow with respect to parameter

- Let $V_t(s)$ be a nonautonomous vector field depending smoothly on a real parameter s. We study dependence of the flow of $V_t(s)$ on the parameter s.
- We write

$$\overrightarrow{\exp} \int_0^t V_\tau(s+\varepsilon)\, d\tau = \overrightarrow{\exp} \int_0^t \left(V_\tau(s) + \delta_{V_\tau}(s,\varepsilon)\right)\, d\tau \tag{9}$$

with the perturbation $\delta_{V_\tau}(s,\varepsilon) = V_\tau(s+\varepsilon) - V_\tau(s)$.
- By the variations formula (6), the previous flow is equal to

$$\overrightarrow{\exp} \int_0^t \left(\overrightarrow{\exp} \int_0^\tau \operatorname{ad} V_\theta(s)\, d\theta\right) \delta_{V_\tau}(s,\varepsilon)\, d\tau \circ \overrightarrow{\exp} \int_0^t V_\tau(s)\, d\tau.$$

- Now we expand in ε:

$$\begin{aligned}
\delta_{V_\tau}(s,\varepsilon) &= \varepsilon \frac{\partial}{\partial s} V_\tau(s) + O(\varepsilon^2), \qquad \varepsilon \to 0, \\
W_\tau(s,\varepsilon) &\stackrel{\text{def}}{=} \left(\overrightarrow{\exp} \int_0^\tau \operatorname{ad} V_\theta(s)\, d\theta \right) \delta_{V_\tau}(s,\varepsilon) \\
&= \varepsilon \left(\overrightarrow{\exp} \int_0^\tau \operatorname{ad} V_\theta(s)\, d\theta \right) \frac{\partial}{\partial s} V_\tau(s) + O(\varepsilon^2), \qquad \varepsilon \to 0,
\end{aligned}$$

thus

$$\begin{aligned}
\overrightarrow{\exp} \int_0^t W_\tau(s,\varepsilon)\, d\tau &= \mathrm{Id} + \int_0^t W_\tau(s,\varepsilon)\, d\tau + O(\varepsilon^2) \\
&= \mathrm{Id} + \varepsilon \int_0^t \left(\overrightarrow{\exp} \int_0^\tau \operatorname{ad} V_\theta(s)\, d\theta \right) \frac{\partial}{\partial s} V_\tau(s)\, d\tau + O(\varepsilon^2).
\end{aligned}$$

- Finally,

$$\overrightarrow{\exp} \int_0^t V_\tau(s+\varepsilon)\, d\tau = \overrightarrow{\exp} \int_0^t W_{s,\tau}(\varepsilon)\, d\tau \circ \overrightarrow{\exp} \int_0^t V_\tau(s)\, d\tau$$
$$= \overrightarrow{\exp} \int_0^t V_\tau(s)\, d\tau$$
$$+ \varepsilon \int_0^t \left(\overrightarrow{\exp} \int_0^\tau \operatorname{ad} V_\theta(s)\, d\theta \right) \frac{\partial}{\partial s} V_\tau(s)\, d\tau \circ \overrightarrow{\exp} \int_0^t V_\tau(s)\, d\tau + O(\varepsilon^2),$$

that is,

$$\frac{\partial}{\partial s} \overrightarrow{\exp} \int_0^t V_\tau(s)\, d\tau = \int_0^t \left(\overrightarrow{\exp} \int_0^\tau \operatorname{ad} V_\theta(s)\, d\theta \right) \frac{\partial}{\partial s} V_\tau(s)\, d\tau \circ \overrightarrow{\exp} \int_0^t V_\tau(s)\, d\tau. \quad (10)$$

- Similarly, we obtain from the variations formula (7) the equality

$$\frac{\partial}{\partial s} \overrightarrow{\exp} \int_0^t V_\tau(s)\, d\tau$$
$$= \overrightarrow{\exp} \int_0^t V_\tau(s)\, d\tau \circ \int_0^t \left(\overrightarrow{\exp} \int_t^\tau \operatorname{ad} V_\theta(s)\, d\theta \right) \frac{\partial}{\partial s} V_\tau(s)\, d\tau. \quad (11)$$

- For an autonomous vector field depending on a parameter $V(s)$, formula (10) takes the form

$$\frac{\partial}{\partial s} e^{tV(s)} = \int_0^t e^{\tau \operatorname{ad} V(s)} \frac{\partial V}{\partial s}\, d\tau \circ e^{tV(s)},$$

and at $t = 1$:

$$\frac{\partial}{\partial s} e^{V(s)} = \int_0^1 e^{\tau \operatorname{ad} V(s)} \frac{\partial V}{\partial s}\, d\tau \circ e^{V(s)}. \quad (12)$$

Proposition 1

Assume that

$$\left[\int_0^t V_\tau \, d\tau, V_t\right] = 0 \qquad \forall t. \tag{13}$$

Then

$$\overrightarrow{\exp} \int_0^t V_\tau \, d\tau = e^{\int_0^t V_\tau \, d\tau} \qquad \forall t. \tag{14}$$

That is, we state that under the commutativity assumption (13), the chronological exponential $\overrightarrow{\exp} \int_0^t V_\tau \, d\tau$ coincides with the flow $Q^t = e^{\int_0^t V_\tau \, d\tau}$ defined as follows:

$$Q^t = Q_1^t,$$
$$\frac{\partial Q_s^t}{\partial s} = \int_0^t V_\tau \, d\tau \circ Q_s^t, \qquad Q_0^t = \mathrm{Id}\,.$$

Proof.

- We show that the exponential in the right-hand side of (14) satisfies the same ODE as the chronological exponential in the left-hand side.
- By (12), we have

$$\frac{d}{d\,t} e^{\int_0^t V_\tau \, d\tau} = \int_0^1 e^{\tau \operatorname{ad} \int_0^t V_\theta \, d\theta} V_t \, d\tau \circ e^{\int_0^t V_\tau \, d\tau}.$$

- In view of equality (13),

$$e^{\tau \operatorname{ad} \int_0^t V_\theta \, d\theta} V_t = V_t,$$

  thus

$$\frac{d}{d\,t} e^{\int_0^t V_\tau \, d\tau} = V_t \circ e^{\int_0^t V_\tau \, d\tau}.$$

- By equality (13), we can permute operators in the right-hand side:

$$\frac{d}{d\,t} e^{\int_0^t V_\tau \, d\tau} = e^{\int_0^t V_\tau \, d\tau} \circ V_t.$$

- Notice the initial condition
$$e^{\int_0^t V_\tau \, d\tau}\Big|_{t=0} = \mathrm{Id}\,.$$
- Now the statement follows since the Cauchy problem for flows
$$\dot{A}_t = A_t \circ V_t, \qquad A_0 = \mathrm{Id}$$
has a unique solution:
$$A_t = e^{\int_0^t V_\tau \, d\tau} = \overrightarrow{\exp} \int_0^t V_\tau \, d\tau.$$
$\square$

- Here we finish our excursion to Chronological Calculus.

Differential 1-forms

Linear forms

- E a real vector space of finite dimension n.
- A *linear form* on E is a linear function $\xi : E \to \mathbb{R}$.
- The set of linear forms on E has a natural structure of a vector space called the *dual space* to E and denoted by E^*.
- If vectors $e_1, \dots, e_n$ form a basis of E, then the corresponding *dual basis* of E^* is formed by the covectors $e_1^*, \dots, e_n^*$ such that

$$\langle e_i^*, e_j \rangle = \delta_{ij}, \qquad i,\ j = 1, \dots n.$$

- So the dual space has the same dimension as the initial one:

$$\dim E^* = n = \dim E.$$

Cotangent bundle

- M a smooth manifold and T_qM its tangent space at a point $q \in M$.
- The space of linear forms on T_qM, i.e., the dual space $(T_qM)^*$ to T_qM, is called the *cotangent space* to M at q and is denoted as T_q^*M.
- The disjoint union of all cotangent spaces is called the *cotangent bundle* of M:

$$T^*M \overset{\mathrm{def}}{=} \bigsqcup_{q \in M} T_q^*M.$$

- The set T^*M has a natural structure of a smooth manifold of dimension $2n$, where $n = \dim M$.
- Local coordinates on T^*M are constructed from local coordinates on M.
- Let $O \subset M$ be a coordinate neighborhood and let

$$\Phi : O \to \mathbb{R}^n, \qquad \Phi(q) = (x_1(q), \ldots, x_n(q)),$$

be a local coordinate system.

- Differentials of the coordinate functions

$$dx_i\big|_q \in T_q^*M, \qquad i = 1, \dots, n, \quad q \in O,$$

form a basis in the cotangent space T_q^*M.

- The dual basis in the tangent space T_qM is formed by the vectors

$$\left.\frac{\partial}{\partial x_i}\right|_q \in T_qM, \qquad i = 1, \dots, n, \quad q \in O,$$

$$\left\langle dx_i, \frac{\partial}{\partial x_j} \right\rangle \equiv \delta_{ij}, \qquad i, \ j = 1, \dots, n.$$

- Any linear form $\xi \in T_q^*M$ can be decomposed via the basis forms:

$$\xi = \sum_{i=1}^{n} \xi_i \, dx_i.$$

- So any covector $\xi \in T^*M$ is characterized by n coordinates $(x_1, \dots, x_n)$ of the point $q \in M$ where ξ is attached, and by n coordinates $(\xi_1, \dots, \xi_n)$ of the linear form ξ in the basis $dx_1, \dots, dx_n$.

- Mappings of the form

$$\xi \mapsto (\xi_1, \ldots, \xi_n;\ x_1, \ldots, x_n)$$

  define local coordinates on the cotangent bundle. Consequently, T^*M is a $2n$-dimensional manifold.
- Coordinates of the form (ξ, x) are called *canonical coordinates* on T^*M.

- If $F : M \to N$ is a smooth mapping between smooth manifolds, then the differential
$$F_* : T_q M \to T_{F(q)} N$$
has the adjoint (dual) mapping
$$F^* \stackrel{\text{def}}{=} (F_*)^* : T^*_{F(q)} N \to T^*_q M$$
defined as follows:
$$F^*\xi = \xi \circ F_*, \qquad \xi \in T^*_{F(q)} N,$$
$$\langle F^*\xi, v \rangle = \langle \xi, F_* v \rangle, \qquad v \in T_q M.$$
- A vector $v \in T_q M$ is pushed forward by the differential F_* to the vector $F_* v \in T_{F(q)} N$, while a covector $\xi \in T^*_{F(q)} N$ is pulled back to the covector $F^*\xi \in T^*_q M$.
- So a smooth mapping $F : M \to N$ between manifolds induces a smooth mapping $F^* : T^*N \to T^*M$ between their cotangent bundles.

Differential 1-forms

- A *differential 1-form* on M is a smooth mapping $q \mapsto \omega_q \in T_q^*M, \quad q \in M$, i.e, a family $\omega = \{\omega_q\}$ of linear forms on the tangent spaces T_qM smoothly depending on the point $q \in M$.
- The set of all differential 1-forms on M has a natural structure of an infinite-dimensional vector space denoted as $\Lambda^1 M$.
- Like linear forms on a vector space are dual objects to vectors of the space, differential forms on a manifold are dual objects to smooth curves in the manifold.
- The pairing operation is the *integral* of a differential 1-form $\omega \in \Lambda^1 M$ along a smooth oriented curve $\gamma : [t_0, t_1] \to M$, defined as follows:

$$\int_\gamma \omega \stackrel{\text{def}}{=} \int_{t_0}^{t_1} \langle \omega_{\gamma(t)}, \dot{\gamma}(t) \rangle \, dt.$$

- The integral of a 1-form along a curve does not change under orientation-preserving smooth reparametrizations of the curve and changes its sign under change of orientation.

Differential k-forms

- A differential k-form on M is an object to integrate over k-dim. surfaces in M.
- Infinitesimally, a k-dimensional surface is presented by its tangent space, i.e., a k-dimensional subspace in T_qM.
- We need a dual object to the set of k-dim. subspaces in the linear space.
- Fix a linear space E.
- A k-dimensional subspace is defined by its basis $v_1, \dots, v_k \in E$.
- The dual objects should be mappings

  $$(v_1, \dots, v_k) \mapsto \omega(v_1, \dots, v_k) \in \mathbb{R}$$

  such that $\omega(v_1, \dots, v_k)$ depend only on the linear hull $\operatorname{span}\{v_1, \dots, v_k\}$ and the oriented volume of the k-dimensional parallelepiped generated by $v_1, \dots, v_k$.
- Moreover, the dependence on the volume should be linear.
- Recall that the ratio of volumes of the parallelepipeds generated by vectors $w_i = \sum_{j=1}^{k} \alpha_{ij} v_j$, $i = 1, \dots, k$, and the vectors $v_1, \dots, v_k$, equals $\det(\alpha_{ij})_{i,j=1}^{k}$, and that determinant of a $k \times k$ matrix is a multilinear skew-symmetric form of the columns of the matrix.

Exterior k-forms

- Let E be a finite-dimensional real vector space, $\dim E = n$, and let $k \in \mathbb{N}$.
- An *exterior k-form* on E is a mapping

  $$\omega : \underbrace{E \times \cdots \times E}_{k \text{ times}} \to \mathbb{R},$$

  which is multilinear:

  $$\omega(v_1, \ldots, \alpha_1 v_i^1 + \alpha_2 v_i^2, \ldots, v_k)$$
  $$= \alpha_1 \omega(v_1, \ldots, v_i^1, \ldots, v_k) + \alpha_2 \omega(v_1, \ldots, v_i^2, \ldots, v_k), \qquad \alpha_1,\ \alpha_2 \in \mathbb{R},$$

  and skew-symmetric:

  $$\omega(v_1, \ldots, v_i, \ldots, v_j, \ldots, v_k) = -\omega(v_1, \ldots, v_j, \ldots, v_i, \ldots, v_k), \quad i,\ j = 1, \ldots, k.$$

- The set of all exterior k-forms on E is denoted by $\Lambda^k E$.
- By the skew-symmetry, any exterior form of order $k > n$ is zero, thus $\Lambda^k E = \{0\}$ for $k > n$.

- Exterior forms can be multiplied by real numbers, and exterior forms of the same order k can be added one with another, so each $\Lambda^k E$ is a vector space.
- We construct a basis of $\Lambda^k E$ after we consider another operation between exterior forms — the exterior product.
- The exterior product of two forms $\omega_1 \in \Lambda^{k_1} E$, $\omega_2 \in \Lambda^{k_2} E$ is an exterior form $\omega_1 \wedge \omega_2$ of order $k_1 + k_2$.
- Given linear 1-forms $\omega_1, \omega_2 \in \Lambda^1 E$, we have a natural (tensor) product for them:

$$\omega_1 \otimes \omega_2 \ : \ (v_1, v_2) \mapsto \omega_1(v_1)\omega_2(v_2), \qquad v_1, v_2 \in E.$$

- The result is a bilinear but not a skew-symmetric form.
- The *exterior product* is the anti-symmetrization of the tensor one:

$$\omega_1 \wedge \omega_2 \ : \ (v_1, v_2) \mapsto \omega_1(v_1)\omega_2(v_2) - \omega_1(v_2)\omega_2(v_1), \qquad v_1, v_2 \in E.$$

- Similarly, the tensor and exterior products of forms $\omega_1 \in \Lambda^{k_1} E$ and $\omega_2 \in \Lambda^{k_2} E$ are the following forms of order $k_1 + k_2$:

$$\begin{aligned}\omega_1 \otimes \omega_2 \ &: \ (v_1, \dots, v_{k_1+k_2}) \mapsto \omega_1(v_1, \dots, v_{k_1})\omega_2(v_{k_1+1}, \dots, v_{k_1+k_2}),\\ \omega_1 \wedge \omega_2 \ &: \ (v_1, \dots, v_{k_1+k_2}) \mapsto\\ &\frac{1}{k_1!\,k_2!}\sum_{\sigma}(-1)^{\nu(\sigma)}\omega_1(v_{\sigma(1)}, \dots, v_{\sigma(k_1)})\omega_2(v_{\sigma(k_1+1)}, \dots, v_{\sigma(k_1+k_2)}),\end{aligned} \tag{15}$$

where the sum is taken over all permutations σ of order $k_1 + k_2$ and $\nu(\sigma)$ is parity of a permutation σ.
- The factor $\frac{1}{k_1!\,k_2!}$ normalizes the sum in (15) since it contains $k_1!\,k_2!$ identically equal terms: e.g., if permutations σ do not mix the first k_1 and the last k_2 arguments, then all terms of the form

$$(-1)^{\nu(\sigma)}\omega_1(v_{\sigma(1)}, \dots, v_{\sigma(k_1)})\omega_2(v_{\sigma(k_1+1)}, \dots, v_{\sigma(k_1+k_2)})$$

are equal to

$$\omega_1(v_1, \dots, v_{k_1})\omega_2(v_{k_1+1}, \dots, v_{k_1+k_2}).$$

- This guarantees the associative property of the exterior product:

$$\omega_1 \wedge (\omega_2 \wedge \omega_3) = (\omega_1 \wedge \omega_2) \wedge \omega_3, \qquad \omega_i \in \Lambda^{k_i} E,$$

- Further, the exterior product is skew-commutative:

$$\omega_2 \wedge \omega_1 = (-1)^{k_1 k_2} \omega_1 \wedge \omega_2, \qquad \omega_i \in \Lambda^{k_i} E.$$

- Let $e_1, \ldots, e_n$ be a basis of the space E and $e_1^*, \ldots, e_n^*$ the corresponding dual basis of E^*.
- If $1 \leq k \leq n$, then the following $C_n^k = \frac{n!}{k!(n-k)!}$ elements form a basis of the space $\Lambda^k E$:

$$e_{i_1}^* \wedge \ldots \wedge e_{i_k}^*, \qquad 1 \leq i_1 < i_2 < \cdots < i_k \leq n.$$

- The equalities

$$(e_{i_1}^* \wedge \ldots \wedge e_{i_k}^*)(e_{i_1}, \ldots, e_{i_k}) = 1,$$
$$(e_{i_1}^* \wedge \ldots \wedge e_{i_k}^*)(e_{j_1}, \ldots, e_{j_k}) = 0, \quad \text{if } (i_1, \ldots, i_k) \neq (j_1, \ldots, j_k)$$

for $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ imply that any k-form $\omega \in \Lambda^k E$ has a unique decomposition of the form

$$\omega = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} \omega_{i_1 \ldots i_k} e_{i_1}^* \wedge \ldots \wedge e_{i_k}^*$$

with

$$\omega_{i_1 \ldots i_k} = \omega(e_{i_1}, \ldots, e_{i_k}).$$

Exercise 1

Show that for any 1-forms $\omega_1, \ldots \omega_p \in \Lambda^1 E$ and any vectors $v_1, \ldots, v_p \in E$ there holds the equality

$$(\omega_1 \wedge \ldots \wedge \omega_p)(v_1, \ldots, v_p) = \det \left(\langle \omega_i, v_j \rangle \right)_{i,j=1}^p . \tag{16}$$

- Notice that the space of n-forms of an n-dimensional space E is one-dimensional.
- Any nonzero n-form on E is called a *volume form*.
- For example, the value of the standard volume form $e_1^* \wedge \ldots \wedge e_n^*$ on an n-tuple of vectors $(v_1, \ldots, v_n)$ is

$$(e_1^* \wedge \ldots \wedge e_n^*)(v_1, \ldots, v_n) = \det\left(\langle e_i^*, v_j \rangle\right)_{i,j=1}^n ,$$

the oriented volume of the parallelepiped generated by the vectors $v_1, \ldots, v_n$.

Differential k-forms

- A *differential k-form* on M is a mapping

$$\omega \;:\; q \mapsto \omega_q \in \Lambda^k T_q M, \qquad q \in M,$$

smooth w.r.t. $q \in M$.

- The set of all differential k-forms on M is denoted by $\Lambda^k M$.
- It is natural to consider smooth functions on M as 0-forms, so $\Lambda^0 M = C^\infty(M)$.
- In local coordinates $(x_1, \ldots, x_n)$ on a domain $O \subset M$, any differential k-form $\omega \in \Lambda^k M$ can be uniquely decomposed as follows:

$$\omega_x = \sum_{i_1 < \cdots < i_k} a_{i_1 \ldots i_k}(x) dx_{i_1} \wedge \ldots \wedge dx_{i_k}, \qquad x \in O, \quad a_{i_1 \ldots i_k} \in C^\infty(O). \tag{17}$$

- Any smooth mapping $F : M \to N$ induces a mapping of differential forms $\widehat{F} : \Lambda^k N \to \Lambda^k M$ in the following way: given a differential k-form $\omega \in \Lambda^k N$, the k-form $\widehat{F}\omega \in \Lambda^k M$ is defined as

$$(\widehat{F}\omega)_q(v_1, \ldots, v_k) = \omega_{F(q)}(F_* v_1, \ldots, F_* v_k), \qquad q \in M, \ v_i \in T_q M.$$

- For 0-forms, pull-back is a substitution of variables:

$$\widehat{F} a(q) = a \circ F(q), \qquad a \in C^\infty(M), \quad q \in M.$$

- The pull-back $\widehat{F}$ is linear w.r.t. forms and preserves the exterior product:

$$\widehat{F}(\omega_1 \wedge \omega_2) = \widehat{F}\omega_1 \wedge \widehat{F}\omega_2.$$

Exercise 2

Prove the composition law for pull-back of differential forms:

$$\widehat{F_2 \circ F_1} = \widehat{F}_1 \circ \widehat{F}_2, \tag{18}$$

where $F_1 : M_1 \to M_2$ and $F_2 : M_2 \to M_3$ are smooth mappings.

- Now we can define the integral of a k-form over an oriented k-dimensional surface.
- Let $\Pi \subset \mathbb{R}^k$ be a k-dimensional open oriented domain and $\Phi : \Pi \to \Phi(\Pi) \subset M$ a diffeomorphism.
- Then the *integral* of a k-form $\omega \in \Lambda^k M$ over the k-dimensional oriented surface $\Phi(\Pi)$ is defined as follows:

$$\int_{\Phi(\Pi)} \omega \quad \stackrel{\text{def}}{=} \quad \int_{\Pi} \widehat{\Phi}\omega,$$

it remains only to define the integral over Π in the right-hand side.

- Since $\widehat{\Phi}\omega \in \Lambda^k\mathbb{R}^k$ is a k-form on $\mathbb{R}^k$, it is expressed via the standard volume form $dx_1 \wedge \ldots \wedge dx_k \in \Lambda^k\mathbb{R}^k$:

$$(\widehat{\Phi}\omega)_x = a(x)\, dx_1 \wedge \cdots \wedge dx_k, \qquad x \in \Pi.$$

- We set

$$\int_{\Pi} \widehat{\Phi}\omega \stackrel{\text{def}}{=} \int_{\Pi} a(x)\, dx_1 \ldots dx_k,$$

a usual multiple integral.

- The integral $\int_{\Phi(\Pi)} \omega$ is defined correctly with respect to orientation-preserving reparametrizations of the surface $\Phi(\Pi)$.
- Although, if a parametrization changes orientation, then the integral changes sign.
- The notion of integral is extended to arbitrary submanifolds as follows.
- Let $N \subset M$ be a k-dimensional submanifold and let $\omega \in \Lambda^k M$.
- Consider a covering of N by coordinate neighborhoods $O_i \subset M$:

$$N = \bigcup_i (N \cap O_i).$$

- Take a partition of unity subordinated to this covering:

$$\alpha_i \in C^\infty(M), \quad \operatorname{supp} \alpha_i \subset O_i, \quad 0 \le \alpha_i \le 1,$$
$$\sum_i \alpha_i \equiv 1.$$

- Then

$$\int_N \omega \overset{\text{def}}{=} \sum_i \int_{N \cap O_i} \alpha_i \omega.$$

- The integral thus defined does not depend upon the choice of partition of unity.

Exterior differential

- Exterior differential of a function (i.e., a 0-form) is a 1-form: if $a \in C^\infty(M) = \Lambda^0 M$, then its differential $d_q a \in T_q^* M$ is the functional (directional derivative)

$$\langle d_q a, v \rangle = va, \qquad v \in T_q M, \tag{19}$$

so $da \in \Lambda^1 M$.

- By the Newton-Leibniz formula, if $\gamma \subset M$ is a smooth oriented curve starting at a point $q_0 \in M$ and terminating at $q_1 \in M$, then

$$\int_\gamma da = a(q_1) - a(q_0).$$

- The right-hand side can be considered as the integral of the function a over the oriented boundary of the curve: $\partial\gamma = q_1 - q_0$, thus

$$\int_\gamma da = \int_{\partial\gamma} a. \tag{20}$$

- In the exposition above, Newton-Leibniz formula (20) comes as a consequence of definition (19) of differential of a function. But one can go the reverse way: if we postulate Newton-Leibniz formula (20) for any smooth curve $\gamma \subset M$ and pass to the limit $q_1 \to q_0$, we necessarily obtain definition (19) of differential of a function.
- Such approach can be realized for higher order differential forms as well.
- Let $\omega \in \Lambda^k M$. We define the *exterior differential*

$$d\omega \in \Lambda^{k+1} M$$

as the differential $(k+1)$-form for which Stokes formula holds:

$$\int_N d\omega = \int_{\partial N} \omega \tag{21}$$

for $(k+1)$-dimensional submanifolds with boundary $N \subset M$ (for simplicity, one can take here N equal to a diffeomorphic image of a $(k+1)$-dimensional polytope).
- The boundary ∂N is oriented by a frame of tangent vectors $e_1, \dots e_k \in T_q(\partial N)$ in such a way that the frame $e_{\mathrm{norm}}, e_1, \dots, e_k \in T_q N$ define a positive orientation of N, where e_{norm} is the outward normal vector to N at q.

- The existence of a form $d\omega$ that satisfies Stokes formula (21) comes from the fact that the mapping $N \mapsto \int_{\partial N} \omega$ is additive w.r.t. domain: if $N = N_1 \cup N_2$, $N_1 \cap N_2 = \partial N_1 \cap \partial N_2$, then

$$\int_{\partial N} \omega = \int_{\partial N_1} \omega + \int_{\partial N_2} \omega$$

  (notice that orientation of the boundaries is coordinated: ∂N_1 and ∂N_2 have mutually opposite orientations at points of their intersection).
- Thus the integral $\int_{\partial N} \omega$ is a kind of measure w.r.t. N, and one can recover $(d\omega)_q$ passing to limit in (21) as the submanifold N contracts to a point q.