Elements of Chronological Calculus-2

(Lecture 4)

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Plan of previous lecture

1. Points, Diffeomorphisms, and Vector Fields
2. Seminorms and $C^\infty(M)$-Topology
3. Families of Functionals and Operators
4. ODEs with discontinuous right-hand side
5. Definition of the right chronological exponential
6. Formal series expansion

Plan of this lecture

1. Estimates and convergence of the series
2. Left chronological exponential
3. Uniqueness for functional and operator ODEs
4. Autonomous vector fields
5. Action of diffeomorphisms on vector fields
6. Commutation of flows
7. Variations formula
8. Derivative of flow with respect to parameter

Definition of the right chronological exponential

- The Cauchy problem $\dot q = V_t(q)$, $q(0) = q_0$, rewritten as a linear equation for Lipschitzian w.r.t. t families of functionals on $C^\infty(M)$:

$$\dot q(t) = q(t) \circ V_t, \qquad q(0) = q_0, \tag{1}$$

is satisfied for the family of functionals

$$q(t, q_0) \,:\, C^\infty(M) \to \mathbb{R}, \qquad q_0 \in M, \quad t \in \mathbb{R}$$

constructed in the previous lecture.
- We prove later that this Cauchy problem has no other solutions.
- Thus the flow defined as

$$P^t \;:\; q_0 \mapsto q(t, q_0) \tag{2}$$

is a unique solution of the operator Cauchy problem $\dot P^t = P^t \circ V_t$, $P^0 = \mathrm{Id}$ (where Id is the identity operator), in the class of Lipschitzian flows on M.
- The flow P^t determined in (2) is called the *right chronological exponential* of the field V_t and is denoted as $P^t = \overrightarrow{\exp} \displaystyle\int_0^t V_\tau \, d\tau$.

Formal series expansion

- Purely formally passing to the limit $n \to \infty$, we obtained a formal series for the solution $q(t)$ to problem $\dot{q}(t) = q(t) \circ V_t$, $q(0) = q_0$:

$$q_0 \circ \left(\mathrm{Id} + \sum_{n=1}^{\infty} \int \cdots \int_{\Delta_n(t)} V_{\tau_n} \circ \cdots \circ V_{\tau_1} \, d\tau_n \, \ldots \, d\tau_1 \right),$$

thus for the solution P^t to operator Cauchy problem $\dot{P}^t = P^t \circ V_t$, $P^0 = \mathrm{Id}$:

$$\mathrm{Id} + \sum_{n=1}^{\infty} \int \cdots \int_{\Delta_n(t)} V_{\tau_n} \circ \cdots \circ V_{\tau_1} \, d\tau_n \, \ldots \, d\tau_1 . \tag{3}$$

Estimates and convergence of the series

- Unfortunately, series (3) never converges on $C^\infty(M)$ in the weak sense (if $V_t \not\equiv 0$): there always exists a smooth function on M, on which it diverges.
- Although, one can show that series (3) gives an asymptotic expansion for the chronological exponential $P^t = \overrightarrow{\exp} \int_0^t V_\tau \, d\tau$.
- There holds the following bound of the remainder term: denote the m-th partial sum of series (3) as $S_m(t) = \mathrm{Id} + \sum_{n=1}^{m-1} \underset{\Delta_n(t)}{\int \cdots \int} V_{\tau_n} \circ \cdots \circ V_{\tau_1} \, d\tau_n \ldots d\tau_1$, then for any $a \in C^\infty(M)$, $s \geq 0$, $K \Subset M$

$$\left\| \left(\overrightarrow{\exp} \int_0^t V_\tau \, d\tau - S_m(t) \right) a \right\|_{s,K}$$
$$\leq C e^{C \int_0^t \|V_\tau\|_{s,K'} \, d\tau} \frac{1}{m!} \left(\int_0^t \|V_\tau\|_{s+m-1,K'} \, d\tau \right)^m \|a\|_{s+m,K'} \tag{4}$$
$$= O(t^m), \qquad t \to 0,$$

where $K' \Subset M$ is some compactum containing K, see [AS].

- It follows from estimate (4) that

$$\left\| \left(\overrightarrow{\exp} \int_0^t \varepsilon V_\tau \, d\tau - S_m^\varepsilon(t) \right) a \right\|_{s,K} = O(\varepsilon^m), \qquad \varepsilon \to 0,$$

where $S_m^\varepsilon(t)$ is the m-th partial sum of series (3) for the field εV_t.

- Thus we have an asymptotic series expansion:

$$\overrightarrow{\exp} \int_0^t V_\tau \, d\tau \approx \mathrm{Id} + \sum_{n=1}^{\infty} \int \cdots \int_{\Delta_n(t)} V_{\tau_n} \circ \cdots \circ V_{\tau_1} \, d\tau_n \ldots d\tau_1. \tag{5}$$

- In the sequel we will use terms of the zeroth, first, and second orders of the series obtained:

$$\overrightarrow{\exp} \int_0^t V_\tau \, d\tau \approx \mathrm{Id} + \int_0^t V_\tau \, d\tau + \iint_{0 \le \tau_2 \le \tau_1 \le t} V_{\tau_2} \circ V_{\tau_1} \, d\tau_2 \, d\tau_1 + \cdots .$$

- We prove now that the asymptotic series converges to the chronological exponential on any normed subspace $L \subset C^\infty(M)$ where V_t is well-defined and bounded:

$$V_t L \subset L, \qquad \|V_t\| = \sup\{\|V_t a\| \mid a \in L, \ \|a\| \leq 1\} < \infty. \tag{6}$$

- We apply operator series (5) to any $a \in L$ and bound terms of the series obtained:

$$a + \sum_{n=1}^{\infty} \int \cdots \int_{\Delta_n(t)} V_{\tau_n} \circ \cdots \circ V_{\tau_1} \, a \, d\tau_n \, \ldots \, d\tau_1 . \tag{7}$$

$$
\begin{aligned}
&\left\| \int_{\Delta_n(t)} \cdots \int V_{\tau_n} \circ \cdots \circ V_{\tau_1} \, a \, d\tau_n \, \ldots \, d\tau_1 \right\| \\
&\quad \leq \int_{0 \leq \tau_n \leq \cdots \leq \tau_1 \leq t} \cdots \int \| V_{\tau_n} \| \cdot \, \cdots \, \cdot \| V_{\tau_1} \| \, d\tau_n \, \ldots \, d\tau_1 \cdot \| a \| \\
&\quad = \int_{0 \leq \tau_{\sigma(n)} \leq \cdots \leq \tau_{\sigma(1)} \leq t} \cdots \int \| V_{\tau_n} \| \cdot \, \cdots \, \cdot \| V_{\tau_1} \| \, d\tau_n \, \ldots \, d\tau_1 \cdot \| a \| \\
&\quad = \frac{1}{n!} \int_0^t \ldots \int_0^t \| V_{\tau_n} \| \cdot \, \cdots \, \cdot \| V_{\tau_1} \| \, d\tau_n \, \ldots \, d\tau_1 \cdot \| a \| \\
&\quad = \frac{1}{n!} \left(\int_0^t \| V_\tau \| \, d\tau \right)^n \cdot \| a \| .
\end{aligned}
$$

- So series (7) is majorized by the exponential series, thus the operator series (5) converges on L.
- Series (7) can be differentiated termwise, thus it satisfies the same ODE as the function $P^t a$:

$$\dot{a}_t = V_t a_t, \qquad a_0 = a.$$

- Consequently,

$$P^t a = a + \sum_{n=1}^{\infty} \int \cdots \int_{\Delta_n(t)} V_{\tau_n} \circ \cdots \circ V_{\tau_1} \, a \, d\tau_n \, \ldots \, d\tau_1 .$$

- So in the case (6) the asymptotic series converges to the chronological exponential and there holds the bound

$$\|P^t a\| \leq e^{\int_0^t \|V_\tau\| \, d\tau} \|a\|, \qquad a \in L.$$

- Moreover, one can show that the bound and convergence hold not only for locally bounded, but also for integrable on $[0, t]$ vector fields: $\displaystyle\int_0^t \|V_\tau\| \, d\tau < \infty$.

- Notice that conditions (6) are satisfied for any finite-dimensional V_t-invariant subspace $L \subset C^\infty(M)$. In particular, this is the case when $M = \mathbb{R}^n$, L is the space of linear functions, and V_t is a linear vector field on $\mathbb{R}^n$.
- If M, V_t, and a are real analytic, then series (7) converges for sufficiently small t.

Left chronological exponential

- Consider the inverse operator $Q^t = (P^t)^{-1}$ to the right chronological exponential $P^t = \overrightarrow{\exp} \int_0^t V_\tau \, d\tau$.
- We find an ODE for the flow Q^t by differentiation of the identity
$$P^t \circ Q^t = \mathrm{Id}\,.$$
- Leibniz rule yields $\dot{P}^t \circ Q^t + P^t \circ \dot{Q}^t = 0$, thus, in view of the ODE for the flow P^t,
$$P^t \circ V_t \circ Q^t + P^t \circ \dot{Q}^t = 0.$$
- We multiply this equality by Q^t from the left and obtain
$$V_t \circ Q^t + \dot{Q}^t = 0.$$

  That is, the flow Q^t is a solution of the Cauchy problem
$$\frac{d}{d\,t} Q^t = -V_t \circ Q^t, \qquad Q^0 = \mathrm{Id}, \tag{8}$$
  which is dual to the Cauchy problem for P^t: $\frac{d}{d\,t} P^t = P^t \circ V_t$, $P^0 = \mathrm{Id}$.

- The flow Q^t is called the *left chronological exponential* and is denoted as

$$Q^t = \overleftarrow{\exp} \int_0^t (-V_\tau)\, d\tau.$$

- We find an asymptotic expansion for the left chronological exponential in the same way as for the right one, by successive substitutions into the right-hand side:

$$\begin{aligned} Q^t &= \mathrm{Id} + \int_0^t (-V_\tau) \circ Q^\tau \, d\tau \\ &= \mathrm{Id} + \int_0^t (-V_\tau)\, d\tau + \iint\limits_{\Delta_2(t)} (-V_{\tau_1}) \circ (-V_{\tau_2}) \circ Q^{\tau_2}\, d\tau_2\, d\tau_1 = \cdots \\ &= \mathrm{Id} + \sum_{n=1}^{m-1} \int\limits_{\Delta_n(t)} \cdots \int (-V_{\tau_1}) \circ \cdots \circ (-V_{\tau_n})\, d\tau_n \ldots d\tau_1 \\ &\quad + \int\limits_{\Delta_m(t)} \cdots \int (-V_{\tau_1}) \circ \cdots \circ (-V_{\tau_m}) \circ Q^{\tau_m}\, d\tau_m \ldots d\tau_1. \end{aligned}$$

- For the left chronological exponential holds an estimate of the remainder term as (4) for the right one, and the series obtained is asymptotic:

$$\overleftarrow{\exp} \int_0^t (-V_\tau)\, d\tau \approx \mathrm{Id} + \sum_{n=1}^{\infty} \int \cdots \int_{\Delta_n(t)} (-V_{\tau_1}) \circ \cdots \circ (-V_{\tau_n})\, d\tau_n \ldots d\tau_1 .$$

- Notice that the reverse arrow in the left chronological exponential $\overleftarrow{\exp}$ corresponds to the reverse order of the operators $(-V_{\tau_1}) \circ \cdots \circ (-V_{\tau_n})$, $\tau_n \leq \ldots \leq \tau_1$.
- The right and left chronological exponentials satisfy the corresponding differential equations:

$$\frac{d}{d\,t}\ \overrightarrow{\exp} \int_0^t V_\tau\, d\tau = \overrightarrow{\exp} \int_0^t V_\tau\, d\tau \circ V_t,$$
$$\frac{d}{d\,t}\ \overleftarrow{\exp} \int_0^t (-V_\tau)\, d\tau = -V_t \circ \overleftarrow{\exp} \int_0^t (-V_\tau)\, d\tau.$$

The directions of arrows correlate with the direction of appearance of the operators V_t and $(-V_t)$ in the right-hand side of these ODEs.

- If the initial value is prescribed at a moment of time $t_0 \neq 0$, then the lower limit of integrals in the chronological exponentials is t_0.
- There holds the following obvious rule for composition of flows:

$$\overrightarrow{\exp} \int_{t_0}^{t_1} V_\tau \, d\tau \circ \overrightarrow{\exp} \int_{t_1}^{t_2} V_\tau \, d\tau = \overrightarrow{\exp} \int_{t_0}^{t_2} V_\tau \, d\tau.$$

- There hold the identities

$$\overrightarrow{\exp} \int_{t_0}^{t_1} V_\tau \, d\tau = \left(\overrightarrow{\exp} \int_{t_1}^{t_0} V_\tau \, d\tau \right)^{-1} = \overleftarrow{\exp} \int_{t_1}^{t_0} (-V_\tau) \, d\tau. \tag{9}$$

- We saw that equation (1) for Lipschitzian families of functionals has a solution $q(t) = q_0 \circ \overrightarrow{\exp} \int_0^t V_\tau \, d\tau$. We can prove now that this equation has no other solutions.

Proposition 1

Let V_t be a complete nonautonomous vector field on M. Then Cauchy problem (1) has a unique solution in the class of Lipschitzian families of functionals on $C^\infty(M)$.

Proof.

Let a Lipschitzian family of functionals q_t be a solution to problem (1). Then

$$\frac{d}{d\,t}\left(q_t \circ (P^t)^{-1}\right) = \frac{d}{d\,t}\left(q_t \circ Q^t\right) = q_t \circ V_t \circ Q^t - q_t \circ V_t \circ Q^t = 0,$$

thus $q_t \circ Q^t \equiv \text{const}$. But $Q^0 = \mathrm{Id}$, consequently, $q_t \circ Q^t \equiv q_0$, hence

$$q_t = q_0 \circ P^t = q_0 \circ \overrightarrow{\exp} \int_0^t V_\tau \, d\tau$$

is a unique solution of Cauchy problem (1). $\square$

Similarly, the both operator equations $\dot{P}^t = P^t \circ V_t$ and $\dot{Q}^t = -V_t \circ Q^t$ have no other solutions in addition to the chronological exponentials.

Autonomous vector fields

- For an *autonomous vector field*

$$V_t \equiv V \in \operatorname{Vec} M,$$

the flow generated by a complete field is called the *exponential* and is denoted as e^{tV}.

- The asymptotic series for the exponential takes the form

$$e^{tV} \approx \sum_{n=0}^{\infty} \frac{t^n}{n!} V^n = \mathrm{Id} + tV + \frac{t^2}{2} V \circ V + \cdots \ ,$$

i.e, it is the standard exponential series.

- The exponential of an autonomous vector field satisfies the ODEs

$$\frac{d}{d\,t} e^{tV} = e^{tV} \circ V = V \circ e^{tV}, \qquad e^{tV}\Big|_{t=0} = \mathrm{Id}\,.$$

- We apply the asymptotic series for exponential to find the Lie bracket of autonomous vector fields $V, W \in \operatorname{Vec} M$.
- We compute the first nonconstant term in the asymptotic expansion at $t = 0$ of the curve:

$$\begin{aligned}
q(t) &= q \circ e^{tV} \circ e^{tW} \circ e^{-tV} \circ e^{-tW} \\
&= q \circ \left(\operatorname{Id} + tV + \frac{t^2}{2} V^2 + \cdots \right) \circ \left(\operatorname{Id} + tW + \frac{t^2}{2} W^2 + \cdots \right) \\
&\qquad \circ \left(\operatorname{Id} - tV + \frac{t^2}{2} V^2 + \cdots \right) \circ \left(\operatorname{Id} - tW + \frac{t^2}{2} W^2 + \cdots \right) \\
&= q \circ \left(\operatorname{Id} + t(V + W) + \frac{t^2}{2} (V^2 + 2V \circ W + W^2) + \cdots \right) \\
&\qquad \circ \left(\operatorname{Id} - t(V + W) + \frac{t^2}{2} (V^2 + 2V \circ W + W^2) + \cdots \right) \\
&= q \circ \left(\operatorname{Id} + t^2 (V \circ W - W \circ V) + \cdots \right) .
\end{aligned}$$

- So the Lie bracket of the vector fields as operators (directional derivatives) in $C^\infty(M)$ is
$$[V, W] = V \circ W - W \circ V.$$
- This proves the formula in local coordinates: if
$$V = \sum_{i=1}^n a_i \frac{\partial}{\partial x_i}, \qquad W = \sum_{i=1}^n b_i \frac{\partial}{\partial x_i}, \qquad a_i,\ b_i \in C^\infty(M),$$
then
$$[V, W] = \sum_{i,j=1}^n \left(a_j \frac{\partial b_i}{\partial x_j} - b_j \frac{\partial a_i}{\partial x_j} \right) \frac{\partial}{\partial x_i} = \frac{dW}{dx} V - \frac{dV}{dx} W.$$
- Similarly,
$$\begin{aligned} q \circ e^{tV} \circ e^{sW} \circ e^{-tV} &= q \circ (\mathrm{Id} + tV + \cdots) \circ (\mathrm{Id} + sW + \cdots) \circ (\mathrm{Id} - tV + \cdots) \\ &= q \circ (\mathrm{Id} + sW + ts[V, W] + \cdots), \end{aligned}$$
and
$$q \circ [V, W] = \frac{\partial^2}{\partial s \partial t}\bigg|_{s=t=0} q \circ e^{tV} \circ e^{sW} \circ e^{-tV}.$$

Action of diffeomorphisms on tangent vectors

- We have already found counterparts to points, diffeomorphisms, and vector fields among functionals and operators on $C^\infty(M)$. Now we consider action of diffeomorphisms on tangent vectors and vector fields.
- Take a tangent vector $v \in T_q M$ and a diffeomorphism $P \in \operatorname{Diff} M$. The tangent vector $P_* v \in T_{P(q)} M$ is the velocity vector of the image of a curve starting from q with the velocity vector v. We claim that

$$P_* v = v \circ P, \qquad v \in T_q M, \quad P \in \operatorname{Diff} M, \tag{10}$$

as functionals on $C^\infty(M)$.
- Take a curve

$$q(t) \in M, \qquad q(0) = q, \quad \left.\frac{d}{d\,t}\right|_{t=0} q(t) = v,$$

then

$$\begin{aligned} P_* v\, a &= \left.\frac{d}{d\,t}\right|_{t=0} a(P(q(t))) = \left(\left.\frac{d}{d\,t}\right|_{t=0} q(t) \right) \circ P a \\ &= v \circ P a, \qquad a \in C^\infty(M). \end{aligned}$$

Action of diffeomorphisms on vector fields

- Now we find expression for $P_* V$, $V \in \operatorname{Vec} M$, as a derivation of $C^\infty(M)$.
- We have

$$\begin{aligned} q \circ P \circ P_* V &= P(q) \circ P_* V = (P_* V)(P(q)) = P_*(V(q)) = V(q) \circ P \\ &= q \circ V \circ P, \quad q \in M, \end{aligned}$$

thus

$$P \circ P_* V = V \circ P,$$

i.e.,

$$P_* V = P^{-1} \circ V \circ P, \qquad P \in \operatorname{Diff} M, \ V \in \operatorname{Vec} M.$$

- So diffeomorphisms act on vector fields as similarities.
- In particular, diffeomorphisms preserve compositions:

$$P_*(V \circ W) = P^{-1} \circ (V \circ W) \circ P = (P^{-1} \circ V \circ P) \circ (P^{-1} \circ W \circ P) = P_* V \circ P_* W,$$

thus Lie brackets of vector fields:

$$P_*[V, W] = P_*(V \circ W - W \circ V) = P_* V \circ P_* W - P_* W \circ P_* V = [P_* V, P_* W].$$

Action of diffeomorphisms on vector fields

- If $B : C^\infty(M) \to C^\infty(M)$ is an automorphism, then the standard algebraic notation for the corresponding similarity is $\operatorname{Ad} B$:

$$(\operatorname{Ad} B)V \stackrel{\text{def}}{=} B \circ V \circ B^{-1}.$$

- That is,

$$P_* = \operatorname{Ad} P^{-1}, \qquad P \in \operatorname{Diff} M.$$

- Now we find an infinitesimal version of the operator Ad.
- Let P^t be a flow on M,

$$P^0 = \mathrm{Id}, \qquad \left.\frac{d}{d\,t}\right|_{t=0} P^t = V \in \operatorname{Vec} M.$$

- Then

$$\left.\frac{d}{d\,t}\right|_{t=0} (P^t)^{-1} = -V,$$

so

$$\begin{aligned} \left.\frac{d}{d\,t}\right|_{t=0} (\operatorname{Ad} P^t)W &= \left.\frac{d}{d\,t}\right|_{t=0} (P^t \circ W \circ (P^t)^{-1}) = V \circ W - W \circ V \\ &= [V, W], \qquad W \in \operatorname{Vec} M. \end{aligned}$$

- Denote

$$\operatorname{ad} V = \operatorname{ad}\left(\left.\frac{d}{d\,t}\right|_{t=0} P^t\right) \stackrel{\text{def}}{=} \left.\frac{d}{d\,t}\right|_{t=0} \operatorname{Ad} P^t,$$

then

$$(\operatorname{ad} V)W = [V, W], \qquad W \in \operatorname{Vec} M.$$

- Differentiation of the equality

$$\operatorname{Ad} P^t\,[X, Y] = [\operatorname{Ad} P^t\, X, \operatorname{Ad} P^t\, Y] \qquad X, Y \in \operatorname{Vec} M,$$

at $t = 0$ gives *Jacobi identity* for Lie bracket of vector fields:

$$(\operatorname{ad} V)[X, Y] = [(\operatorname{ad} V)X, Y] + [X, (\operatorname{ad} V)Y],$$

which may also be written as

$$[V, [X, Y]] = [[V, X], Y] + [X, [V, Y]], \qquad V, X, Y \in \operatorname{Vec} M,$$

or, in a symmetric way

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0, \qquad X, Y, Z \in \operatorname{Vec} M. \tag{11}$$

- The set $\mathrm{Vec}\, M$ is a vector space with an additional operation — Lie bracket, which has the properties:
  (1) bilinearity:
$$
\begin{aligned}
&[\alpha X + \beta Y, Z] = \alpha[X, Z] + \beta[Y, Z], \\
&[X, \alpha Y + \beta Z] = \alpha[X, Y] + \beta[X, Z], \qquad X, Y, Z \in \mathrm{Vec}\, M, \quad \alpha, \beta \in \mathbb{R},
\end{aligned}
$$
  (2) skew-symmetry:
$$
[X, Y] = -[Y, X], \qquad X, Y \in \mathrm{Vec}\, M,
$$
  (3) Jacobi identity (11).
- In other words, the set $\mathrm{Vec}\, M$ of all smooth vector fields on a smooth manifold M forms a *Lie algebra*.

- Consider the flow $P^t = \overrightarrow{\exp} \int_0^t V_\tau \, d\tau$ of a nonautonomous vector field V_t. We find an ODE for the family of operators $\operatorname{Ad} P^t = (P^t)_*^{-1}$ on the Lie algebra $\operatorname{Vec} M$.

$$\begin{aligned} \frac{d}{d\,t}(\operatorname{Ad} P^t)X &= \frac{d}{d\,t}\left(P^t \circ X \circ (P^t)^{-1}\right) \\ &= P^t \circ V_t \circ X \circ (P^t)^{-1} - P^t \circ X \circ V_t \circ (P^t)^{-1} \\ &= (\operatorname{Ad} P^t)[V_t, X] = (\operatorname{Ad} P^t) \operatorname{ad} V_t \, X, \qquad X \in \operatorname{Vec} M. \end{aligned}$$

- Thus the family of operators $\operatorname{Ad} P^t$ satisfies the ODE

$$\frac{d}{d\,t} \operatorname{Ad} P^t = (\operatorname{Ad} P^t) \circ \operatorname{ad} V_t \tag{12}$$

with the initial condition

$$\operatorname{Ad} P^0 = \operatorname{Id}. \tag{13}$$

- So the family $\operatorname{Ad} P^t$ is an invertible solution for the Cauchy problem

$$\dot{A}_t = A_t \circ \operatorname{ad} V_t, \quad A_0 = \operatorname{Id}$$

for operators $A_t : \operatorname{Vec} M \to \operatorname{Vec} M$.

- We can apply the same argument as for the analogous Cauchy problem for flows to derive the asymptotic expansion

$$\operatorname{Ad} P^t \approx \operatorname{Id} + \int_0^t \operatorname{ad} V_\tau \, d\tau + \cdots \\ + \underset{\Delta_n(t)}{\int \cdots \int} \operatorname{ad} V_{\tau_n} \circ \cdots \circ \operatorname{ad} V_{\tau_1} \, d\tau_n \ldots d\tau_1 + \cdots \tag{14}$$

then prove uniqueness of the solution, and justify the following notation:

$$\overrightarrow{\exp} \int_0^t \operatorname{ad} V_\tau \, d\tau \overset{\text{def}}{=} \operatorname{Ad} P^t = \operatorname{Ad}\left(\overrightarrow{\exp} \int_0^t V_\tau \, d\tau \right).$$

- Similar identities for the left chronological exponential are

$$\overleftarrow{\exp} \int_0^t \operatorname{ad}(-V_\tau) \, d\tau \overset{\text{def}}{=} \operatorname{Ad}\left(\overleftarrow{\exp} \int_0^t (-V_\tau) \, d\tau \right) \\ \approx \operatorname{Id} + \sum_{n=1}^{\infty} \underset{\Delta_n(t)}{\int \cdots \int} (-\operatorname{ad} V_{\tau_1}) \circ \cdots \circ (-\operatorname{ad} V_{\tau_n}) \, d\tau_n \ldots d\tau_1.$$

- For the asymptotic series (14), there holds an estimate of the remainder term similar to the estimate for the flow P^t.
- Denote the partial sum

$$T_m = \mathrm{Id} + \sum_{n=1}^{m-1} \underset{\Delta_n(t)}{\int \cdots \int} \mathrm{ad}\, V_{\tau_n} \circ \cdots \circ \mathrm{ad}\, V_{\tau_1}\, d\tau_n \ldots d\tau_1,$$

then for any $X \in \mathrm{Vec}\, M$, $s \geq 0$, $K \Subset M$

$$\begin{aligned}
&\left\| \left(\mathrm{Ad}\, \overrightarrow{\exp} \int_0^t V_\tau\, d\tau - T_m \right) X \right\|_{s,K} \\
&\qquad \leq C_1 e^{C_1 \int_0^t \|V_\tau\|_{s+1,K'}\, d\tau} \frac{1}{m!} \left(\int_0^t \|V_\tau\|_{s+m,K'}\, d\tau \right)^m \|X\|_{s+m,K'} \qquad (15) \\
&\qquad = O(t^m), \qquad t \to 0,
\end{aligned}$$

where $K' \Subset M$ is some compactum containing K.

- For autonomous vector fields, we denote

$$e^{t \operatorname{ad} V} \stackrel{\text{def}}{=} \operatorname{Ad} e^{tV},$$

thus the family of operators $e^{t \operatorname{ad} V} : \operatorname{Vec} M \to \operatorname{Vec} M$ is the unique solution to the problem

$$\dot{A}_t = A_t \circ \operatorname{ad} V, \qquad A_0 = \operatorname{Id},$$

which admits the asymptotic expansion

$$e^{t \operatorname{ad} V} \approx \operatorname{Id} + t \operatorname{ad} V + \frac{t^2}{2} \operatorname{ad}^2 V + \cdots .$$

- Let $P \in \operatorname{Diff} M$, and let V_t be a nonautonomous vector field on M. Then

$$P \circ \overrightarrow{\exp} \int_0^t V_\tau \, d\tau \circ P^{-1} = \overrightarrow{\exp} \int_0^t \operatorname{Ad} P \, V_\tau \, d\tau \tag{16}$$

since the both parts satisfy the same operator Cauchy problem.

Commutation of flows

Let $V_t \in \operatorname{Vec} M$ be a nonautonomous vector field and $P^t = \overrightarrow{\exp} \int_0^t V_\tau \, d\tau$ the corresponding flow. We are interested in the question: under what conditions the flow P^t preserves a vector field $W \in \operatorname{Vec} M$.

Proposition 2

$P^t_* W = W \quad \forall t \quad \Leftrightarrow \quad [V_t, W] = 0 \quad \forall t.$

Proof.

$$
\begin{aligned}
\frac{d}{d\,t} (P_t)^{-1}_* \, W &= \frac{d}{d\,t} \operatorname{Ad} P^t W = \left(\frac{d}{d\,t} \; \overrightarrow{\exp} \int_0^t \operatorname{ad} V_\tau \, d\tau \right) W \\
&= \left(\overrightarrow{\exp} \int_0^t \operatorname{ad} V_\tau \, d\tau \circ \operatorname{ad} V_\tau \right) W = \left(\overrightarrow{\exp} \int_0^t \operatorname{ad} V_\tau \, d\tau \right) [V_t, W] \\
&= (P^t)^{-1}_* [V_t, W],
\end{aligned}
$$

thus $(P^t)^{-1}_* W \equiv W$ if and only if $[V_t, W] \equiv 0$. $\square$

- In general, flows do not commute, neither for nonautonomous vector fields V_t, W_t:

$$\overrightarrow{\exp} \int_0^{t_1} V_\tau \, d\tau \circ \overrightarrow{\exp} \int_0^{t_2} W_\tau \, d\tau \neq \overrightarrow{\exp} \int_0^{t_2} W_\tau \, d\tau \circ \overrightarrow{\exp} \int_0^{t_1} V_\tau \, d\tau,$$

nor for autonomous vector fields V, W:

$$e^{t_1 V} \circ e^{t_2 W} \neq e^{t_2 W} \circ e^{t_1 V}.$$

Proposition 3

In the autonomous case, commutativity of flows is equivalent to commutativity of vector fields: if $V, W \in \operatorname{Vec} M$, *then*

$$e^{t_1 V} \circ e^{t_2 W} = e^{t_2 W} \circ e^{t_1 V}, \quad t_1, t_2 \in \mathbb{R}, \qquad \Leftrightarrow \qquad [V, W] = 0.$$

Proof.

Necessity:

$$\frac{d^2}{dt^2} q \circ e^{tV} \circ e^{tW} \circ e^{-tV} \circ e^{-tW} = q \circ 2[V, W].$$

Sufficiency. We have $\left(\operatorname{Ad} e^{t_1 V}\right) W = e^{t_1 \operatorname{ad} V} W = W$. Taking into account equality (16), we obtain

$$e^{t_1 V} \circ e^{t_2 W} \circ e^{-t_1 V} = e^{t_2 \left(\operatorname{Ad} e^{t_1 V}\right) W} = e^{t_2 W}.$$

□

Variations formula

- Consider an ODE of the form

$$\dot{q} = V_t(q) + W_t(q). \tag{17}$$

  We think of V_t as an initial vector field and W_t as its perturbation.
- Our aim is to find a formula for the flow Q^t of the new field $V_t + W_t$ as a perturbation of the flow $P^t = \overrightarrow{\exp} \int_0^t V_\tau \, d\tau$ of the initial field V_t.
- In other words, we wish to have a decomposition of the form

$$Q^t = \overrightarrow{\exp} \int_0^t (V_\tau + W_\tau) \, d\tau = C_t \circ P^t.$$

- We proceed as in the method of variation of parameters; we substitute the previous expression to ODE (17):

$$
\begin{aligned}
\frac{d}{d\,t} Q^t &= Q^t \circ (V_t + W_t) \\
&= \dot{C}_t \circ P^t + C_t \circ P^t \circ V_t \\
&= \dot{C}_t \circ P^t + Q^t \circ V_t,
\end{aligned}
$$

cancel the common term $Q^t \circ V_t$:

$$Q^t \circ W_t = \dot{C}_t \circ P^t,$$

and write down the ODE for the unknown flow C_t:

$$
\begin{aligned}
\dot{C}_t &= Q^t \circ W_t \circ \left(P^t\right)^{-1} \\
&= C_t \circ P^t \circ W_t \circ \left(P^t\right)^{-1} \\
&= C_t \circ \left(\operatorname{Ad} P^t\right) W_t \\
&= C_t \circ \left(\overrightarrow{\exp} \int_0^t \operatorname{ad} V_\tau \, d\tau\right) W_t, \qquad C_0 = \operatorname{Id}.
\end{aligned}
$$

- This operator Cauchy problem is of the form $\dot{C}^t = C^t \circ V_t,\ C^0 = \mathrm{Id}$, thus it has a unique solution:

$$C_t = \overrightarrow{\exp} \int_0^t \left(\overrightarrow{\exp} \int_0^\tau \operatorname{ad} V_\theta \, d\theta \right) W_\tau \, d\tau.$$

- Hence we obtain the required decomposition of the perturbed flow:

$$\overrightarrow{\exp} \int_0^t (V_\tau + W_\tau) \, d\tau = \overrightarrow{\exp} \int_0^t \left(\overrightarrow{\exp} \int_0^\tau \operatorname{ad} V_\theta \, d\theta \right) W_\tau \, d\tau \circ \overrightarrow{\exp} \int_0^t V_\tau \, d\tau. \tag{18}$$

- This equality is called the *variations formula*.
- It can be written as follows:

$$\overrightarrow{\exp} \int_0^t (V_\tau + W_\tau) \, d\tau = \overrightarrow{\exp} \int_0^t (\operatorname{Ad} P^\tau) \, W_\tau \, d\tau \circ P^t.$$

- So the perturbed flow is a composition of the initial flow P^t with the flow of the perturbation W_t twisted by P^t.