Differential Forms and Symplectic Geometry-2
(Lecture 8)

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Reminder: Plan of previous lecture

1. Differential 1-forms
2. Differential k-forms
3. Exterior differential

Plan of this lecture

1. Lie derivative of differential forms
2. Liouville form and symplectic form
3. Hamiltonian vector fields

Lie derivative of differential forms

- The "infinitesimal version" of the pull-back $\widehat{P}$ of a differential form by a flow P is given by the following operation.
- *Lie derivative* of a differential form $\omega \in \Lambda^k M$ along a vector field $f \in \operatorname{Vec} M$ is the differential form $L_f\omega \in \Lambda^k M$ defined as follows:

$$L_f\omega \stackrel{\text{def}}{=} \left.\frac{d}{d\,\varepsilon}\right|_{\varepsilon=0} \widehat{e^{\varepsilon f}}\omega. \tag{1}$$

- Since

$$\widehat{e^{tf}}(\omega_1 \wedge \omega_2) = \widehat{e^{tf}}\omega_1 \wedge \widehat{e^{tf}}\omega_2,$$

Lie derivative L_f is a derivation of the algebra of differential forms:

$$L_f(\omega_1 \wedge \omega_2) = (L_f\omega_1) \wedge \omega_2 + \omega_1 \wedge L_f\omega_2.$$

- Further, we have

$$\widehat{e^{tf}} \circ d = d \circ \widehat{e^{tf}},$$

thus

$$L_f \circ d = d \circ L_f.$$

- For 0-forms, Lie derivative is just the directional derivative:

$$L_f a = fa, \qquad a \in C^\infty(M),$$

since $\widehat{e^{tf}} a = a \circ e^{tf}$ is a substitution of variables.
- Now we obtain a useful formula for the action of Lie derivative on differential forms of an arbitrary order.
- Consider, along with exterior differential

$$d : \Lambda^k M \to \Lambda^{k+1} M$$

the *interior product* of a differential form ω with a vector field $f \in \operatorname{Vec} M$:

$$i_f : \Lambda^k M \to \Lambda^{k-1} M,$$
$$(i_f \omega)(v_1, \dots, v_{k-1}) \stackrel{\text{def}}{=} \omega(f, v_1, \dots, v_{k-1}), \qquad \omega \in \Lambda^k M, \; v_i \in T_q M,$$

which acts as substitution of f for the first argument of ω. By definition, for 0-order forms

$$i_f a = 0, \qquad a \in \Lambda^0 M.$$

- Interior product is an antiderivation, as well as the exterior differential:

$$i_f(\omega_1 \wedge \omega_2) = (i_f\omega_1) \wedge \omega_2 + (-1)^{k_1}\omega_1 \wedge i_f\omega_2, \qquad \omega_i \in \Lambda^{k_i} M.$$

- Now we prove that Lie derivative of a differential form of an arbitrary order can be computed by the following formula:

$$L_f = d \circ i_f + i_f \circ d \tag{2}$$

called *Cartan's formula*, for short "$L = di + id$".

- Notice first of all that the right-hand side in (2) has the required order:

$$d \circ i_f + i_f \circ d \ : \ \Lambda^k M \to \Lambda^k M.$$

- Further, $d \circ i_f + i_f \circ d$ is a derivation as it is obtained from two antiderivations.

- Moreover, this derivation commutes with differential:

$$d \circ (d \circ i_f + i_f \circ d) = d \circ i_f \circ d,$$
$$(d \circ i_f + i_f \circ d) \circ d = d \circ i_f \circ d.$$

- Now we check the formula $L = di + id$ on 0-forms: if $a \in \Lambda^0 M$, then

$$(d \circ i_f)a = 0,$$
$$(i_f \circ d)a = \langle da, f \rangle = fa = L_f a.$$

  So the formula $L = di + id$ holds for 0-forms.
- The properties of the mappings L_f and $d \circ i_f + i_f \circ d$ established and the coordinate representation of differential forms reduce the general case of k-forms to the case of 0-forms.
- Cartan's formula $L = di + id$ is proved for k-forms.

- The differential definition (1) of Lie derivative can be integrated, i.e., there holds the following equality on $\Lambda^k M$:

$$\widehat{\left(\overrightarrow{\exp} \int_0^t f_\tau \, d\tau \right)} = \overrightarrow{\exp} \int_0^t L_{f_\tau} \, d\tau, \qquad (3)$$

in the following sense.

- Denote the flow $P_{t_0}^{t_1} = \overrightarrow{\exp} \int_{t_0}^{t_1} f_\tau \, d\tau$ of a nonautonomous vector field f_τ on M.
- The family of operators on differential forms $\widehat{P_0^t} : \Lambda^k M \to \Lambda^k M$ is a unique solution of the Cauchy problem

$$\frac{d}{d\,t} \widehat{P_0^t} = \widehat{P_0^t} \circ L_{f_t}, \qquad \widehat{P_0^t} \Big|_{t=0} = \mathrm{Id}, \qquad (4)$$

compare with Cauchy problems for the flow P_0^t and for the family of operators $\mathrm{Ad}\, P_0^t$, and this solution is denoted as

$$\overrightarrow{\exp} \int_0^t L_{f_\tau} \, d\tau \stackrel{\mathrm{def}}{=} \widehat{P_0^t} = \widehat{\left(\overrightarrow{\exp} \int_0^t f_\tau \, d\tau \right)} .$$

- In order to verify the ODE in (4), we prove first the following equality for operators on forms:

$$\left.\frac{d}{d\,\varepsilon}\right|_{\varepsilon=0} \widehat{P_t^{t+\varepsilon}}\omega = L_{f_t}\omega, \qquad \omega \in \Lambda^k M. \tag{5}$$

- This equality is straightforward for 0-order forms:

$$\left.\frac{d}{d\,\varepsilon}\right|_{\varepsilon=0} \widehat{P_t^{t+\varepsilon}} a = \left.\frac{d}{d\,\varepsilon}\right|_{\varepsilon=0} a \circ P_t^{t+\varepsilon} = f_t a = L_{f_t} a, \qquad a \in C^\infty(M).$$

- Further, the both operators $\left.\frac{d}{d\,\varepsilon}\right|_{\varepsilon=0} \widehat{P_t^{t+\varepsilon}}$ and L_{f_t} commute with d and satisfy the Leibniz rule w.r.t. product of a function with a differential form.
- Then equality (5) follows for forms of arbitrary order, as in the proof of Cartan's formula.

- Now we easily verify the ODE in (4):

$$\frac{d}{d\,t}\widehat{P_0^t} = \frac{d}{d\,\varepsilon}\Big|_{\varepsilon=0} \widehat{P_0^{t+\varepsilon}} = \frac{d}{d\,\varepsilon}\Big|_{\varepsilon=0} \widehat{\left(P_t^{t+\varepsilon} \circ P_0^t\right)}$$

by the composition rule for pull-back of differential forms

$$= \frac{d}{d\,\varepsilon}\Big|_{\varepsilon=0} \widehat{P_0^t} \circ \widehat{P_t^{t+\varepsilon}} = \widehat{P_0^t} \circ \frac{d}{d\,\varepsilon}\Big|_{\varepsilon=0} \widehat{P_t^{t+\varepsilon}}$$

$$= \widehat{P_0^t} \circ L_{f_t}.$$

Exercise 1

Prove uniqueness for Cauchy problem (4).

- For an autonomous vector field $f \in \operatorname{Vec} M$, equality (3) takes the form

$$\widehat{e^{tf}} = e^{tL_f}.$$

- Notice that the Lie derivatives of differential forms L_f and vector fields $(-\operatorname{ad} f)$ are in a certain sense dual one to another, see equality (6) below.
- That is, the function

$$\langle \omega, X \rangle \ : \ q \mapsto \langle \omega_q, X(q) \rangle, \qquad q \in M,$$

defines a pairing of $\Lambda^1 M$ and $\operatorname{Vec} M$ over $C^\infty(M)$. Then the equality

$$\langle \widehat{P}\omega, X \rangle = P\langle \omega, \operatorname{Ad} P^{-1} X \rangle, \qquad P \in \operatorname{Diff} M,\ X \in \operatorname{Vec} M,\ \omega \in \Lambda^1 M,$$

has an infinitesimal version of the form

$$\langle L_Y \omega, X \rangle = Y\langle \omega, X \rangle - \langle \omega, (\operatorname{ad} Y)X \rangle, \qquad X,\ Y \in \operatorname{Vec} M,\ \ \omega \in \Lambda^1 M. \qquad (6)$$

- Taking into account Cartan's formula $L = di + id$, we immediately obtain the following important equality:

$$d\omega(Y, X) = Y\langle \omega, X \rangle - X\langle \omega, Y \rangle - \langle \omega, [Y, X] \rangle, \quad X,\ Y \in \operatorname{Vec} M,\ \ \omega \in \Lambda^1 M. \ (7)$$

- We have already seen that the cotangent bundle $T^*M = \cup_{q \in M} T_q^*M$ of an n-dimensional manifold M is a $2n$-dimensional manifold. Any local coordinates $x = (x_1, \ldots, x_n)$ on M determine canonical local coordinates on T^*M of the form $(\xi, x) = (\xi_1, \ldots, \xi_n; x_1, \ldots, x_n)$ in which any covector $\lambda \in T_{q_0}^*M$ has the decomposition $\lambda = \sum_{i=1}^n \xi_i \, dx_i|_{q_0}$.
- The *"tautological" 1-form* (or *Liouville 1-form*) on the cotangent bundle

$$s \in \Lambda^1(T^*M)$$

is defined as follows.
- Let $\lambda \in T^*M$ be a point in the cotangent bundle and $w \in T_\lambda(T^*M)$ a tangent vector to T^*M at λ.
- Denote by π the canonical projection from T^*M to M:

$$\pi \; : \; T^*M \to M,$$
$$\pi \; : \; \lambda \mapsto q, \qquad \lambda \in T_q^*M.$$

- Differential of π is a linear mapping

$$\pi_* : T_\lambda(T^*M) \to T_qM, \qquad q = \pi(\lambda).$$

- The tautological 1-form s at the point λ acts on the tangent vector w in the following way:

$$\langle s_\lambda, w\rangle \stackrel{\text{def}}{=} \langle \lambda, \pi_* w\rangle.$$

- That is, we project the vector $w \in T_\lambda(T^*M)$ to the vector $\pi_* w \in T_qM$, and then act by the covector $\lambda \in T_q^*M$.
- So

$$s_\lambda \stackrel{\text{def}}{=} \lambda \circ \pi_*.$$

- The title "tautological" is explained by the coordinate representation of the form s.
- In canonical coordinates (ξ, x) on T^*M, we have:

$$\lambda = \sum_{i=1}^{n} \xi_i dx_i, \tag{8}$$

$$w = \sum_{i=1}^{n} \alpha_i \frac{\partial}{\partial \xi_i} + \beta_i \frac{\partial}{\partial x_i}.$$

- The projection written in canonical coordinates

$$\pi \ : \ (\xi, x) \mapsto x$$

is a linear mapping, its differential acts as follows:

$$\pi_* \left(\frac{\partial}{\partial \xi_i} \right) = 0, \qquad i = 1, \dots, n,$$

$$\pi_* \left(\frac{\partial}{\partial x_i} \right) = \frac{\partial}{\partial x_i}, \qquad i = 1, \dots, n.$$

- Thus
$$\pi_* w = \sum_{i=1}^{n} \beta_i \frac{\partial}{\partial x_i},$$
consequently,
$$\langle s_\lambda, w\rangle = \langle \lambda, \pi_* w\rangle = \sum_{i=1}^{n} \xi_i \beta_i.$$
- But $\beta_i = \langle dx_i, w\rangle$, so the form s has in coordinates (ξ, x) exactly the same expression
$$s_\lambda = \sum_{i=1}^{n} \xi_i dx_i \tag{9}$$
as the covector λ, see (8).
- Although, definition of the form s does not depend on any coordinates.

Remark 1

In mechanics, the tautological form s is denoted as $p\,dq$.

- Consider the exterior differential of the 1-form s:

$$\sigma \stackrel{\text{def}}{=} ds.$$

- The differential 2-form $\sigma \in \Lambda^2(T^*M)$ is called the *canonical symplectic structure* on T^*M.
- In canonical coordinates, we obtain from (9):

$$\sigma = \sum_{i=1}^{n} d\xi_i \wedge dx_i. \tag{10}$$

- This expression shows that the form σ is nondegenerate, i.e., the bilinear skew-symmetric form

$$\sigma_\lambda \,:\, T_\lambda(T^*M) \times T_\lambda(T^*M) \to \mathbb{R}$$

has no kernel:

$$\sigma(w, \cdot) = 0 \quad \Rightarrow \quad w = 0, \qquad\qquad w \in T_\lambda(T^*M).$$

- In the following basis in the tangent space $T_\lambda(T^*M)$

$$\frac{\partial}{\partial x_1}, \frac{\partial}{\partial \xi_1}, \dots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial \xi_n},$$

the form σ_λ has the block matrix
$$\begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & 1 \\ & & & -1 & 0 \end{pmatrix}.$$

- The form σ is closed: $d\sigma = 0$ since it is exact: $\sigma = ds$, and $d \circ d = 0$.

Remarks

(1) A closed nondegenerate exterior differential 2-form on a $2n$-dimensional manifold is called a *symplectic structure*. A manifold with a symplectic structure is called a *symplectic manifold*. The cotangent bundle T^*M with the canonical symplectic structure σ is the most important example of a symplectic manifold.
(2) In mechanics, the 2-form σ is known as the form $dp \wedge dq$.

Hamiltonian vector fields

- Due to the symplectic structure $\sigma \in \Lambda^2(T^*M)$, we can develop the Hamiltonian formalism on T^*M.
- A *Hamiltonian* is an arbitrary smooth function on the cotangent bundle:

$$h \in C^\infty(T^*M).$$

- To any Hamiltonian h, we associate the *Hamiltonian vector field*

$$\vec{h} \in \mathrm{Vec}(T^*M)$$

  by the rule:

$$\sigma_\lambda(\cdot, \vec{h}) = d_\lambda h, \qquad \lambda \in T^*M. \tag{11}$$

- In terms of the interior product $i_v\omega(\cdot,\cdot) = \omega(v,\cdot)$, the Hamiltonian vector field is a vector field $\vec{h}$ that satisfies

$$i_{\vec{h}}\sigma = -dh.$$

- Since the symplectic form σ is nondegenerate, the mapping

$$w \mapsto \sigma_\lambda(\cdot, w)$$

is a linear isomorphism

$$T_\lambda(T^*M) \to T^*_\lambda(T^*M),$$

thus the Hamiltonian vector field $\vec{h}$ in (11) exists and is uniquely determined by the Hamiltonian function h.

- In canonical coordinates (ξ, x) on T^*M we have

$$dh = \sum_{i=1}^{n} \left(\frac{\partial h}{\partial \xi_i} d\xi_i + \frac{\partial h}{\partial x_i} dx_i \right),$$

then in view of (10)

$$\vec{h} = \sum_{i=1}^{n} \left(\frac{\partial h}{\partial \xi_i} \frac{\partial}{\partial x_i} - \frac{\partial h}{\partial x_i} \frac{\partial}{\partial \xi_i} \right). \tag{12}$$

- So the *Hamiltonian system* of ODEs corresponding to h

$$\dot{\lambda} = \vec{h}(\lambda), \qquad \lambda \in T^*M,$$

reads in canonical coordinates as follows:

$$\begin{cases} \dot{x}_i = \dfrac{\partial h}{\partial \xi_i}, & i = 1, \ldots, n, \\ \dot{\xi}_i = -\dfrac{\partial h}{\partial x_i}, & i = 1, \ldots, n. \end{cases}$$

- The Hamiltonian function can depend on a parameter: h_t, $t \in \mathbb{R}$. Then the nonautonomous Hamiltonian vector field $\vec{h}_t$, $t \in \mathbb{R}$ is defined in the same way as in the autonomous case.
- The flow of a Hamiltonian system preserves the symplectic form σ.

Proposition 1.1

Let $\vec{h}_t$ be a nonautonomous Hamiltonian vector field on T^*M. Then

$$\widehat{\left(\overrightarrow{\exp}\int_0^t \vec{h}_\tau \, d\tau\right)} \sigma = \sigma.$$

Proof:

- In view of equality (3), we have

$$\widehat{\left(\overrightarrow{\exp}\int_0^t \vec{h}_\tau \, d\tau\right)} = \overrightarrow{\exp}\int_0^t L_{\vec{h}_\tau} \, d\tau,$$

  thus the statement of this proposition can be rewritten as $L_{\vec{h}_t}\sigma = 0$.
- But this Lie derivative is easily computed by Cartan's formula:

$$L_{\vec{h}_t}\sigma = i_{\vec{h}_t} \circ \underbrace{d\sigma}_{=0} + d \circ \underbrace{i_{\vec{h}_t}\sigma}_{=-dh_t} = -d \circ dh_t = 0.$$

- Moreover, there holds a local converse statement: if a flow preserves σ, then it is locally Hamiltonian.
- Indeed,

$$\widehat{\left(\overrightarrow{\exp} \int_0^t f_\tau \, d\tau\right)} \sigma = \sigma \quad \Leftrightarrow \quad L_{f_t}\sigma = 0,$$

further

$$L_{f_t}\sigma = i_{f_t} \circ \underbrace{d\sigma}_{=0} + d \circ i_{f_t}\sigma,$$

thus

$$L_{f_t}\sigma = 0 \quad \Leftrightarrow \quad d \circ i_{f_t}\sigma = 0.$$

- If the form $i_{f_t}\sigma$ is closed, then it is locally exact (Poincaré's Lemma), i.e., there exists a Hamiltonian h_t such that locally $f_t = \vec{h}_t$.
- Essentially, only Hamiltonian flows preserve σ (globally, "multi-valued Hamiltonians" can appear).
- If a manifold M is simply connected, then there holds a global statement: a flow on T^*M is Hamiltonian if and only if it preserves the symplectic structure.

- The *Poisson bracket* of Hamiltonians $a, b \in C^\infty(T^*M)$ is a Hamiltonian
$$\{a, b\} \in C^\infty(T^*M)$$
defined in one of the following equivalent ways:
$$\{a, b\} = \vec{a}b = \langle db, \vec{a}\rangle = \sigma(\vec{a}, \vec{b}) = -\sigma(\vec{b}, \vec{a}) = -\vec{b}a.$$
- It is obvious that Poisson bracket is bilinear and skew-symmetric:
$$\{a, b\} = -\{b, a\}.$$
- In canonical coordinates (ξ, x) on T^*M,
$$\{a, b\} = \sum_{i=1}^{n} \left(\frac{\partial a}{\partial \xi_i} \frac{\partial b}{\partial x_i} - \frac{\partial a}{\partial x_i} \frac{\partial b}{\partial \xi_i} \right). \tag{13}$$
- Leibniz rule for Poisson bracket easily follows from definition:
$$\{a, bc\} = \{a, b\}c + b\{a, c\}$$
(here bc is the usual pointwise product of functions b and c).

- Symplectomorphisms of cotangent bundle preserve Hamiltonian vector fields; the action of a symplectomorphism $P \in \mathrm{Diff}(T^*M)$, $\widehat{P}\sigma = \sigma$, on a Hamiltonian vector field $\vec{h}$ reduces to the action of P on the Hamiltonian function as substitution of variables:

$$\mathrm{Ad}\, P\, \vec{h} = \overrightarrow{Ph}\ .$$

- This follows from the chain

$$\sigma\left(X, \mathrm{Ad}\, P\, \vec{h}\right) = \widehat{P}\sigma\left(X, \mathrm{Ad}\, P\, \vec{h}\right) = P\sigma\left(\mathrm{Ad}\, P^{-1} X, \vec{h}\right)$$
$$= P\langle dh, \mathrm{Ad}\, P^{-1} X\rangle = X(Ph) = \sigma\left(X, \overrightarrow{Ph}\right), \qquad X \in \mathrm{Vec}(T^*M).$$

- In particular, a Hamiltonian flow transforms a Hamiltonian vector field into a Hamiltonian vector field:

$$\mathrm{Ad}\, P^t\, \vec{b}_t = \overrightarrow{P^t b_t}, \qquad P^t = \overrightarrow{\exp} \int_0^t \vec{a}_\tau\, d\tau. \tag{14}$$

- Infinitesimally, this equality implies Jacobi identity for Poisson bracket.

Proposition 1.2

$$\{a, \{b, c\}\} + \{b, \{c, a\}\} + \{c, \{a, b\}\} = 0, \qquad a, b, c \in C^\infty(T^*M). \tag{15}$$

Proof:

- Any symplectomorphism $P \in \mathrm{Diff}(T^*M)$, $\widehat{P}\sigma = \sigma$, preserves Poisson brackets:

$$P\{b, c\} = P\sigma\left(\vec{b}, \vec{c}\right) = \widehat{P}\sigma\left(\mathrm{Ad}\, P\, \vec{b}, \mathrm{Ad}\, P\, \vec{c}\right) = \sigma\left(\overrightarrow{Pb}, \overrightarrow{Pc}\right) = \{Pb, Pc\}.$$

- Taking $P = e^{t\vec{a}}$ and differentiating at $t = 0$, we come to Jacobi identity:

$$\{a, \{b, c\}\} = \{\{a, b\}, c\} + \{b, \{a, c\}\}.$$

- So the space of all Hamiltonians $C^\infty(T^*M)$ forms a Lie algebra with Poisson bracket as a product.
- The correspondence

$$a \mapsto \vec{a}, \qquad a \in C^\infty(T^*M), \tag{16}$$

  is a homomorphism from the Lie algebra of Hamiltonians to the Lie algebra of Hamiltonian vector fields on M. This follows from the next statement.

Corollary 1

$\overrightarrow{\{a, b\}} = [\vec{a}, \vec{b}]$ *for any Hamiltonians* $a, b \in C^\infty(T^*M)$.

Proof:

- Jacobi identity can be rewritten as

$$\{\{a, b\}, c\} = \{a, \{b, c\}\} - \{b, \{a, c\}\},$$

  i.e.,

$$\overrightarrow{\{a, b\}}\; c = \vec{a} \circ \vec{b}\, c - \vec{b} \circ \vec{a}\, c = [\vec{a}, \vec{b}]\, c, \qquad c \in C^\infty(T^*M).$$

- It is easy to see from the coordinate representation (12) that the kernel of the mapping $a \mapsto \vec{a}$ consists of constant functions, i.e., this is isomorphism up to constants.
- On the other hand, this homomorphism is far from being onto all vector fields on T^*M.
- Indeed, a general vector field on T^*M is locally defined by arbitrary $2n$ smooth real functions of $2n$ variables, while a Hamiltonian vector field is determined by just one real function of $2n$ variables, a Hamiltonian.

Theorem 2 (Nöther)

A function $a \in C^\infty(T^*M)$ is an integral of a Hamiltonian system of ODEs

$$\dot{\lambda} = \vec{h}(\lambda), \qquad \lambda \in T^*M, \tag{17}$$

i.e.,

$$e^{t\vec{h}} a = a \qquad t \in \mathbb{R},$$

if and only if it Poisson-commutes with the Hamiltonian:

$$\{a, h\} = 0.$$

Proof:

- $e^{t\vec{h}} a \equiv a \Leftrightarrow 0 = \vec{h} a = \{h, a\}$.

Corollary 3

$e^{t\vec{h}} h = h$, i.e., any Hamiltonian $h \in C^\infty(T^*M)$ is an integral of the corresponding Hamiltonian system (17).

- Further, Jacobi identity for Poisson brackets implies that the set of integrals of the Hamiltonian system (17) forms a Lie algebra with respect to Poisson brackets.

Corollary 4

$\{h, a\} = \{h, b\} = 0 \Rightarrow \{h, \{a, b\}\} = 0.$

Remark 2

The Hamiltonian formalism developed generalizes for arbitrary symplectic manifolds.