Differential Forms and Symplectic Geometry
(Lecture 7)

Yuri Sachkov

Program Systems Institute
Russian Academy of Sciences
Pereslavl-Zalessky, Russia
yusachkov@gmail.com

«Elements of Control Theory»

Lecture course in Program Systems Institute, Pereslavl-Zalessky

23 May 2023

Reminder: Plan of previous lectures

1. Optimal Control Problem: Statement end existence of solutions
2. Chronological calculus

Plan of this lecture

1. Differential 1-forms
2. Differential k-forms
3. Exterior differential

Differential 1-forms

Linear forms

- E a real vector space of finite dimension n.
- A *linear form* on E is a linear function $\xi : E \to \mathbb{R}$.
- The set of linear forms on E has a natural structure of a vector space called the *dual space* to E and denoted by E^*.
- If vectors $e_1, \ldots, e_n$ form a basis of E, then the corresponding *dual basis* of E^* is formed by the covectors $e_1^*, \ldots, e_n^*$ such that

$$\langle e_i^*, e_j \rangle = \delta_{ij}, \qquad i,\ j = 1, \ldots n.$$

- So the dual space has the same dimension as the initial one:

$$\dim E^* = n = \dim E.$$

Cotangent bundle

- M a smooth manifold and T_qM its tangent space at a point $q \in M$.
- The space of linear forms on T_qM, i.e., the dual space $(T_qM)^*$ to T_qM, is called the *cotangent space* to M at q and is denoted as T_q^*M.
- The disjoint union of all cotangent spaces is called the *cotangent bundle* of M:

$$T^*M \stackrel{\text{def}}{=} \bigsqcup_{q \in M} T_q^*M.$$

- The set T^*M has a natural structure of a smooth manifold of dimension $2n$, where $n = \dim M$.
- Local coordinates on T^*M are constructed from local coordinates on M.
- Let $O \subset M$ be a coordinate neighborhood and let

$$\Phi : O \to \mathbb{R}^n, \qquad \Phi(q) = (x_1(q), \dots, x_n(q)),$$

be a local coordinate system.

- Differentials of the coordinate functions

$$dx_i\big|_q \in T_q^*M, \qquad i = 1, \dots, n, \quad q \in O,$$

form a basis in the cotangent space T_q^*M.
- The dual basis in the tangent space T_qM is formed by the vectors

$$\left.\frac{\partial}{\partial x_i}\right|_q \in T_qM, \qquad i = 1, \dots, n, \quad q \in O,$$

$$\left\langle dx_i, \frac{\partial}{\partial x_j} \right\rangle \equiv \delta_{ij}, \qquad i,\ j = 1, \dots, n.$$

- Any linear form $\xi \in T_q^*M$ can be decomposed via the basis forms:

$$\xi = \sum_{i=1}^{n} \xi_i \, dx_i.$$

- So any covector $\xi \in T^*M$ is characterized by n coordinates $(x_1, \dots, x_n)$ of the point $q \in M$ where ξ is attached, and by n coordinates $(\xi_1, \dots, \xi_n)$ of the linear form ξ in the basis $dx_1, \dots, dx_n$.

- Mappings of the form
$$\xi \mapsto (\xi_1, \ldots, \xi_n; x_1, \ldots, x_n)$$
define local coordinates on the cotangent bundle. Consequently, T^*M is a $2n$-dimensional manifold.
- Coordinates of the form (ξ, x) are called *canonical coordinates* on T^*M.

- If $F : M \to N$ is a smooth mapping between smooth manifolds, then the differential

$$F_* : T_q M \to T_{F(q)} N$$

has the adjoint (dual) mapping

$$F^* \stackrel{\text{def}}{=} (F_*)^* : T^*_{F(q)} N \to T^*_q M$$

defined as follows:

$$\begin{aligned} F^* \xi &= \xi \circ F_*, & \xi &\in T^*_{F(q)} N, \\ \langle F^* \xi, v \rangle &= \langle \xi, F_* v \rangle, & v &\in T_q M. \end{aligned}$$

- A vector $v \in T_q M$ is pushed forward by the differential F_* to the vector $F_* v \in T_{F(q)} N$, while a covector $\xi \in T^*_{F(q)} N$ is pulled back to the covector $F^* \xi \in T^*_q M$.
- So a smooth mapping $F : M \to N$ between manifolds induces a smooth mapping $F^* : T^* N \to T^* M$ between their cotangent bundles.

Differential 1-forms

- A *differential 1-form* on M is a smooth mapping $q \mapsto \omega_q \in T_q^*M, \;\; q \in M$, i.e, a family $\omega = \{\omega_q\}$ of linear forms on the tangent spaces T_qM smoothly depending on the point $q \in M$.
- The set of all differential 1-forms on M has a natural structure of an infinite-dimensional vector space denoted as $\Lambda^1 M$.
- Like linear forms on a vector space are dual objects to vectors of the space, differential forms on a manifold are dual objects to smooth curves in the manifold.
- The pairing operation is the *integral* of a differential 1-form $\omega \in \Lambda^1 M$ along a smooth oriented curve $\gamma : [t_0, t_1] \to M$, defined as follows:

$$\int_\gamma \omega \stackrel{\text{def}}{=} \int_{t_0}^{t_1} \langle \omega_{\gamma(t)}, \dot{\gamma}(t) \rangle \, dt.$$

- The integral of a 1-form along a curve does not change under orientation-preserving smooth reparametrizations of the curve and changes its sign under change of orientation.

Differential k-forms

- A differential k-form on M is an object to integrate over k-dim. surfaces in M.
- Infinitesimally, a k-dimensional surface is presented by its tangent space, i.e., a k-dimensional subspace in T_qM.
- We need a dual object to the set of k-dim. subspaces in the linear space.
- Fix a linear space E.
- A k-dimensional subspace is defined by its basis $v_1, \ldots, v_k \in E$.
- The dual objects should be mappings

$$(v_1, \ldots, v_k) \mapsto \omega(v_1, \ldots, v_k) \in \mathbb{R}$$

  such that $\omega(v_1, \ldots, v_k)$ depend only on the linear hull $\text{span}\{v_1, \ldots, v_k\}$ and the oriented volume of the k-dimensional parallelepiped generated by $v_1, \ldots, v_k$.
- Moreover, the dependence on the volume should be linear.
- Recall that the ratio of volumes of the parallelepipeds generated by vectors $w_i = \sum_{j=1}^{k} \alpha_{ij} v_j$, $i = 1, \ldots, k$, and the vectors $v_1, \ldots, v_k$, equals $\det(\alpha_{ij})_{i,j=1}^{k}$, and that determinant of a $k \times k$ matrix is a multilinear skew-symmetric form of the columns of the matrix.

Exterior k-forms

- Let E be a finite-dimensional real vector space, $\dim E = n$, and let $k \in \mathbb{N}$.
- An *exterior k-form* on E is a mapping

$$\omega : \underbrace{E \times \cdots \times E}_{k \text{ times}} \to \mathbb{R},$$

which is multilinear:

$$\begin{aligned}&\omega(v_1, \ldots, \alpha_1 v_i^1 + \alpha_2 v_i^2, \ldots, v_k) \\ &\qquad = \alpha_1 \omega(v_1, \ldots, v_i^1, \ldots, v_k) + \alpha_2 \omega(v_1, \ldots, v_i^2, \ldots, v_k), \qquad \alpha_1,\ \alpha_2 \in \mathbb{R},\end{aligned}$$

and skew-symmetric:

$$\omega(v_1, \ldots, v_i, \ldots, v_j, \ldots, v_k) = -\omega(v_1, \ldots, v_j, \ldots, v_i, \ldots, v_k), \quad i,\ j = 1, \ldots, k.$$

- The set of all exterior k-forms on E is denoted by $\Lambda^k E$.
- By the skew-symmetry, any exterior form of order $k > n$ is zero, thus $\Lambda^k E = \{0\}$ for $k > n$.

- Exterior forms can be multiplied by real numbers, and exterior forms of the same order k can be added one with another, so each $\Lambda^k E$ is a vector space.
- We construct a basis of $\Lambda^k E$ after we consider another operation between exterior forms — the exterior product.
- The exterior product of two forms $\omega_1 \in \Lambda^{k_1} E$, $\omega_2 \in \Lambda^{k_2} E$ is an exterior form $\omega_1 \wedge \omega_2$ of order $k_1 + k_2$.
- Given linear 1-forms $\omega_1, \omega_2 \in \Lambda^1 E$, we have a natural (tensor) product for them:

$$\omega_1 \otimes \omega_2 \ : \ (v_1, v_2) \mapsto \omega_1(v_1)\omega_2(v_2), \qquad v_1, v_2 \in E.$$

- The result is a bilinear but not a skew-symmetric form.
- The *exterior product* is the anti-symmetrization of the tensor one:

$$\omega_1 \wedge \omega_2 \ : \ (v_1, v_2) \mapsto \omega_1(v_1)\omega_2(v_2) - \omega_1(v_2)\omega_2(v_1), \qquad v_1, v_2 \in E.$$

- Similarly, the tensor and exterior products of forms $\omega_1 \in \Lambda^{k_1} E$ and $\omega_2 \in \Lambda^{k_2} E$ are the following forms of order $k_1 + k_2$:

$$\begin{aligned} \omega_1 \otimes \omega_2 \ &: \ (v_1, \dots, v_{k_1+k_2}) \mapsto \omega_1(v_1, \dots, v_{k_1})\omega_2(v_{k_1+1}, \dots, v_{k_1+k_2}), \\ \omega_1 \wedge \omega_2 \ &: \ (v_1, \dots, v_{k_1+k_2}) \mapsto \\ &\frac{1}{k_1!\, k_2!} \sum_{\sigma} (-1)^{\nu(\sigma)} \omega_1(v_{\sigma(1)}, \dots, v_{\sigma(k_1)}) \omega_2(v_{\sigma(k_1+1)}, \dots, v_{\sigma(k_1+k_2)}), \end{aligned} \tag{1}$$

  where the sum is taken over all permutations σ of order $k_1 + k_2$ and $\nu(\sigma)$ is parity of a permutation σ.
- The factor $\frac{1}{k_1!\, k_2!}$ normalizes the sum in (1) since it contains $k_1!\, k_2!$ identically equal terms: e.g., if permutations σ do not mix the first k_1 and the last k_2 arguments, then all terms of the form

$$(-1)^{\nu(\sigma)} \omega_1(v_{\sigma(1)}, \dots, v_{\sigma(k_1)}) \omega_2(v_{\sigma(k_1+1)}, \dots, v_{\sigma(k_1+k_2)})$$

  are equal to

$$\omega_1(v_1, \dots, v_{k_1}) \omega_2(v_{k_1+1}, \dots, v_{k_1+k_2}).$$

- This guarantees the associative property of the exterior product:

$$\omega_1 \wedge (\omega_2 \wedge \omega_3) = (\omega_1 \wedge \omega_2) \wedge \omega_3, \qquad \omega_i \in \Lambda^{k_i} E,$$

- Further, the exterior product is skew-commutative:

$$\omega_2 \wedge \omega_1 = (-1)^{k_1 k_2} \omega_1 \wedge \omega_2, \qquad \omega_i \in \Lambda^{k_i} E.$$

- Let $e_1, \ldots, e_n$ be a basis of the space E and $e_1^*, \ldots, e_n^*$ the corresponding dual basis of E^*.
- If $1 \leq k \leq n$, then the following $C_n^k = \frac{n!}{k!(n-k)!}$ elements form a basis of the space $\Lambda^k E$:

$$e_{i_1}^* \wedge \ldots \wedge e_{i_k}^*, \qquad 1 \leq i_1 < i_2 < \cdots < i_k \leq n.$$

- The equalities

$$(e^*_{i_1} \wedge \ldots \wedge e^*_{i_k})(e_{i_1}, \ldots, e_{i_k}) = 1,$$
$$(e^*_{i_1} \wedge \ldots \wedge e^*_{i_k})(e_{j_1}, \ldots, e_{j_k}) = 0, \quad \text{if } (i_1, \ldots, i_k) \neq (j_1, \ldots, j_k)$$

for $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ imply that any k-form $\omega \in \Lambda^k E$ has a unique decomposition of the form

$$\omega = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} \omega_{i_1 \ldots i_k} e^*_{i_1} \wedge \ldots \wedge e^*_{i_k}$$

with

$$\omega_{i_1 \ldots i_k} = \omega(e_{i_1}, \ldots, e_{i_k}).$$

Exercise 1

Show that for any 1-forms $\omega_1, \ldots \omega_p \in \Lambda^1 E$ and any vectors $v_1, \ldots, v_p \in E$ there holds the equality

$$(\omega_1 \wedge \ldots \wedge \omega_p)(v_1, \ldots, v_p) = \det\left(\langle \omega_i, v_j \rangle\right)^p_{i,j=1}. \tag{2}$$

- Notice that the space of n-forms of an n-dimensional space E is one-dimensional.
- Any nonzero n-form on E is called a *volume form*.
- For example, the value of the standard volume form $e_1^* \wedge \ldots \wedge e_n^*$ on an n-tuple of vectors $(v_1, \ldots, v_n)$ is

$$(e_1^* \wedge \ldots \wedge e_n^*)(v_1, \ldots, v_n) = \det\left(\langle e_i^*, v_j \rangle\right)_{i,j=1}^{n},$$

the oriented volume of the parallelepiped generated by the vectors $v_1, \ldots, v_n$.

Differential k-forms

- A *differential k-form* on M is a mapping

$$\omega \;:\; q \mapsto \omega_q \in \Lambda^k T_q M, \qquad q \in M,$$

  smooth w.r.t. $q \in M$.
- The set of all differential k-forms on M is denoted by $\Lambda^k M$.
- It is natural to consider smooth functions on M as 0-forms, so $\Lambda^0 M = C^\infty(M)$.
- In local coordinates $(x_1, \ldots, x_n)$ on a domain $O \subset M$, any differential k-form $\omega \in \Lambda^k M$ can be uniquely decomposed as follows:

$$\omega_x = \sum_{i_1 < \cdots < i_k} a_{i_1 \ldots i_k}(x) dx_{i_1} \wedge \ldots \wedge dx_{i_k}, \qquad x \in O, \quad a_{i_1 \ldots i_k} \in C^\infty(O). \tag{3}$$

- Any smooth mapping $F : M \to N$ induces a mapping of differential forms $\widehat{F} : \Lambda^k N \to \Lambda^k M$ in the following way: given a differential k-form $\omega \in \Lambda^k N$, the k-form $\widehat{F}\omega \in \Lambda^k M$ is defined as

$$(\widehat{F}\omega)_q(v_1, \ldots, v_k) = \omega_{F(q)}(F_* v_1, \ldots, F_* v_k), \qquad q \in M, \ v_i \in T_q M.$$

- For 0-forms, pull-back is a substitution of variables:

$$\widehat{F} a(q) = a \circ F(q), \qquad a \in C^\infty(M), \quad q \in M.$$

- The pull-back $\widehat{F}$ is linear w.r.t. forms and preserves the exterior product:

$$\widehat{F}(\omega_1 \wedge \omega_2) = \widehat{F}\omega_1 \wedge \widehat{F}\omega_2.$$

Exercise 2

Prove the composition law for pull-back of differential forms:

$$\widehat{F_2 \circ F_1} = \widehat{F_1} \circ \widehat{F_2}, \tag{4}$$

where $F_1 : M_1 \to M_2$ and $F_2 : M_2 \to M_3$ are smooth mappings.

- Now we can define the integral of a k-form over an oriented k-dimensional surface.
- Let $\Pi \subset \mathbb{R}^k$ be a k-dimensional open oriented domain and $\Phi : \Pi \to \Phi(\Pi) \subset M$ a diffeomorphism.
- Then the *integral* of a k-form $\omega \in \Lambda^k M$ over the k-dimensional oriented surface $\Phi(\Pi)$ is defined as follows:

$$\int_{\Phi(\Pi)} \omega \stackrel{\text{def}}{=} \int_{\Pi} \widehat{\Phi}\omega,$$

it remains only to define the integral over Π in the right-hand side.
- Since $\widehat{\Phi}\omega \in \Lambda^k\mathbb{R}^k$ is a k-form on $\mathbb{R}^k$, it is expressed via the standard volume form $dx_1 \wedge \ldots \wedge dx_k \in \Lambda^k\mathbb{R}^k$:

$$(\widehat{\Phi}\omega)_x = a(x)\, dx_1 \wedge \cdots \wedge dx_k, \qquad x \in \Pi.$$

- We set

$$\int_{\Pi} \widehat{\Phi}\omega \stackrel{\text{def}}{=} \int_{\Pi} a(x)\, dx_1 \ldots dx_k,$$

a usual multiple integral.

- The integral $\int_{\Phi(\Pi)} \omega$ is defined correctly with respect to orientation-preserving reparametrizations of the surface $\Phi(\Pi)$.
- Although, if a parametrization changes orientation, then the integral changes sign.
- The notion of integral is extended to arbitrary submanifolds as follows.
- Let $N \subset M$ be a k-dimensional submanifold and let $\omega \in \Lambda^k M$.
- Consider a covering of N by coordinate neighborhoods $O_i \subset M$:

$$N = \bigcup_i (N \cap O_i).$$

- Take a partition of unity subordinated to this covering:

$$\alpha_i \in C^\infty(M), \quad \operatorname{supp} \alpha_i \subset O_i, \quad 0 \le \alpha_i \le 1,$$
$$\sum_i \alpha_i \equiv 1.$$

- Then

$$\int_N \omega \overset{\text{def}}{=} \sum_i \int_{N \cap O_i} \alpha_i \omega.$$

- The integral thus defined does not depend upon the choice of partition of unity.

Exterior differential

- Exterior differential of a function (i.e., a 0-form) is a 1-form: if $a \in C^\infty(M) = \Lambda^0 M$, then its differential $d_q a \in T_q^* M$ is the functional (directional derivative)
$$\langle d_q a, v \rangle = va, \qquad v \in T_q M, \tag{5}$$
so $da \in \Lambda^1 M$.
- By the Newton-Leibniz formula, if $\gamma \subset M$ is a smooth oriented curve starting at a point $q_0 \in M$ and terminating at $q_1 \in M$, then
$$\int_\gamma da = a(q_1) - a(q_0).$$
- The right-hand side can be considered as the integral of the function a over the oriented boundary of the curve: $\partial\gamma = q_1 - q_0$, thus
$$\int_\gamma da = \int_{\partial\gamma} a. \tag{6}$$

- In the exposition above, Newton-Leibniz formula (6) comes as a consequence of definition (5) of differential of a function. But one can go the reverse way: if we postulate Newton-Leibniz formula (6) for any smooth curve $\gamma \subset M$ and pass to the limit $q_1 \to q_0$, we necessarily obtain definition (5) of differential of a function.
- Such approach can be realized for higher order differential forms as well.
- Let $\omega \in \Lambda^k M$. We define the *exterior differential*

  $$d\omega \in \Lambda^{k+1} M$$

  as the differential $(k+1)$-form for which Stokes formula holds:

  $$\int_N d\omega = \int_{\partial N} \omega \qquad (7)$$

  for $(k+1)$-dimensional submanifolds with boundary $N \subset M$ (for simplicity, one can take here N equal to a diffeomorphic image of a $(k+1)$-dimensional polytope).
- The boundary ∂N is oriented by a frame of tangent vectors $e_1, \dots e_k \in T_q(\partial N)$ in such a way that the frame $e_{\text{norm}}, e_1, \dots, e_k \in T_q N$ define a positive orientation of N, where e_{norm} is the outward normal vector to N at q.

- The existence of a form $d\omega$ that satisfies Stokes formula (7) comes from the fact that the mapping $N \mapsto \int_{\partial N} \omega$ is additive w.r.t. domain: if $N = N_1 \cup N_2$, $N_1 \cap N_2 = \partial N_1 \cap \partial N_2$, then

$$\int_{\partial N} \omega = \int_{\partial N_1} \omega + \int_{\partial N_2} \omega$$

  (notice that orientation of the boundaries is coordinated: ∂N_1 and ∂N_2 have mutually opposite orientations at points of their intersection).
- Thus the integral $\int_{\partial N} \omega$ is a kind of measure w.r.t. N, and one can recover $(d\omega)_q$ passing to limit in (7) as the submanifold N contracts to a point q.

- We recall some basic properties of exterior differential.
- First of all, it is obvious from the Stokes formula that $d : \Lambda^k M \to \Lambda^{k+1} M$ is a linear operator.
- Further, if $F : M \to N$ is a diffeomorphism, then

$$d\widehat{F}\omega = \widehat{F}d\omega, \qquad \omega \in \Lambda^k N. \tag{8}$$

- Indeed, if $W \subset M$, then

$$\int_{F(W)} \omega = \int_W \widehat{F}\omega, \qquad \omega \in \Lambda^k N,$$

thus

$$\begin{aligned} \int_W d\widehat{F}\omega &= \int_{\partial W} \widehat{F}\omega = \int_{F(\partial W)} \omega = \int_{\partial F(W)} \omega = \int_{F(W)} d\omega \\ &= \int_W \widehat{F}d\omega, \end{aligned}$$

and equality (8) follows.

- Another basic property of exterior differential is given by the equality

$$d \circ d = 0,$$

which follows since $\partial(\partial N) = \emptyset$ for any submanifold with boundary $N \subset M$.

- Exterior differential is an antiderivation:

$$d(\omega_1 \wedge \omega_2) = (d\omega_1) \wedge \omega_2 + (-1)^{k_1} \omega_1 \wedge d\omega_2, \qquad \omega_i \in \Lambda^{k_i} M,$$

this equality is dual to the formula of boundary $\partial(N_1 \times N_2)$.

- In local coordinates exterior differential is computed as follows: if

$$\omega = \sum_{i_1 < \dots < i_k} a_{i_1 \dots i_k} dx_{i_1} \wedge \dots \wedge dx_{i_k}, \qquad a_{i_1 \dots i_k} \in C^\infty,$$

then

$$d\omega = \sum_{i_1 < \dots < i_k} (da_{i_1 \dots i_k}) \wedge dx_{i_1} \wedge \dots \wedge dx_{i_k},$$

this formula is forced by above properties of differential forms.