Elements of Chronological Calculus-1

(Lecture 4)

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Reminder: Plan of previous lecture

1. Smooth manifolds
2. Tangent space and tangent vector
3. Ordinary differential equations on manifolds

Plan of this lecture

1. Points, Diffeomorphisms, and Vector Fields
2. Seminorms and $C^\infty(M)$-Topology
3. Families of Functionals and Operators

Points, Diffeomorphisms, and Vector Fields

- We identify points, diffeomorphisms, and vector fields on the manifold M with functionals and operators on the algebra $C^\infty(M)$ of all smooth real-valued functions on M.
- Addition, multiplication, and product with constants are defined in the *algebra* $C^\infty(M)$, as usual, pointwise: if $a, b \in C^\infty(M)$, $q \in M$, $\alpha \in \mathbb{R}$, then

$$\begin{aligned}(a + b)(q) &= a(q) + b(q),\\ (a \cdot b)(q) &= a(q) \cdot b(q),\\ (\alpha \cdot a)(q) &= \alpha \cdot a(q).\end{aligned}$$

- Any *point* $q \in M$ defines a *linear functional*

$$\widehat{q} : C^\infty(M) \to \mathbb{R}, \qquad \widehat{q}a = a(q), \ a \in C^\infty(M).$$

- The functionals $\widehat{q}$ are homomorphisms of the algebras $C^\infty(M)$ and $\mathbb{R}$:

$$\widehat{q}(a+b) = \widehat{q}a + \widehat{q}b, \qquad a,\ b \in C^\infty(M),$$
$$\widehat{q}(a \cdot b) = (\widehat{q}a) \cdot (\widehat{q}b), \qquad a,\ b \in C^\infty(M),$$
$$\widehat{q}(\alpha \cdot a) = \alpha \cdot \widehat{q}a, \qquad \alpha \in \mathbb{R},\ a \in C^\infty(M).$$

- So to any point $q \in M$, there corresponds a nontrivial *homomorphism of algebras* $\widehat{q} : C^\infty(M) \to \mathbb{R}$. It turns out that there exists an inverse correspondence.

Proposition 1

Let $\varphi : C^\infty(M) \to \mathbb{R}$ be a nontrivial homomorphism of algebras. Then there exists a point $q \in M$ such that $\varphi = \widehat{q}$.

Proof.

A.A. Agrachev, Yu.L. Sachkov, *Control theory from the geometric viewpoint.* Springer-Verlag, 2004. □

- Not only the manifold M can be reconstructed as a set from the algebra $C^\infty(M)$. One can recover topology on M from the weak topology in the space of functionals on $C^\infty(M)$:

$$\lim_{n\to\infty} q_n = q \quad \text{if and only if} \quad \lim_{n\to\infty} \widehat{q}_n a = \widehat{q} a \quad \forall a \in C^\infty(M).$$

- Moreover, the smooth structure on M is also recovered from $C^\infty(M)$, actually, "by definition": a real function on the set $\{\widehat{q} \mid q \in M\}$ is smooth if and only if it has a form $\widehat{q} \mapsto \widehat{q}a$ for some $a \in C^\infty(M)$.
- Any *diffeomorphism* $P : M \to M$ defines an *automorphism of the algebra* $C^\infty(M)$:

$$\widehat{P} : C^\infty(M) \to C^\infty(M), \qquad \widehat{P} \in \mathrm{Aut}(C^\infty(M)),$$
$$(\widehat{P}a)(q) = a(P(q)), \qquad q \in M, \quad a \in C^\infty(M),$$

i.e., $\widehat{P}$ acts as a change of variables in a function a.

- Conversely, any automorphism of $C^\infty(M)$ has such a form.

Proposition 2

Any automorphism $A : C^\infty(M) \to C^\infty(M)$ has a form of $\widehat{P}$ for some $P \in \operatorname{Diff} M$.

Proof.

Let $A \in \operatorname{Aut}(C^\infty(M))$. Take any point $q \in M$. Then the composition

$$\widehat{q} \circ A : C^\infty(M) \to \mathbb{R}$$

is a nonzero homomorphism of algebras, thus it has the form $\widehat{q_1}$ for some $q_1 \in M$. We denote $q_1 = P(q)$ and obtain

$$\widehat{q} \circ A = \widehat{P(q)} = \widehat{q} \circ \widehat{P} \qquad \forall q \in M,$$

i.e.,

$$A = \widehat{P},$$

and P is the required diffeomorphism. □

- Now we characterize *tangent vectors* to M as *functionals* on $C^\infty(M)$.
- Tangent vectors to M are velocity vectors to curves in M, and points of M are identified with linear functionals on $C^\infty(M)$; thus we should obtain linear functionals on $C^\infty(M)$, but not homomorphisms into $\mathbb{R}$.
- To understand, which functionals on $C^\infty(M)$ correspond to tangent vectors to M, take a smooth curve $q(t)$ of points in M. Then the corresponding curve of functionals $\widehat{q}(t) = \widehat{q(t)}$ on $C^\infty(M)$ satisfies the multiplicative rule

$$\widehat{q}(t)(a \cdot b) = \widehat{q}(t)a \cdot \widehat{q}(t)b, \qquad a,\ b \in C^\infty(M).$$

- We differentiate this equality at $t = 0$ and obtain that the velocity vector to the curve of functionals

$$\xi \stackrel{\text{def}}{=} \left.\frac{d\,\widehat{q}}{d\,t}\right|_{t=0}, \qquad \xi : C^\infty(M) \to \mathbb{R},$$

satisfies the Leibniz rule:

$$\xi(ab) = \xi(a)b(q(0)) + a(q(0))\xi(b).$$

- Consequently, to each tangent vector $v \in T_q M$ we should put into correspondence a linear functional
$$\xi : C^\infty(M) \to \mathbb{R}$$
such that
$$\xi(ab) = (\xi a)b(q) + a(q)(\xi b), \qquad a,\ b \in C^\infty(M). \tag{1}$$
- But there is a linear functional $\xi = \widehat{v}$ naturally related to any tangent vector $v \in T_q M$, the directional derivative along v:
$$\widehat{v}a = \left.\frac{d}{d\,t}\right|_{t=0} a(q(t)), \qquad q(0) = q, \quad \dot{q}(0) = v,$$
and such functional satisfies Leibniz rule (1).
- Now we show that this rule characterizes exactly directional derivatives.

Proposition 3

Let $\xi : C^\infty(M) \to \mathbb{R}$ *be a linear functional that satisfies Leibniz rule* (1) *for some point* $q \in M$*. Then* $\xi = \widehat{v}$ *for some tangent vector* $v \in T_qM$*.*

Proof.

- Notice first of all that any functional ξ that meets Leibniz rule (1) is local, i.e., it depends only on values of functions in an arbitrarily small neighborhood $O_q \subset M$ of the point q:

$$\tilde{a}\big|_{O_q} = a\big|_{O_q} \quad \Rightarrow \quad \xi\tilde{a} = \xi a, \qquad a, \ \tilde{a} \in C^\infty(M).$$

- Indeed, take a cut function $b \in C^\infty(M)$ such that $b\big|_{M \setminus O_q} \equiv 1$ and $b(q) = 0$. Then $(\tilde{a} - a)b = \tilde{a} - a$, thus

$$\xi(\tilde{a} - a) = \xi((\tilde{a} - a)b) = \xi(\tilde{a} - a)\, b(q) + (\tilde{a} - a)(q)\, \xi b = 0.$$

- So the statement of the proposition is local, and we prove it in coordinates.
- Let $(x_1, \dots, x_n)$ be local coordinates on M centered at the point q. We have to prove that there exist $\alpha_1, \dots, \alpha_n \in \mathbb{R}$ such that $\xi = \sum_{i=1}^n \alpha_i \left. \frac{\partial}{\partial x_i} \right|_0$.

- First of all,
$$\xi(1)=\xi(1\cdot 1)=(\xi 1)\cdot 1+1\cdot(\xi 1)=2\xi(1),$$
thus $\xi(1)=0$. By linearity, $\xi(\text{const})=0$.
- In order to find the action of ξ on an arbitrary smooth function, we expand it by the Hadamard Lemma:
$$a(x)=a(0)+\sum_{i=1}^{n}\int_0^1\frac{\partial a}{\partial x_i}(tx)x_i\,dt=a(0)+\sum_{i=1}^{n}b_i(x)x_i,$$
where $b_i(x)=\int_0^1\frac{\partial a}{\partial x_i}(tx)\,dt$ are smooth functions.
- Now
$$\xi a=\sum_{i=1}^{n}\xi(b_ix_i)=\sum_{i=1}^{n}((\xi b_i)x_i(0)+b_i(0)(\xi x_i))=\sum_{i=1}^{n}\alpha_i\frac{\partial a}{\partial x_i}(0),$$
where we denote $\alpha_i=\xi x_i$ and make use of the equality $b_i(0)=\dfrac{\partial a}{\partial x_i}(0)$. □

- So *tangent vectors* $v \in T_qM$ can be identified with directional derivatives $\widehat{v} : C^\infty(M) \to \mathbb{R}$, i.e., *linear functionals that meet Leibniz rule* (1).
- Now we characterize *vector fields* on M. A smooth vector field on M is a family of tangent vectors $v_q \in T_qM$, $q \in M$, such that for any $a \in C^\infty(M)$ the mapping $q \mapsto v_q a$, $q \in M$, is a smooth function on M.
- To a smooth vector field $V \in \operatorname{Vec} M$ there corresponds a *linear operator*

  $$\widehat{V} : C^\infty(M) \to C^\infty(M)$$

  that satisfies the Leibniz rule

  $$\widehat{V}(ab) = (\widehat{V}a)b + a(\widehat{V}b), \qquad a,\ b \in C^\infty(M),$$

  the directional derivative (Lie derivative) along V.
- A linear operator on an algebra meeting the Leibniz rule is called a *derivation* of the algebra, so the Lie derivative $\widehat{V}$ is a derivation of the algebra $C^\infty(M)$.

- We show that the correspondence between smooth vector fields on M and derivations of the algebra $C^\infty(M)$ is invertible.

Proposition 4

Any derivation of the algebra $C^\infty(M)$ is the directional derivative along some smooth vector field on M.

Proof.

Let $D : C^\infty(M) \to C^\infty(M)$ be a derivation. Take any point $q \in M$. We show that the linear functional

$$d_q \stackrel{\text{def}}{=} \widehat{q} \circ D : C^\infty(M) \to \mathbb{R}$$

is a directional derivative at the point q, i.e., satisfies Leibniz rule (1):

$$\begin{aligned} d_q(ab) &= \widehat{q}(D(ab)) = \widehat{q}((Da)b + a(Db)) = \widehat{q}(Da)b(q) + a(q)\widehat{q}(Db) = \\ &\quad (d_q a)b(q) + a(q)(d_q b), \qquad a,\ b \in C^\infty(M). \end{aligned}$$

□

- So we can identify points $q \in M$, diffeomorphisms $P \in \operatorname{Diff} M$, and vector fields $V \in \operatorname{Vec} M$ with nontrivial homomorphisms $\widehat{q} : C^\infty(M) \to \mathbb{R}$, automorphisms $\widehat{P} : C^\infty(M) \to C^\infty(M)$, and derivations $\widehat{V} : C^\infty(M) \to C^\infty(M)$ respectively.
- For example, we can write a point $P(q)$ in the operator notation as $\widehat{q} \circ \widehat{P}$.
- Moreover, in the sequel we omit hats and write $q \circ P$. This does not cause ambiguity: if q is to the right of P, then q is a point, P a diffeomorphism, and $P(q)$ is the value of the diffeomorphism P at the point q. And if q is to the left of P, then q is a homomorphism, P an automorphism, and $q \circ P$ a homomorphism of $C^\infty(M)$.
- Similarly, $V(q) \in T_q M$ is the value of the vector field V at the point q, and $q \circ V : C^\infty(M) \to \mathbb{R}$ is the directional derivative along the vector $V(q)$.

Seminorms and $C^\infty(M)$-Topology

- We introduce seminorms and topology on the space $C^\infty(M)$.
- By Whitney's Theorem, a smooth manifold M can be properly embedded into a Euclidean space $\mathbb{R}^N$ for sufficiently large N. Denote by h_i, $i = 1, \ldots, N$, the smooth vector field on M that is the orthogonal projection from $\mathbb{R}^N$ to M of the constant basis vector field $\frac{\partial}{\partial x_i} \in \mathrm{Vec}(\mathbb{R}^N)$. So we have N vector fields $h_1, \ldots, h_N \in \mathrm{Vec}\, M$ that span the tangent space $T_q M$ at each point $q \in M$.
- We define the family of seminorms $\| \cdot \|_{s,K}$ on the space $C^\infty(M)$ in the following way:

$$\|a\|_{s,K} = \sup \left\{ |h_{i_l} \circ \cdots \circ h_{i_1} a(q)| \mid q \in K,\ 1 \leq i_1, \ldots, i_l \leq N,\ 0 \leq l \leq s \right\},$$
$$a \in C^\infty(M), \quad s \geq 0, \quad K \Subset M.$$

- This family of seminorms defines a topology on $C^\infty(M)$.

- A local base of this topology is given by the subsets

$$\left\{ a \in C^\infty(M) \mid \|a\|_{n,K_n} < \frac{1}{n} \right\}, \qquad n \in \mathbb{N},$$

where K_n, $n \in \mathbb{N}$, is a chained system of compacta that cover M:

$$K_n \subset K_{n+1}, \qquad \bigcup_{n=1}^{\infty} K_n = M.$$

- This topology on $C^\infty(M)$ does not depend on embedding of M into $\mathbb{R}^N$. It is called the *topology of uniform convergence of all derivatives on compacta*, or just *$C^\infty(M)$-topology*.
- This topology turns $C^\infty(M)$ into a Fréchet space (a complete, metrizable, locally convex topological vector space).
- A sequence of functions $a_k \in C^\infty(M)$ converges to $a \in C^\infty(M)$ as $k \to \infty$ if and only if

$$\lim_{k\to\infty} \|a_k - a\|_{s,K} = 0 \quad \forall\ s \geq 0,\ K \Subset M.$$

- For vector fields $V \in \operatorname{Vec} M$, we define the seminorms

$$\| V \|_{s,K} = \sup \left\{ \| Va \|_{s,K} \mid \| a \|_{s+1,K} = 1 \right\}, \qquad s \geq 0, \quad K \Subset M. \tag{2}$$

- One can prove that any vector field $V \in \operatorname{Vec} M$ has finite seminorms $\| V \|_{s,K}$, and that there holds an estimate of the action of a diffeomorphism $P \in \operatorname{Diff} M$ on a function $a \in C^\infty(M)$:

$$\| Pa \|_{s,K} \leq C_{s,P} \| a \|_{s,P(K)}, \qquad s \geq 0, \quad K \Subset M. \tag{3}$$

- Thus vector fields and diffeomorphisms are linear *continuous* operators on the topological vector space $C^\infty(M)$.

Families of Functionals and Operators

- In the sequel we will often consider *one-parameter families* of points, diffeomorphisms, and vector fields that satisfy various regularity properties (e.g. differentiability or absolute continuity) with respect to the parameter.
- Since we treat points as functionals, and diffeomorphisms and vector fields as operators on $C^\infty(M)$, we can introduce regularity properties for them in the weak sense, via the corresponding properties for one-parameter families of functions

$$t \mapsto a_t, \qquad a_t \in C^\infty(M), \quad t \in \mathbb{R}.$$

- So we start from definitions for families of functions.
- *Continuity* and *differentiability* of a family of functions a_t w.r.t. parameter t are defined in a standard way since $C^\infty(M)$ is a topological vector space.

- A family of functions a_t is called *measurable* w.r.t. t if the real function $t \mapsto a_t(q)$ is measurable for any $q \in M$. A measurable family a_t is called *locally integrable* if

$$\int_{t_0}^{t_1} \|a_t\|_{s,K}\, dt < \infty \qquad \forall\ s \geq 0, \quad K \Subset M, \quad t_0,\ t_1 \in \mathbb{R}.$$

- A family a_t is called *absolutely continuous* w.r.t. t if

$$a_t = a_{t_0} + \int_{t_0}^{t} b_\tau\, d\tau$$

for some locally integrable family of functions b_t.

- A family a_t is called *Lipschitzian* w.r.t. t if

$$\|a_t - a_\tau\|_{s,K} \leq C_{s,K}|t-\tau| \qquad \forall s \geq 0, \quad K \Subset M, \quad t,\ \tau \in \mathbb{R},$$

and *locally bounded* w.r.t. t if

$$\|a_t\|_{s,K} \leq C_{s,K,I}, \qquad \forall\ s \geq 0, \quad K \Subset M, \quad I \Subset \mathbb{R}, \quad t \in I,$$

where $C_{s,K}$ and $C_{s,K,I}$ are some constants depending on s, K, and I.

- Now we can define regularity properties of families of functionals and operators on $C^\infty(M)$.
- A family of linear functionals or linear operators on $C^\infty(M)$

$$t \mapsto A_t, \qquad t \in \mathbb{R},$$

has some regularity property (i.e., is *continuous*, *differentiable*, *measurable*, *locally integrable*, *absolutely continuous*, *Lipschitzian*, *locally bounded* w.r.t. t) if the family

$$t \mapsto A_t a, \qquad t \in \mathbb{R},$$

has the same property for any $a \in C^\infty(M)$.
- A locally bounded w.r.t. t family of vector fields

$$t \mapsto V_t, \qquad V_t \in \operatorname{Vec} M, \quad t \in \mathbb{R},$$

is called a *nonautonomous vector field*, or simply a *vector field*, on M.
- An absolutely continuous w.r.t. t family of diffeomorphisms

$$t \mapsto P^t, \qquad P^t \in \operatorname{Diff} M, \quad t \in \mathbb{R},$$

is called a *flow* on M.

- So, for a nonautonomous vector field V_t, the family of functions $t \mapsto V_t a$ is locally integrable for any $a \in C^\infty(M)$.
- Similarly, for a flow P^t, the family of functions $(P^t a)(q) = a(P^t(q))$ is absolutely continuous w.r.t. t for any $a \in C^\infty(M)$.
- Integrals of measurable locally integrable families, and derivatives of differentiable families are also defined in the weak sense:

$$\int_{t_0}^{t_1} A_t \, dt \ : a \mapsto \int_{t_0}^{t_1} (A_t a) \, dt, \qquad a \in C^\infty(M),$$

$$\frac{d}{d\,t} A_t \ : a \mapsto \frac{d}{d\,t}(A_t a), \qquad a \in C^\infty(M).$$

- One can show that if A_t and B_t are continuous families of operators on $C^\infty(M)$ which are differentiable at t_0, then the family $A_t \circ B_t$ is continuous, moreover, differentiable at t_0, and satisfies the Leibniz rule:

$$\left.\frac{d}{d\,t}\right|_{t_0} (A_t \circ B_t) = \left(\left.\frac{d}{d\,t}\right|_{t_0} A_t \right) \circ B_{t_0} + A_{t_0} \circ \left(\left.\frac{d}{d\,t}\right|_{t_0} B_t \right).$$

- If families A_t and B_t of operators are absolutely continuous, then the composition $A_t \circ B_t$ is absolutely continuous as well, the same is true for composition of functionals with operators.
- For an absolute continuous family of functions a_t, the family $A_t a_t$ is also absolutely continuous, and the Leibniz rule holds for it as well.