Elements of Chronological Calculus-2: Chronological Exponential

(Lecture 5)

Yuri Sachkov

Program Systems Institute
Russian Academy of Sciences
Pereslavl-Zalessky, Russia
yusachkov@gmail.com

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Reminder: Plan of previous lecture

1. Points, Diffeomorphisms, and Vector Fields
2. Seminorms and $C^\infty(M)$-Topology
3. Families of Functionals and Operators

Plan of this lecture

1. ODEs with discontinuous right-hand side
2. Definition of the right chronological exponential
3. Formal series expansion
4. Estimates and convergence of the series
5. Left chronological exponential
6. Uniqueness for functional and operator ODEs
7. Autonomous vector fields

ODEs with discontinuous right-hand side

- We consider a *nonautonomous ordinary differential equation* of the form

$$\dot{q} = V_t(q), \qquad q(0) = q_0, \tag{1}$$

where V_t is a nonautonomous vector field on M, and study the flow determined by this field.

- We denote by $\dot{q}$ the derivative $\frac{d\,q}{d\,t}$, so equation (1) reads in the expanded form as

$$\frac{d\,q(t)}{d\,t} = V_t(q(t)).$$

- To obtain local solutions to the Cauchy problem (1) on a manifold M, we reduce it to a Cauchy problem in a Euclidean space.
- Choose local coordinates $x = (x^1, \dots, x^n)$ in a neighborhood O_{q_0} of the point q_0:

$$\Phi : O_{q_0} \subset M \to O_{x_0} \subset \mathbb{R}^n, \qquad \Phi : q \mapsto x,$$
$$\Phi(q_0) = x_0.$$

- In these coordinates, the field V_t reads

$$(\Phi_* V_t)(x) = \widetilde{V}_t(x) = \sum_{i=1}^{n} v_i(t,x) \frac{\partial}{\partial x^i}, \qquad x \in O_{x_0}, \quad t \in \mathbb{R}, \tag{2}$$

and problem (1) takes the form

$$\dot{x} = \widetilde{V}_t(x), \quad x(0) = x_0, \qquad x \in O_{x_0} \subset \mathbb{R}^n. \tag{3}$$

- Since the nonautonomous vector field $V_t \in \operatorname{Vec} M$ is locally bounded, the components $v_i(t,x)$, $i = 1, \dots, n$, of its coordinate representation (2) are:
  (1) measurable and locally bounded w.r.t. t for any fixed $x \in O_{x_0}$,
  (2) smooth w.r.t. x for any fixed $t \in \mathbb{R}$,
  (3) differentiable in x with locally bounded partial derivatives:

$$\left| \frac{\partial v_i}{\partial x}(t,x) \right| \le C_{I,K}, \qquad t \in I \Subset \mathbb{R},\ x \in K \Subset O_{x_0},\ i = 1, \dots, n.$$

- By the classical Carathéodory Theorem, the Cauchy problem (3) has a unique solution, i.e., a vector-function $x(t, x_0)$, Lipschitzian w.r.t. t and smooth w.r.t. x_0, and such that:
  1. ODE (3) is satisfied for almost all t,
  2. initial condition holds: $x(0, x_0) = x_0$.
- Then the pull-back of this solution from $\mathbb{R}^n$ to M

$$q(t, q_0) = \Phi^{-1}(x(t, x_0)),$$

  is a solution to problem (1) in M.
- The mapping $q(t, q_0)$ is Lipschitzian w.r.t. t and smooth w.r.t. q_0, it satisfies almost everywhere the ODE and the initial condition in (1).
- For any $q_0 \in M$, the solution $q(t, q_0)$ to the Cauchy problem (1) can be continued to a maximal interval $t \in J_{q_0} \subset \mathbb{R}$ containing the origin and depending on q_0.
- We will assume that the solutions $q(t, q_0)$ are defined for all $q_0 \in M$ and all $t \in \mathbb{R}$, i.e., $J_{q_0} = \mathbb{R}$ for any $q_0 \in M$. Then the nonautonomous field V_t is called *complete*.
- This holds, e.g., when all the fields V_t, $t \in \mathbb{R}$, vanish outside of a common compactum in M (in this case we say that the nonautonomous vector field V_t has a *compact support*).

Definition of the right chronological exponential

- The Cauchy problem $\dot{q} = V_t(q)$, $q(0) = q_0$, rewritten as a linear equation for Lipschitzian w.r.t. t families of functionals on $C^\infty(M)$:

$$\dot{q}(t) = q(t) \circ V_t, \qquad q(0) = q_0, \tag{4}$$

is satisfied for the family of functionals

$$q(t, q_0) : C^\infty(M) \to \mathbb{R}, \qquad q_0 \in M, \quad t \in \mathbb{R}$$

constructed in the previous subsection.

- We prove later that this Cauchy problem has no other solutions.
- Thus the flow defined as

$$P^t : q_0 \mapsto q(t, q_0) \tag{5}$$

is a unique solution of the operator Cauchy problem $\dot{P}^t = P^t \circ V_t$, $P^0 = \mathrm{Id}$ (where Id is the identity operator), in the class of Lipschitzian flows on M.

- The flow P^t determined in (5) is called the *right chronological exponential* of the field V_t and is denoted as $P^t = \overrightarrow{\exp} \displaystyle\int_0^t V_\tau \, d\tau$.

Formal series expansion

- We rewrite differential equation in (4) as an integral one:

$$q(t) = q_0 + \int_0^t q(\tau) \circ V_\tau \, d\tau \tag{6}$$

then substitute this expression for $q(t)$ into the right-hand side

$$= q_0 + \int_0^t \left(q_0 + \int_0^{\tau_1} q(\tau_2) \circ V_{\tau_2} \, d\tau_2 \right) \circ V_{\tau_1} \, d\tau_1$$
$$= q_0 \circ \left(\mathrm{Id} + \int_0^t V_\tau \, dt \right) + \iint\limits_{0 \le \tau_2 \le \tau_1 \le t} q(\tau_2) \circ V_{\tau_2} \circ V_{\tau_1} \, d\tau_2 \, d\tau_1,$$

repeat this procedure iteratively, and obtain the decomposition:

$$
\begin{aligned}
q(t) = q_0 \circ \Bigg(\mathrm{Id} + \int_0^t V_\tau \, d\tau + \iint\limits_{\Delta_2(t)} V_{\tau_2} \circ V_{\tau_1} \, d\tau_2 \, d\tau_1 + \ldots + \\
\int\limits_{\Delta_n(t)} \cdots \int V_{\tau_n} \circ \cdots \circ V_{\tau_1} \, d\tau_n \, \ldots \, d\tau_1 \Bigg) + \\
\int\limits_{\Delta_{n+1}(t)} \cdots \int q(\tau_{n+1}) \circ V_{\tau_{n+1}} \circ \cdots \circ V_{\tau_1} \, d\tau_{n+1} \, \ldots \, d\tau_1 .
\end{aligned} \tag{7}
$$

- Here

$$
\Delta_n(t) = \{(\tau_1, \ldots, \tau_n) \in \mathbb{R}^n \mid 0 \leq \tau_n \leq \cdots \leq \tau_1 \leq t\}
$$

is the n-dimensional simplex.

- Purely formally passing in (7) to the limit $n \to \infty$, we obtain a formal series for the solution $q(t)$ to problem (4):

$$q_0 \circ \left(\mathrm{Id} + \sum_{n=1}^{\infty} \int \cdots \int_{\Delta_n(t)} V_{\tau_n} \circ \cdots \circ V_{\tau_1} \, d\tau_n \, \ldots \, d\tau_1 \right),$$

thus for the solution P^t to our Cauchy problem:

$$\mathrm{Id} + \sum_{n=1}^{\infty} \int \cdots \int_{\Delta_n(t)} V_{\tau_n} \circ \cdots \circ V_{\tau_1} \, d\tau_n \, \ldots \, d\tau_1. \tag{8}$$

Estimates and convergence of the series

- Unfortunately, series (8) never converge on $C^\infty(M)$ in the weak sense (if $V_t \not\equiv 0$): there always exists a smooth function on M, on which they diverge.
- Although, one can show that series (8) gives an asymptotic expansion for the chronological exponential $P^t = \overrightarrow{\exp} \displaystyle\int_0^t V_\tau \, d\tau$.
- There holds the following bound of the remainder term: denote the m-th partial sum of series (8) as $S_m(t) = \mathrm{Id} + \sum_{n=1}^{m-1} \underset{\Delta_n(t)}{\int \cdots \int} V_{\tau_n} \circ \cdots \circ V_{\tau_1} \, d\tau_n \ldots d\tau_1$, then for any $a \in C^\infty(M)$, $s \geq 0$, $K \Subset M$

$$
\begin{aligned}
&\left\| \left(\overrightarrow{\exp} \int_0^t V_\tau \, d\tau - S_m(t) \right) a \right\|_{s,K} \\
&\qquad \leq C e^{C \int_0^t \|V_\tau\|_{s,K'} \, d\tau} \frac{1}{m!} \left(\int_0^t \|V_\tau\|_{s+m-1,K'} \, d\tau \right)^m \|a\|_{s+m,K'} \qquad (9) \\
&\qquad = O(t^m), \qquad t \to 0,
\end{aligned}
$$

where $K' \Subset M$ is some compactum containing K, see the proof in [AS].

- It follows from estimate (9) that

$$\left\| \left(\overrightarrow{\exp} \int_0^t \varepsilon V_\tau \, d\tau - S_m^\varepsilon(t) \right) a \right\|_{s,K} = O(\varepsilon^m), \qquad \varepsilon \to 0,$$

where $S_m^\varepsilon(t)$ is the m-th partial sum of series (8) for the field εV_t.

- Thus we have an asymptotic series expansion:

$$\overrightarrow{\exp} \int_0^t V_\tau \, d\tau \approx \mathrm{Id} + \sum_{n=1}^{\infty} \int \cdots \int_{\Delta_n(t)} V_{\tau_n} \circ \cdots \circ V_{\tau_1} \, d\tau_n \ldots d\tau_1 . \tag{10}$$

- In the sequel we will use terms of the zeroth, first, and second orders of the series obtained:

$$\overrightarrow{\exp} \int_0^t V_\tau \, d\tau \approx \mathrm{Id} + \int_0^t V_\tau \, d\tau + \iint_{0 \le \tau_2 \le \tau_1 \le t} V_{\tau_2} \circ V_{\tau_1} \, d\tau_2 \, d\tau_1 + \cdots .$$

- We prove now that the asymptotic series converges to the chronological exponential on any normed subspace $L \subset C^\infty(M)$ where V_t is well-defined and bounded:

$$V_t L \subset L, \qquad \|V_t\| = \sup \{ \|V_t a\| \mid a \in L, \ \|a\| \leq 1 \} < \infty. \tag{11}$$

- We apply operator series (10) to any $a \in L$ and bound terms of the series obtained:

$$a + \sum_{n=1}^{\infty} \int \cdots \int_{\Delta_n(t)} V_{\tau_n} \circ \cdots \circ V_{\tau_1} \, a \, d\tau_n \, \dots \, d\tau_1 . \tag{12}$$

$$
\begin{aligned}
&\left\| \int_{\Delta_n(t)} \cdots \int V_{\tau_n} \circ \cdots \circ V_{\tau_1}\, a \, d\tau_n \, \ldots \, d\tau_1 \right\| \\
&\quad \leq \int_{0 \leq \tau_n \leq \cdots \leq \tau_1 \leq t} \cdots \int \|V_{\tau_n}\| \cdot \, \cdots \, \cdot \|V_{\tau_1}\| \, d\tau_n \, \ldots \, d\tau_1 \cdot \|a\| \\
&\quad = \int_{0 \leq \tau_{\sigma(n)} \leq \cdots \leq \tau_{\sigma(1)} \leq t} \cdots \int \|V_{\tau_n}\| \cdot \, \cdots \, \cdot \|V_{\tau_1}\| \, d\tau_n \, \ldots \, d\tau_1 \cdot \|a\| \\
&\quad = \frac{1}{n!} \int_0^t \ldots \int_0^t \|V_{\tau_n}\| \cdot \, \cdots \, \cdot \|V_{\tau_1}\| \, d\tau_n \, \ldots \, d\tau_1 \cdot \|a\| \\
&\quad = \frac{1}{n!} \left(\int_0^t \|V_\tau\| \, d\tau \right)^n \cdot \|a\|.
\end{aligned}
$$

- So series (12) is majorized by the exponential series, thus the operator series (10) converges on L.
- Series (12) can be differentiated termwise, thus it satisfies the same ODE as the function $P^t a$:

$$\dot{a}_t = V_t a_t, \qquad a_0 = a.$$

- Consequently,

$$P^t a = a + \sum_{n=1}^{\infty} \int \cdots \int_{\Delta_n(t)} V_{\tau_n} \circ \cdots \circ V_{\tau_1} \, a \, d\tau_n \ldots d\tau_1.$$

- So in the case (11) the asymptotic series converges to the chronological exponential and there holds the bound

$$\|P^t a\| \leq e^{\int_0^t \|V_\tau\| \, d\tau} \|a\|, \qquad a \in L.$$

- Moreover, one can show that the bound and convergence hold not only for locally bounded, but also for integrable on $[0, t]$ vector fields: $\displaystyle\int_0^t \|V_\tau\| \, d\tau < \infty$.

- Notice that conditions (11) are satisfied for any finite-dimensional V_t-invariant subspace $L \subset C^\infty(M)$. In particular, this is the case when $M = \mathbb{R}^n$, L is the space of linear vector fields, and V_t is a linear vector field on $\mathbb{R}^n$.
- If M, V_t, and a are real analytic, then series (12) converges for sufficiently small t.

Left chronological exponential

- Consider the inverse operator $Q^t = (P^t)^{-1}$ to the right chronological exponential $P^t = \overrightarrow{\exp} \int_0^t V_\tau \, d\tau$.
- We find an ODE for the flow Q^t by differentiation of the identity
$$P^t \circ Q^t = \mathrm{Id}\,.$$
- Leibniz rule yields $\dot{P}^t \circ Q^t + P^t \circ \dot{Q}^t = 0$, thus, in view of the ODE for the flow P^t,
$$P^t \circ V_t \circ Q^t + P^t \circ \dot{Q}^t = 0.$$
- We multiply this equality by Q^t from the left and obtain
$$V_t \circ Q^t + \dot{Q}^t = 0.$$

  That is, the flow Q^t is a solution of the Cauchy problem
$$\frac{d}{d\,t} Q^t = -V_t \circ Q^t, \qquad Q^0 = \mathrm{Id}, \tag{13}$$
  which is dual to the Cauchy problem for P^t.

- The flow Q^t is called the *left chronological exponential* and is denoted as

$$Q^t = \overleftarrow{\exp} \int_0^t (-V_\tau)\, d\tau.$$

- We find an asymptotic expansion for the left chronological exponential in the same way as for the right one, by successive substitutions into the right-hand side:

$$\begin{aligned} Q^t &= \mathrm{Id} + \int_0^t (-V_\tau) \circ Q^\tau \, d\tau \\ &= \mathrm{Id} + \int_0^t (-V_\tau)\, d\tau + \iint\limits_{\Delta_2(t)} (-V_{\tau_1}) \circ (-V_{\tau_2}) \circ Q^{\tau_2} \, d\tau_2 \, d\tau_1 = \cdots \\ &= \mathrm{Id} + \sum_{n=1}^{m-1} \int\limits_{\Delta_n(t)} \cdots \int (-V_{\tau_1}) \circ \cdots \circ (-V_{\tau_n}) \, d\tau_n \ldots d\tau_1 \\ &\quad + \int\limits_{\Delta_m(t)} \cdots \int (-V_{\tau_1}) \circ \cdots \circ (-V_{\tau_m}) \circ Q^{\tau_m} \, d\tau_m \ldots d\tau_1 . \end{aligned}$$

- For the left chronological exponential holds an estimate of the remainder term as (9) for the right one, and the series obtained is asymptotic:

$$\overleftarrow{\exp} \int_0^t (-V_\tau)\, d\tau \approx \mathrm{Id} + \sum_{n=1}^{\infty} \int \cdots \int_{\Delta_n(t)} (-V_{\tau_1}) \circ \cdots \circ (-V_{\tau_n})\, d\tau_n \ldots d\tau_1 .$$

- Notice that the reverse arrow in the left chronological exponential $\overleftarrow{\exp}$ corresponds to the reverse order of the operators $(-V_{\tau_1}) \circ \cdots \circ (-V_{\tau_n})$, $\tau_n \leq \ldots \leq \tau_1$.
- The right and left chronological exponentials satisfy the corresponding differential equations:

$$\frac{d}{d\,t}\ \overrightarrow{\exp} \int_0^t V_\tau\, d\tau = \overrightarrow{\exp} \int_0^t V_\tau\, d\tau \circ V_t,$$

$$\frac{d}{d\,t}\ \overleftarrow{\exp} \int_0^t (-V_\tau)\, d\tau = -V_t \circ \overleftarrow{\exp} \int_0^t (-V_\tau)\, d\tau.$$

The directions of arrows correlate with the direction of appearance of the operators V_t and $(-V_t)$ in the right-hand side of these ODEs.

- If the initial value is prescribed at a moment of time $t_0 \neq 0$, then the lower limit of integrals in the chronological exponentials is t_0.
- There holds the following obvious rule for composition of flows:

$$\overrightarrow{\exp} \int_{t_0}^{t_1} V_\tau \, d\tau \circ \overrightarrow{\exp} \int_{t_1}^{t_2} V_\tau \, d\tau = \overrightarrow{\exp} \int_{t_0}^{t_2} V_\tau \, d\tau.$$

- There hold the identities

$$\overrightarrow{\exp} \int_{t_0}^{t_1} V_\tau \, d\tau = \left(\overrightarrow{\exp} \int_{t_1}^{t_0} V_\tau \, d\tau \right)^{-1} = \overleftarrow{\exp} \int_{t_1}^{t_0} (-V_\tau) \, d\tau. \tag{14}$$

- We saw that equation (4) for Lipschitzian families of functionals has a solution $q(t) = q_0 \circ \overrightarrow{\exp} \int_0^t V_\tau \, d\tau$. We can prove now that this equation has no other solutions.

Proposition 1

Let V_t be a complete nonautonomous vector field on M. Then Cauchy problem (4) has a unique solution in the class of Lipschitzian families of functionals on $C^\infty(M)$.

Proof.

Let a Lipschitzian family of functionals q_t be a solution to problem (4). Then

$$\frac{d}{d\,t}\left(q_t \circ (P^t)^{-1}\right) = \frac{d}{d\,t}\left(q_t \circ Q^t\right) = q_t \circ V_t \circ Q^t - q_t \circ V_t \circ Q^t = 0,$$

thus $q_t \circ Q^t \equiv \text{const}$. But $Q_0 = \mathrm{Id}$, consequently, $q_t \circ Q^t \equiv q_0$, hence

$$q_t = q_0 \circ P^t = q_0 \circ \overrightarrow{\exp} \int_0^t V_\tau \, d\tau$$

is a unique solution of Cauchy problem (4). □

Similarly, the both operator equations $\dot{P}^t = P^t \circ V_t$ and $\dot{Q}^t = -V_t \circ Q^t$ have no other solutions in addition to the chronological exponentials.

Autonomous vector fields

- For an *autonomous vector field*

$$V_t \equiv V \in \operatorname{Vec} M,$$

the flow generated by a complete field is called the *exponential* and is denoted as e^{tV}.

- The asymptotic series for the exponential takes the form

$$e^{tV} \approx \sum_{n=0}^{\infty} \frac{t^n}{n!} V^n = \mathrm{Id} + tV + \frac{t^2}{2} V \circ V + \cdots \ ,$$

i.e, it is the standard exponential series.

- The exponential of an autonomous vector field satisfies the ODEs

$$\frac{d}{d\,t} e^{tV} = e^{tV} \circ V = V \circ e^{tV}, \qquad e^{tV}\Big|_{t=0} = \mathrm{Id}\,.$$

- We apply the asymptotic series for exponential to find the Lie bracket of autonomous vector fields $V, W \in \operatorname{Vec} M$.
- We compute the first nonconstant term in the asymptotic expansion at $t = 0$ of the curve:

$$\begin{aligned}
q(t) &= q \circ e^{tV} \circ e^{tW} \circ e^{-tV} \circ e^{-tW} \\
&= q \circ \left(\mathrm{Id} + tV + \frac{t^2}{2} V^2 + \cdots\right) \circ \left(\mathrm{Id} + tW + \frac{t^2}{2} W^2 + \cdots\right) \\
&\qquad \circ \left(\mathrm{Id} - tV + \frac{t^2}{2} V^2 + \cdots\right) \circ \left(\mathrm{Id} - tW + \frac{t^2}{2} W^2 + \cdots\right) \\
&= q \circ \left(\mathrm{Id} + t(V + W) + \frac{t^2}{2}(V^2 + 2V \circ W + W^2) + \cdots\right) \\
&\qquad \circ \left(\mathrm{Id} - t(V + W) + \frac{t^2}{2}(V^2 + 2V \circ W + W^2) + \cdots\right) \\
&= q \circ \left(\mathrm{Id} + t^2(V \circ W - W \circ V) + \cdots\right) .
\end{aligned}$$

- So the Lie bracket of the vector fields as operators (directional derivatives) in $C^\infty(M)$ is
$$[V,W] = V \circ W - W \circ V.$$
- This proves the formula in local coordinates: if
$$V = \sum_{i=1}^n a_i \frac{\partial}{\partial x_i}, \qquad W = \sum_{i=1}^n b_i \frac{\partial}{\partial x_i}, \qquad a_i,\ b_i \in C^\infty(M),$$
then
$$[V,W] = \sum_{i,j=1}^n \left(a_j \frac{\partial b_i}{\partial x_j} - b_j \frac{\partial a_i}{\partial x_j} \right) \frac{\partial}{\partial x_i} = \frac{dW}{dx} V - \frac{dV}{dx} W.$$
- Similarly,
$$\begin{aligned} q \circ e^{tV} \circ e^{sW} \circ e^{-tV} &= q \circ (\mathrm{Id} + tV + \cdots) \circ (\mathrm{Id} + sW + \cdots) \circ (\mathrm{Id} - tV + \cdots) \\ &= q \circ (\mathrm{Id} + sW + ts[V,W] + \cdots), \end{aligned}$$
and
$$q \circ [V,W] = \left. \frac{\partial^2}{\partial s \partial t} \right|_{s=t=0} q \circ e^{tV} \circ e^{sW} \circ e^{-tV}.$$