Time-Optimal Problem. Ordinary differential equations on manifolds

(Lecture 2)

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«Elements of Optimal Control»

Lecture course in Steklov Mathematical Institute, Moscow

22 September 2023

Plan of previous lecture

1. Optimal Control Problem Statement
2. Lebesgue measurable sets and functions
3. Lebesgue integral
4. Carathéodory ODEs
5. Reduction of Optimal Control Problem to Study of Attainable Sets
6. Filippov's theorem: Compactness of Attainable Sets

Plan of this lecture

1. Time-Optimal Problem
2. Smooth manifolds
3. Tangent space and tangent vector
4. Ordinary differential equations on manifolds

Optimal Control Problem Statement

$$\dot{q} = f_u(q), \qquad q \in M, \quad u \in U \subset \mathbb{R}^m, \tag{1}$$

$$q(0) = q_0\,, \tag{2}$$

$$q(t_1) = q_1\,, \tag{3}$$

$$J(u) = \int_0^{t_1} \varphi(q, u) dt \to \min\,. \tag{4}$$

Existence of optimal trajectories for problems with fixed t_1

Theorem 1

Let $q_1 \in \mathcal{A}_{q_0}(t_1)$. If $\widehat{\mathcal{A}}_{(0,q_0)}(t_1)$ is compact, then there exists an optimal trajectory in the problem (1)–(4) with the fixed terminal time t_1.

Theorem 2 (Filippov)

Let the space of control parameters $U \Subset \mathbb{R}^m$ be compact. Let there exist a compact $K \Subset M$ such that $f_u(q) = 0$ for $q \notin K$, $u \in U$. Moreover, let the velocity sets

$$f_U(q) = \{f_u(q) \mid u \in U\} \subset T_q M, \qquad q \in M,$$

be convex. Then the attainable sets $\mathcal{A}_{q_0}(t)$ and $\mathcal{A}_{q_0}^t$ are compact for all $q_0 \in M$, $t > 0$.

A priori bound in $\mathbb{R}^n$

- For control systems on $M = \mathbb{R}^n$, there exist well-known sufficient conditions for completeness of vector fields.
- If the right-hand side grows at infinity sublinearly, i.e.,

$$|f_u(x)| \ \leq C(1 + |x|), \qquad x \in \mathbb{R}^n, \quad u \in U, \tag{5}$$

  for some constant C, then the nonautonomous vector fields $f_u(x)$ are complete (here $|x| = \sqrt{x_1^2 + \cdots + x_n^2}$ is the norm of a point $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$).
- These conditions provide an a priori bound for solutions: any solution $x(t)$ of the control system

$$\dot{x} = f_u(x), \qquad x \in \mathbb{R}^n, \quad u \in U, \tag{6}$$

  with the right-hand side satisfying (5) admits the bound

$$|x(t)| \leq e^{2Ct} \left(|x(0)| + 1\right), \qquad t \geq 0.$$

Compactness of attainable sets in $\mathbb{R}^n$

- Filippov's theorem plus the previous remark imply the following sufficient condition for compactness of attainable sets for systems in $\mathbb{R}^n$.

Corollary 3

Let system (6) *have a compact space of control parameters $U \Subset \mathbb{R}^m$ and convex velocity sets $f_U(x)$, $x \in \mathbb{R}^n$.*
Suppose moreover that the right-hand side of the system satisfies a sublinear bound of the form (5).
Then the attainable sets $\mathcal{A}_{x_0}(t)$ and $\mathcal{A}_{x_0}^t$ are compact for all $x_0 \in \mathbb{R}^n$, $t > 0$.

Time-optimal problem

- Given a pair of points $q_0 \in M$ and $q_1 \in \mathcal{A}_{q_0}$, the *time-optimal problem* consists in minimizing the time of motion from q_0 to q_1 via admissible controls of control system (1):

$$\min_u \{t_1 \mid q_u(t_1) = q_1\}. \tag{7}$$

- That is, we consider the optimal control problem with the integrand $\varphi(q, u) \equiv 1$ and free terminal time t_1.
- Reduction of optimal control problems to the study of attainable sets and Filippov's Theorem yield the following existence result.

Corollary 4

Under the hypotheses of Filippov's Theorem 2*, time-optimal problem* (1), (7) *has a solution for any points* $q_0 \in M$, $q_1 \in \mathcal{A}_{q_0}$.

Example of a time-optimal problem: Stopping a train

Given:

- material point of mass $m > 0$ with coordinate $x \in \mathbb{R}$
- force F bounded by the absolute value by $F_{\max} > 0$
- initial position x_0 and initial velocity $\dot{x}_0$ of the material point

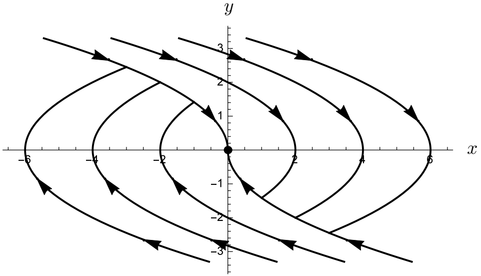
Find:

- force F that steers the point to the origin with zero velocity, for a minimal time.

$$
\begin{aligned}
&\dot{x}_1 = x_2, \qquad (x_1, x_2) \in \mathbb{R}^2,\\
&\dot{x}_2 = u, \qquad |u| \leq 1,\\
&(x_1, x_2)(0) = (x_0, \dot{x}_0), \qquad (x_1, x_2)(t_1) = (0, 0),\\
&t_1 \to \min .
\end{aligned}
$$

Example: Stopping a train

- Trajectories of the system with a constant control $u \neq 0$ are the parabolas $\frac{x_2^2}{2} = ux_1 + C$:



- Now it is visually obvious that $(0,0) \in \mathcal{A}_{(x_1,x_2)}$ for any $(x_1,x_2) \in \mathbb{R}^2$.
- The set of control parameters $U = [-1,1]$ is compact, the set of admissible velocity vectors $f(x,U) = \{(x_2,u) \mid u \in [-1,1]\}$ is convex for any $x \in \mathbb{R}^2$, and the right-hand side of the control system has sublinear growth: $|f(x,u)| \leq C(|x|+1)$.
- All hypotheses of the Filippov theorem are satisfied, thus optimal control exists.

"Smooth" (manifold, mapping, vector field etc.) means C^∞.

Definition 5

A subset $M \subset \mathbb{R}^n$ is called a *smooth k-dimensional submanifold* of $\mathbb{R}^n$, $k \le n$, if any point $x \in M$ has a neighbourhood O_x in $\mathbb{R}^n$ in which M is described in one of the following ways:

(1) there exists a smooth vector-function

$$F \ : \ O_x \to \mathbb{R}^{n-k}, \qquad \operatorname{rank} \left.\frac{dF}{dx}\right|_q = n-k$$

such that

$$O_x \cap M = F^{-1}(0);$$

(2) there exists a smooth vector-function

$$f\ :\ V_0 \to \mathbb{R}^n$$

from a neighbourhood of the origin $0 \in V_0 \subset \mathbb{R}^k$ such that

$$f(0) = x, \qquad \operatorname{rank} \left.\frac{d\,f}{d\,x}\right|_0 = k,$$

$$O_x \cap M = f(V_0)$$

and $f\ :\ V_0 \to O_x \cap M$ is a homeomorphism;

(3) there exists a smooth vector-function

$$\Phi\ :\ O_x \to O_0 \subset \mathbb{R}^n$$

onto a neighbourhood of the origin $0 \in O_0 \subset \mathbb{R}^n$ such that

$$\operatorname{rank} \left.\frac{d\,\Phi}{d\,x}\right|_x = n,$$

$$\Phi(O_x \cap M) = \mathbb{R}^k \cap O_0.$$

- There are two topologically different one-dimensional manifolds: the line $\mathbb{R}^1$ and the circle S^1.
- The sphere S^2 and the torus $\mathbb{T}^2 = S^1 \times S^1$ are two-dimensional manifolds.
- The torus can be viewed as a sphere with a handle. Any compact orientable two-dimensional manifold is topologically a sphere with g handles, $g = 0, 1, 2, \ldots$ is the genus of the manifold.
- Smooth manifolds appear naturally already in the basic analysis. For example, the circle S^1 and the torus $\mathbb{T}^2$ are natural domains of periodic and doubly periodic functions respectively. On the sphere S^2, it is convenient to consider restriction of homogeneous functions of 3 variables.

Abstract manifold

Definition 6

A *smooth k-dimensional manifold* M is a Hausdorff paracompact topological space endowed with a smooth structure: M is covered by a system of open subsets

$$M = \cup_\alpha O_\alpha$$

called coordinate neighbourhoods, in each of which is defined a homeomorphism

$$\Phi_\alpha \ : \ O_\alpha \to \mathbb{R}^k$$

called a local coordinate system such that all compositions

$$\Phi_\beta \circ \Phi_\alpha^{-1} \ : \Phi_\alpha(O_\alpha \cap O_\beta) \subset \mathbb{R}^k \to \Phi_\beta(O_\alpha \cap O_\beta) \subset \mathbb{R}^k$$

are diffeomorphisms, see fig. 1.

Coordinate system in smooth manifold M

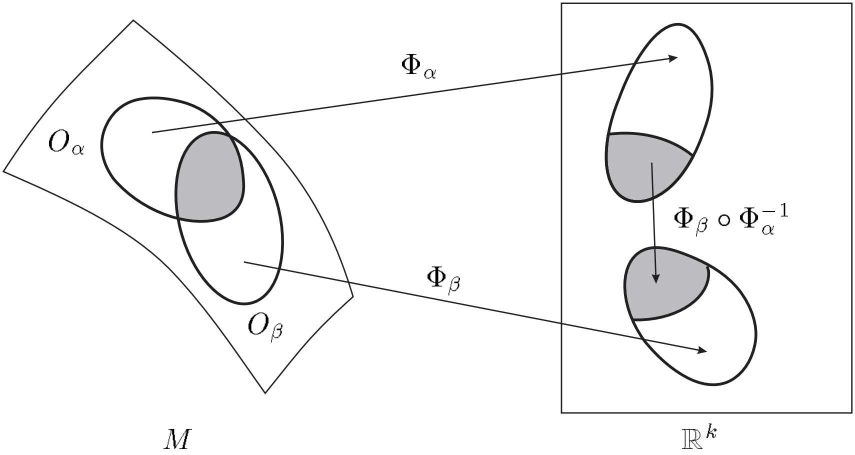


Figure: Coordinate system in smooth manifold M

- As a rule, we denote points of a smooth manifold by q, and its coordinate representation in a local coordinate system by x:

$$q \in M, \qquad \Phi_\alpha : O_\alpha \to \mathbb{R}^k, \quad x = \Phi(q) \in \mathbb{R}^k.$$

- For a smooth submanifold in $\mathbb{R}^n$, the abstract Definition 6 holds. Conversely, any connected smooth abstract manifold can be considered as a smooth submanifold in $\mathbb{R}^n$. Before precise formulation of this statement, we give two definitions.

Definition 7

Let M and N be k- and l-dimensional smooth manifolds respectively. A continuous mapping $f : M \to N$ is called *smooth* if it is smooth in coordinates. That is, let $M = \cup_\alpha O_\alpha$ and $N = \cup_\beta V_\beta$ be coverings of M and N by coordinate neighbourhoods and $\Phi_\alpha : O_\alpha \to \mathbb{R}^k$, $\Psi_\beta : V_\beta \to \mathbb{R}^l$ the corresponding coordinate mappings. Then all

$$\Psi_\beta \circ f \circ \Phi_\alpha^{-1} : \Phi_\alpha(O_\alpha \cap f^{-1}(V_\beta)) \subset \mathbb{R}^k \to \Psi_\beta(f(O_\alpha) \cap V_\beta) \subset \mathbb{R}^l$$

should be smooth.

Definition 8

A smooth manifold M is called *diffeomorphic* to a smooth manifold N if there exists a homeomorphism

$$f : M \to N$$

such that both f and its inverse f^{-1} are smooth mappings. Such mapping f is called a *diffeomorphism*.

The set of all diffeomorphisms $f : M \to M$ of a smooth manifold M is denoted by $\mathrm{Diff}\, M$.

Definition 9

A smooth mapping $f : M \to N$ is called an *embedding* of M into N if $f : M \to f(M)$ is a diffeomorphism. A mapping $f : M \to N$ is called *proper* if $f^{-1}(K)$ is compact for any compactum $K \Subset N$.

Theorem 10 (Whitney)

Any smooth connected k-dimensional manifold can be properly embedded into $\mathbb{R}^{2k+1}$.

Tangent space of a submanifold in $\mathbb{R}^n$

Definition 11

Let M be a smooth k-dimensional submanifold of $\mathbb{R}^n$ and $x \in M$ its point. Then the *tangent space* to M at the point x is a k-dimensional linear subspace $T_x M \subset \mathbb{R}^n$ defined as follows for cases (1)–(3) of Definition 5:

$$(1) \qquad T_x M = \operatorname{Ker} \left.\frac{dF}{dx}\right|_x ,$$

$$(2) \qquad T_x M = \operatorname{Im} \left.\frac{df}{dx}\right|_0 ,$$

$$(3) \qquad T_x M = \left(\left.\frac{d\Phi}{dx}\right|_x \right)^{-1} \mathbb{R}^k .$$

Remark 1

The tangent space is a coordinate-invariant object since smooth changes of variables lead to linear transformations of the tangent space.

Tangent vector to an abstract manifold

Definition 12

Let $\gamma(\cdot)$ be a smooth curve in a smooth manifold M starting from a point $q \in M$:

$$\gamma \ : \ (-\varepsilon, \varepsilon) \to M \text{ a smooth mapping}, \qquad \gamma(0) = q.$$

The *tangent vector* $\left.\frac{d\,\gamma}{d\,t}\right|_{t=0} = \dot{\gamma}(0)$ to the curve $\gamma(\cdot)$ at the point q is the equivalence class of all smooth curves in M starting from q and having the same 1-st order Taylor polynomial as $\gamma(\cdot)$, for any coordinate system in a neighbourhood of q.

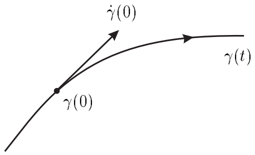


Figure: Tangent vector $\dot{\gamma}(0)$

Tangent space to an abstract manifold

Definition 13

The *tangent space* to a smooth manifold M at a point $q \in M$ is the set of all tangent vectors to all smooth curves in M starting at q:

$$T_q M = \left\{ \left. \frac{d\,\gamma}{d\,t} \right|_{t=0} \mid \gamma \ : \ (-\varepsilon, \varepsilon) \to M \text{ smooth}, \gamma(0) = q \right\}.$$

Remark 2

Let M be a smooth k-dimensional manifold and $q \in M$. Then the tangent space $T_q M$ has a natural structure of a linear k-dimensional space.

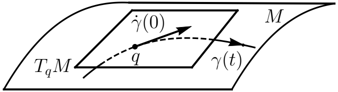


Figure: Tangent space $T_q M$

Dynamical system

Denote by $\mathrm{Vec}\, M$ the set of all smooth vector fields on a smooth manifold M.

Definition 14

A *smooth dynamical system*, or an *ordinary differential equation* (*ODE*), on a smooth manifold M is an equation of the form $\frac{dq}{dt} = V(q), \qquad q \in M$, or, equivalently, $\dot{q} = V(q), \qquad q \in M$, where $V(q)$ is a smooth vector field on M.
A solution to this system is a smooth mapping $\gamma : I \to M$, where $I \subset \mathbb{R}$ is an interval, such that $\frac{d\gamma}{dt} = V(\gamma(t)) \quad \forall\, t \in I$.

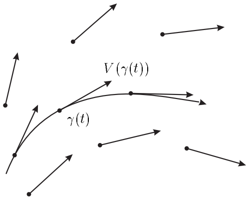


Figure: Solution to ODE $\dot{q} = V(q)$

Differential of a smooth mapping

Definition 15

Let $\Phi : M \to N$ be a smooth mapping between smooth manifolds M and N. The *differential* of Φ at a point $q \in M$ is a linear mapping

$$D_q\Phi : T_qM \to T_{\Phi(q)}N$$

defined as follows:

$$D_q\Phi\left(\left.\frac{d\,\gamma}{d\,t}\right|_{t=0}\right) = \left.\frac{d}{d\,t}\right|_{t=0} \Phi(\gamma(t)),$$

where

$$\gamma : (-\varepsilon, \varepsilon) \subset \mathbb{R} \to M, \qquad \gamma(0) = q,$$

is a smooth curve in M starting at q.

Action of diffeomorphisms on vector fields

- Let $V \in \operatorname{Vec} M$ be a vector field on M and

$$\dot{q} = V(q) \tag{8}$$

the corresponding ODE.

- To find the action of a diffeomorphism

$$\Phi : M \to N, \qquad \Phi : q \mapsto x = \Phi(q)$$

on the vector field $V(q)$, take a solution $q(t)$ of (8) and compute the ODE satisfied by the image $x(t) = \Phi(q(t))$:

$$\dot{x}(t) = \frac{d}{d\,t}\Phi(q(t)) = (D_q\Phi)\,\dot{q}(t) = (D_q\Phi)\,V(q(t)) = (D_{\Phi^{-1}(x)}\Phi)\,V(\Phi^{-1}(x(t))).$$

- So the required ODE is

$$\dot{x} = \left(D_{\Phi^{-1}(x)}\Phi\right) V(\Phi^{-1}(x)). \tag{9}$$

The right-hand side here is the transformed vector field on N induced by the diffeomorphism Φ:

$$(\Phi_* V)(x) \stackrel{\text{def}}{=} \left(D_{\Phi^{-1}(x)}\Phi\right) V(\Phi^{-1}(x)).$$

- The notation Φ_{*q} is used, along with $D_q\Phi$, for differential of a mapping Φ at a point q.
- In general, a smooth mapping Φ induces transformation of tangent vectors, not of vector fields.
- In order that $D\Phi$ transform vector fields to vector fields, Φ should be a diffeomorphism.

Theorem 16

Consider a smooth ODE

$$\dot{q} = V(q), \qquad q \in M \subset \mathbb{R}^n, \tag{10}$$

on a smooth submanifold M of $\mathbb{R}^n$. For any initial point $q_0 \in M$, there exists a unique solution

$$q(t, q_0), \qquad t \in (a, b), \quad a < 0 < b,$$

of equation (10) with the initial condition $q(0, q_0) = q_0$, defined on a sufficiently short interval (a, b). The mapping

$$(t, q_0) \mapsto q(t, q_0)$$

is smooth. In particular, the domain (a, b) of the solution $q(\cdot, q_0)$ can be chosen smoothly depending on q_0.

Proof.

We prove the theorem by reduction to its classical analogue in $\mathbb{R}^n$.
The statement of the theorem is local. We rectify the submanifold M in the neighbourhood of the point q_0:

$$\Phi \,:\, O_{q_0} \subset \mathbb{R}^n \to O_0 \subset \mathbb{R}^n,$$
$$\Phi(O_{q_0} \cap M) = \mathbb{R}^k.$$

Consider the restriction $\varphi = \Phi|_M$. Then a curve $q(t)$ in M is a solution to (10) if and only if its image $x(t) = \varphi(q(t))$ in $\mathbb{R}^k$ is a solution to the induced system:

$$\dot{x} = \Phi_* V(x), \qquad x \in \mathbb{R}^k.$$

$\square$

Theorem 17

Let $M \subset \mathbb{R}^n$ be a smooth submanifold and let

$$\dot{q} = V(q), \qquad q \in \mathbb{R}^n, \tag{11}$$

be a system of ODEs in $\mathbb{R}^n$ such that

$$q \in M \;\Rightarrow\; V(q) \in T_q M.$$

Then for any initial point $q_0 \in M$, the corresponding solution $q(t, q_0)$ to (11) with $q(0, q_0) = q_0$ belongs to M for all sufficiently small $|t|$.

Proof.

Consider the restricted vector field:

$$f = V|_M.$$

By the existence theorem for M, the system

$$\dot{q} = f(q), \qquad q \in M,$$

has a solution $q(t, q_0)$, $q(0, q_0) = q_0$, with

$$q(t, q_0) \in M \quad \text{for small } |t|. \tag{12}$$

On the other hand, the curve $q(t, q_0)$ is a solution of (11) with the same initial condition. Then inclusion (12) proves the theorem. $\square$

Definition 18

A vector field $V \in \operatorname{Vec} M$ is called *complete*, if for all $q_0 \in M$ the solution $q(t, q_0)$ of the Cauchy problem

$$\dot{q} = V(q), \qquad q(0, q_0) = q_0 \tag{13}$$

is defined for all $t \in \mathbb{R}$.

Example 19

The vector field $V(x) = x$ is complete on $\mathbb{R}$, as well as on $\mathbb{R} \setminus \{0\}$, $(-\infty, 0)$, $(0, +\infty)$, and $\{0\}$, but not complete on other submanifolds of $\mathbb{R}$.
The vector field $V(x) = x^2$ is not complete on any submanifolds of $\mathbb{R}$ except $\{0\}$.

Proposition 1

Suppose that there exists $\varepsilon > 0$ *such that for any* $q_0 \in M$ *the solution* $q(t, q_0)$ *to Cauchy problem* (13) *is defined for* $t \in (-\varepsilon, \varepsilon)$. *Then the vector field* $V(q)$ *is complete.*

Remark 3

In this proposition it is required that there exists $\varepsilon > 0$ common for all initial points $q_0 \in M$. In general, ε may be not bounded away from zero for all $q_0 \in M$. E.g., for the vector field $V(x) = x^2$ we have $\varepsilon \to 0$ as $x_0 \to \infty$.

Proof.
Suppose that the hypothesis of the proposition is true. Then we can introduce the following family of mappings in M:

$$P^t : M \to M, \qquad t \in (-\varepsilon, \varepsilon),$$
$$P^t : q_0 \mapsto q(t, q_0).$$

$P^t(q_0)$ is the shift of a point $q_0 \in M$ along the trajectory of the vector field $V(q)$ for time t.
By Theorem 16, all mappings P^t are smooth. Moreover, the family $\{ P^t \mid t \in (-\varepsilon, \varepsilon) \}$ is a smooth family of mappings.
A very important property of this family is that it forms a local one-parameter group, i.e.,

$$P^t(P^s(q)) = P^s(P^t(q)) = P^{t+s}(q), \quad q \in M, \quad t, \ s, \ t+s \in (-\varepsilon, \varepsilon).$$

Indeed, the both curves in M:

$$t \mapsto P^t(P^s(q)) \quad \text{and} \quad t \mapsto P^{t+s}(q)$$

satisfy the ODE $\dot{q} = V(q)$ with the same initial value $P^0(P^s(q)) = P^{0+s}(q) = P^s(q)$. By uniqueness, $P^t(P^s(q)) = P^{t+s}(q)$. The equality for $P^s(P^t(q))$ is obtained by switching t and s.
So we have the following local group properties of the mappings P^t:

$$\begin{aligned}
&P^t \circ P^s = P^s \circ P^t = P^{t+s}, \qquad t,\ s,\ t+s \in (-\varepsilon, \varepsilon),\\
&P^0 = \mathrm{Id},\\
&P^{-t} \circ P^t = P^t \circ P^{-t} = \mathrm{Id}, \qquad t \in (-\varepsilon, \varepsilon),\\
&P^{-t} = \left(P^t\right)^{-1}, \qquad t \in (-\varepsilon, \varepsilon).
\end{aligned}$$

In particular, all P^t are diffeomorphisms.

Now we extend the mappings P^t for all $t \in \mathbb{R}$. Any $t \in \mathbb{R}$ can be represented as

$$t = \frac{\varepsilon}{2}K + \tau, \qquad 0 \le \tau < \frac{\varepsilon}{2}, \quad K = 0, \pm 1, \pm 2, \dots .$$

We set

$$P^t \stackrel{\text{def}}{=} P^\tau \circ \underbrace{P^{\pm\varepsilon/2} \circ \cdots \circ P^{\pm\varepsilon/2}}_{|K| \text{ times}}, \qquad \pm = \operatorname{sgn} t.$$

Then the curve

$$t \mapsto P^t(q_0), \qquad t \in \mathbb{R},$$

is a solution to Cauchy problem (13). $\square$

The flow of a vector field

Definition 20

For a complete vector field $V \in \operatorname{Vec} M$, the mapping

$$t \mapsto P^t, \qquad t \in \mathbb{R},$$

is called the *flow* generated by V.

Example 21

The linear vector field $V(x) = Ax$, $x \in \mathbb{R}^n$, has the flow $P^t = e^{tA} = \sum_{k=0}^{\infty} \frac{(tA)^k}{k!}$.
By this reason the flow of any complete vector field $V \in \operatorname{Vec} M$ is denoted as $P^t = e^{tV}$.

Remark 4

It is useful to imagine a vector field $V \in \operatorname{Vec} M$ as a field of velocity vectors of a moving liquid in M. Then the flow P^t takes any particle of the liquid from a position $q \in M$ and transfers it for a time $t \in \mathbb{R}$ to the position $P^t(q) \in M$.

Sufficient conditions for completeness of a vector field

Proposition 2

Let $K \subset M$ be a compact subset, and let $V \in \operatorname{Vec} M$. Then there exists $\varepsilon_K > 0$ such that for all $q_0 \in K$ the solution $q(t, q_0)$ to Cauchy problem (13) *is defined for all $t \in (-\varepsilon_K, \varepsilon_K)$.*

Proof.

By Theorem 16, domain of the solution $q(t, q_0)$ can be chosen continuously depending on q_0. The diameter of this domain has a positive infimum $2\varepsilon_K$ for q_0 in the compact set K. $\square$

Corollary 22

If a smooth manifold M is compact, then any vector field $V \in \operatorname{Vec} M$ is complete.

Corollary 23

Suppose that a vector field $V \in \operatorname{Vec} M$ has a compact support:

$$\operatorname{supp} V \stackrel{\text{def}}{=} \overline{\{\, q \in M \mid V(q) \neq 0 \,\}} \text{ is compact.}$$

Then V is complete.

Proof.

Indeed, by Proposition 2, there exists $\varepsilon > 0$ such that all trajectories of V starting in $\operatorname{supp} V$ are defined for $t \in (-\varepsilon, \varepsilon)$. But $V|_{M \setminus \operatorname{supp} V} = 0$, and all trajectories of V starting outside of $\operatorname{supp} V$ are constant, thus defined for all $t \in \mathbb{R}$. By Proposition 1, the vector field V is complete. $\square$

Remark 5

If we are interested in the behaviour of (trajectories of) a vector field $V \in \operatorname{Vec} M$ in a compact subset $K \subset M$, we can suppose that V is complete. Indeed, take an open neighbourhood O_K of K with the compact closure $\overline{O_K}$. We can find a function $a \in C^\infty(M)$ such that

$$a(q) = \begin{cases} 1, & q \in K, \\ 0, & q \in M \setminus O_K. \end{cases}$$

Then the vector field $a(q)V(q) \in \operatorname{Vec} M$ is complete since it has a compact support. On the other hand, in K the vector fields $a(q)V(q)$ and $V(q)$ coincide, thus have the same trajectories.