Smooth manifolds and vector fields

(Lecture 3)

Yuri Sachkov

Program Systems Institute
Russian Academy of Sciences
Pereslavl-Zalessky, Russia
yusachkov@gmail.com

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Reminder: Plan of previous lecture

1. Banach-Tarski Paradox
2. Reduction of Optimal Control Problem to Study of Attainable Sets
3. Filippov's theorem: Compactness of Attainable Sets
4. Time-Optimal Problem

Plan of this lecture

1. Smooth manifolds
2. Tangent space and tangent vector
3. Ordinary differential equations on manifolds

Smooth manifolds

"Smooth" (manifold, mapping, vector field etc.) means C^∞.

Definition 1

A subset $M \subset \mathbb{R}^n$ is called a *smooth k-dimensional submanifold* of $\mathbb{R}^n$, $k \leq n$, if any point $x \in M$ has a neighbourhood O_x in $\mathbb{R}^n$ in which M is described in one of the following ways:

(1) there exists a smooth vector-function

$$F \ : \ O_x \to \mathbb{R}^{n-k}, \qquad \operatorname{rank} \left.\frac{dF}{dx}\right|_q = n-k$$

such that

$$O_x \cap M = F^{-1}(0);$$

(2) there exists a smooth vector-function

$$f \;:\; V_0 \to \mathbb{R}^n$$

from a neighbourhood of the origin $0 \in V_0 \subset \mathbb{R}^k$ such that

$$f(0) = x, \qquad \operatorname{rank} \left.\frac{d\,f}{d\,x}\right|_0 = k,$$

$$O_x \cap M = f(V_0)$$

and $f \;:\; V_0 \to O_x \cap M$ is a homeomorphism;

(3) there exists a smooth vector-function

$$\Phi \;:\; O_x \to O_0 \subset \mathbb{R}^n$$

onto a neighbourhood of the origin $0 \in O_0 \subset \mathbb{R}^n$ such that

$$\operatorname{rank} \left.\frac{d\,\Phi}{d\,x}\right|_x = n,$$

$$\Phi(O_x \cap M) = \mathbb{R}^k \cap O_0.$$

- There are two topologically different one-dimensional manifolds: the line $\mathbb{R}^1$ and the circle S^1.
- The sphere S^2 and the torus $\mathbb{T}^2 = S^1 \times S^1$ are two-dimensional manifolds.
- The torus can be viewed as a sphere with a handle. Any compact orientable two-dimensional manifold is topologically a sphere with g handles, $g = 0, 1, 2, \dots$ is the genus of the manifold.
- Smooth manifolds appear naturally already in the basic analysis. For example, the circle S^1 and the torus $\mathbb{T}^2$ are natural domains of periodic and doubly periodic functions respectively. On the sphere S^2, it is convenient to consider restriction of homogeneous functions of 3 variables.

Abstract manifold

Definition 2

A *smooth k-dimensional manifold* M is a Hausdorff paracompact topological space endowed with a smooth structure: M is covered by a system of open subsets

$$M = \cup_\alpha O_\alpha$$

called coordinate neighbourhoods, in each of which is defined a homeomorphism

$$\Phi_\alpha \ : \ O_\alpha \to \mathbb{R}^k$$

called a local coordinate system such that all compositions

$$\Phi_\beta \circ \Phi_\alpha^{-1} \ : \Phi_\alpha(O_\alpha \cap O_\beta) \subset \mathbb{R}^k \to \Phi_\beta(O_\alpha \cap O_\beta) \subset \mathbb{R}^k$$

are diffeomorphisms, see fig. 1.

Coordinate system in smooth manifold M

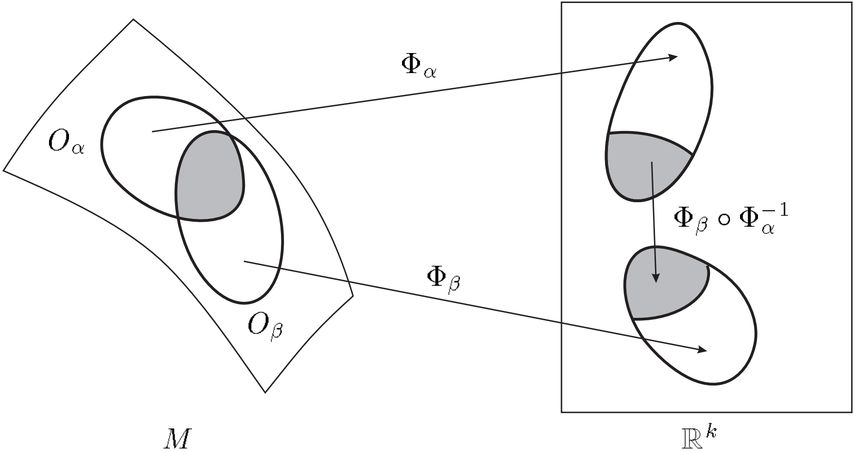

Figure: Coordinate system in smooth manifold M

- As a rule, we denote points of a smooth manifold by q, and its coordinate representation in a local coordinate system by x:

$$q \in M, \qquad \Phi_\alpha : O_\alpha \to \mathbb{R}^k, \quad x = \Phi(q) \in \mathbb{R}^k.$$

- For a smooth submanifold in $\mathbb{R}^n$, the abstract Definition 2 holds. Conversely, any connected smooth abstract manifold can be considered as a smooth submanifold in $\mathbb{R}^n$. Before precise formulation of this statement, we give two definitions.

Definition 3

Let M and N be k- and l-dimensional smooth manifolds respectively. A continuous mapping $f : M \to N$ is called *smooth* if it is smooth in coordinates. That is, let $M = \cup_\alpha O_\alpha$ and $N = \cup_\beta V_\beta$ be coverings of M and N by coordinate neighbourhoods and $\Phi_\alpha : O_\alpha \to \mathbb{R}^k$, $\Psi_\beta : V_\beta \to \mathbb{R}^l$ the corresponding coordinate mappings. Then all

$$\Psi_\beta \circ f \circ \Phi_\alpha^{-1} : \Phi_\alpha(O_\alpha \cap f^{-1}(V_\beta)) \subset \mathbb{R}^k \to \Psi_\beta(f(O_\alpha) \cap V_\beta) \subset \mathbb{R}^l$$

should be smooth.

Definition 4

A smooth manifold M is called *diffeomorphic* to a smooth manifold N if there exists a homeomorphism

$$f : M \to N$$

such that both f and its inverse f^{-1} are smooth mappings. Such mapping f is called a *diffeomorphism*.

The set of all diffeomorphisms $f : M \to M$ of a smooth manifold M is denoted by Diff M.

Definition 5

A smooth mapping $f : M \to N$ is called an *embedding* of M into N if $f : M \to f(M)$ is a diffeomorphism. A mapping $f : M \to N$ is called *proper* if $f^{-1}(K)$ is compact for any compactum $K \Subset N$.

Theorem 6 (Whitney)

Any smooth connected k-dimensional manifold can be properly embedded into $\mathbb{R}^{2k+1}$.

Tangent space of a submanifold in $\mathbb{R}^n$

Definition 7

Let M be a smooth k-dimensional submanifold of $\mathbb{R}^n$ and $x \in M$ its point. Then the *tangent space* to M at the point x is a k-dimensional linear subspace $T_x M \subset \mathbb{R}^n$ defined as follows for cases (1)–(3) of Definition 1:

$$(1) \qquad T_x M = \operatorname{Ker} \left. \frac{dF}{dx} \right|_x ,$$

$$(2) \qquad T_x M = \operatorname{Im} \left. \frac{df}{dx} \right|_0 ,$$

$$(3) \qquad T_x M = \left(\left. \frac{d\Phi}{dx} \right|_x \right)^{-1} \mathbb{R}^k .$$

Remark 1

The tangent space is a coordinate-invariant object since smooth changes of variables lead to linear transformations of the tangent space.

Tangent vector to an abstract manifold

Definition 8

Let $\gamma(\cdot)$ be a smooth curve in a smooth manifold M starting from a point $q \in M$:

$$\gamma \ : \ (-\varepsilon, \varepsilon) \to M \text{ a smooth mapping,} \qquad \gamma(0) = q.$$

The *tangent vector* $\left.\frac{d\,\gamma}{d\,t}\right|_{t=0} = \dot{\gamma}(0)$ to the curve $\gamma(\cdot)$ at the point q is the equivalence class of all smooth curves in M starting from q and having the same 1-st order Taylor polynomial as $\gamma(\cdot)$, for any coordinate system in a neighbourhood of q.

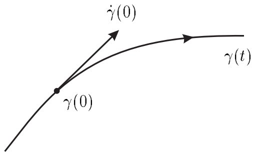

Figure: Tangent vector $\dot{\gamma}(0)$

Tangent space to an abstract manifold

Definition 9

The *tangent space* to a smooth manifold M at a point $q \in M$ is the set of all tangent vectors to all smooth curves in M starting at q:

$$T_qM = \left\{ \left.\frac{d\,\gamma}{d\,t}\right|_{t=0} \mid \gamma \ : \ (-\varepsilon, \varepsilon) \to M \text{ smooth}, \gamma(0) = q \right\}.$$

Remark 2

Let M be a smooth k-dimensional manifold and $q \in M$. Then the tangent space T_qM has a natural structure of a linear k-dimensional space.

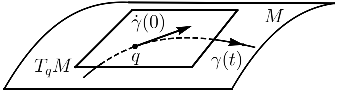

Figure: Tangent space T_qM

Dynamical system

Denote by $\mathrm{Vec}\, M$ the set of all smooth vector fields on a smooth manifold M.

Definition 10

A *smooth dynamical system*, or an *ordinary differential equation* (*ODE*), on a smooth manifold M is an equation of the form $\frac{dq}{dt} = V(q), \qquad q \in M$, or, equivalently, $\dot{q} = V(q), \qquad q \in M$, where $V(q)$ is a smooth vector field on M.

A solution to this system is a smooth mapping $\gamma : I \to M$, where $I \subset \mathbb{R}$ is an interval, such that $\frac{d\gamma}{dt} = V(\gamma(t)) \quad \forall\, t \in I$.

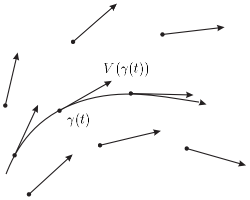

Figure: Solution to ODE $\dot{q} = V(q)$

Differential of a smooth mapping

Definition 11

Let $\Phi : M \to N$ be a smooth mapping between smooth manifolds M and N. The *differential* of Φ at a point $q \in M$ is a linear mapping

$$D_q\Phi : T_qM \to T_{\Phi(q)}N$$

defined as follows:

$$D_q\Phi\left(\left.\frac{d\,\gamma}{d\,t}\right|_{t=0}\right) = \left.\frac{d}{d\,t}\right|_{t=0}\Phi(\gamma(t)),$$

where

$$\gamma : (-\varepsilon, \varepsilon) \subset \mathbb{R} \to M, \qquad \gamma(0) = q,$$

is a smooth curve in M starting at q.

Action of diffeomorphisms on vector fields

- Let $V \in \operatorname{Vec} M$ be a vector field on M and

$$\dot{q} = V(q) \tag{1}$$

  the corresponding ODE.

- To find the action of a diffeomorphism

$$\Phi : M \to N, \qquad \Phi : q \mapsto x = \Phi(q)$$

  on the vector field $V(q)$, take a solution $q(t)$ of (1) and compute the ODE satisfied by the image $x(t) = \Phi(q(t))$:

$$\dot{x}(t) = \frac{d}{d\,t}\Phi(q(t)) = (D_q\Phi)\,\dot{q}(t) = (D_q\Phi)\,V(q(t)) = (D_{\Phi^{-1}(x)}\Phi)\,V(\Phi^{-1}(x(t))).$$

- So the required ODE is

$$\dot{x} = \left(D_{\Phi^{-1}(x)}\Phi\right) V(\Phi^{-1}(x)). \tag{2}$$

The right-hand side here is the transformed vector field on N induced by the diffeomorphism Φ:

$$(\Phi_* V)(x) \stackrel{\text{def}}{=} \left(D_{\Phi^{-1}(x)}\Phi\right) V(\Phi^{-1}(x)).$$

- The notation Φ_{*q} is used, along with $D_q\Phi$, for differential of a mapping Φ at a point q.
- In general, a smooth mapping Φ induces transformation of tangent vectors, not of vector fields.
- In order that $D\Phi$ transform vector fields to vector fields, Φ should be a diffeomorphism.

Theorem 12

Consider a smooth ODE

$$\dot{q} = V(q), \qquad q \in M \subset \mathbb{R}^n, \tag{3}$$

on a smooth submanifold M of $\mathbb{R}^n$. For any initial point $q_0 \in M$, there exists a unique solution

$$q(t, q_0), \qquad t \in (a, b), \quad a < 0 < b,$$

of equation (3) with the initial condition $q(0, q_0) = q_0$, defined on a sufficiently short interval (a, b). The mapping

$$(t, q_0) \mapsto q(t, q_0)$$

is smooth. In particular, the domain (a, b) of the solution $q(\cdot, q_0)$ can be chosen smoothly depending on q_0.

Proof.

We prove the theorem by reduction to its classical analogue in $\mathbb{R}^n$.
The statement of the theorem is local. We rectify the submanifold M in the neighbourhood of the point q_0:

$$\begin{aligned} \Phi \, : \, O_{q_0} \subset \mathbb{R}^n \to O_0 \subset \mathbb{R}^n, \\ \Phi(O_{q_0} \cap M) = \mathbb{R}^k. \end{aligned}$$

Consider the restriction $\varphi = \Phi|_M$. Then a curve $q(t)$ in M is a solution to (3) if and only if its image $x(t) = \varphi(q(t))$ in $\mathbb{R}^k$ is a solution to the induced system:

$$\dot{x} = \Phi_* V(x), \qquad x \in \mathbb{R}^k.$$

□

Theorem 13

Let $M \subset \mathbb{R}^n$ be a smooth submanifold and let

$$\dot{q} = V(q), \qquad q \in \mathbb{R}^n, \tag{4}$$

be a system of ODEs in $\mathbb{R}^n$ such that

$$q \in M \;\Rightarrow\; V(q) \in T_q M.$$

Then for any initial point $q_0 \in M$, the corresponding solution $q(t, q_0)$ to (4) with $q(0, q_0) = q_0$ belongs to M for all sufficiently small $|t|$.

Proof.

Consider the restricted vector field:

$$f = V|_M.$$

By the existence theorem for M, the system

$$\dot{q} = f(q), \qquad q \in M,$$

has a solution $q(t, q_0)$, $q(0, q_0) = q_0$, with

$$q(t, q_0) \in M \quad \text{for small } |t|. \tag{5}$$

On the other hand, the curve $q(t, q_0)$ is a solution of (4) with the same initial condition. Then inclusion (5) proves the theorem. □

Complete vector fields

Definition 14

A vector field $V \in \operatorname{Vec} M$ is called *complete*, if for all $q_0 \in M$ the solution $q(t, q_0)$ of the Cauchy problem

$$\dot{q} = V(q), \qquad q(0, q_0) = q_0 \tag{6}$$

is defined for all $t \in \mathbb{R}$.

Example 15

The vector field $V(x) = x$ is complete on $\mathbb{R}$, as well as on $\mathbb{R} \setminus \{0\}$, $(-\infty, 0)$, $(0, +\infty)$, and $\{0\}$, but not complete on other submanifolds of $\mathbb{R}$.
The vector field $V(x) = x^2$ is not complete on any submanifolds of $\mathbb{R}$ except $\{0\}$.

Proposition 1

Suppose that there exists $\varepsilon > 0$ *such that for any* $q_0 \in M$ *the solution* $q(t, q_0)$ *to Cauchy problem* (6) *is defined for* $t \in (-\varepsilon, \varepsilon)$. *Then the vector field* $V(q)$ *is complete.*

Remark 3

In this proposition it is required that there exists $\varepsilon > 0$ common for all initial points $q_0 \in M$. In general, ε may be not bounded away from zero for all $q_0 \in M$. E.g., for the vector field $V(x) = x^2$ we have $\varepsilon \to 0$ as $x_0 \to \infty$.

Proof.
Suppose that the hypothesis of the proposition is true. Then we can introduce the following family of mappings in M:

$$P^t : M \to M, \qquad t \in (-\varepsilon, \varepsilon),$$
$$P^t : q_0 \mapsto q(t, q_0).$$

$P^t(q_0)$ is the shift of a point $q_0 \in M$ along the trajectory of the vector field $V(q)$ for time t.
By Theorem 12, all mappings P^t are smooth. Moreover, the family $\{ P^t \mid t \in (-\varepsilon, \varepsilon) \}$ is a smooth family of mappings.
A very important property of this family is that it forms a local one-parameter group, i.e.,

$$P^t(P^s(q)) = P^s(P^t(q)) = P^{t+s}(q), \quad q \in M, \quad t,\ s,\ t+s \in (-\varepsilon, \varepsilon).$$

Indeed, the both curves in M:

$$t \mapsto P^t(P^s(q)) \quad \text{and} \quad t \mapsto P^{t+s}(q)$$

satisfy the ODE $\dot{q} = V(q)$ with the same initial value $P^0(P^s(q)) = P^{0+s}(q) = P^s(q)$. By uniqueness, $P^t(P^s(q)) = P^{t+s}(q)$. The equality for $P^s(P^t(q))$ is obtained by switching t and s.
So we have the following local group properties of the mappings P^t:

$$\begin{aligned} & P^t \circ P^s = P^s \circ P^t = P^{t+s}, \qquad t,\ s,\ t+s \in (-\varepsilon, \varepsilon), \\ & P^0 = \mathrm{Id}, \\ & P^{-t} \circ P^t = P^t \circ P^{-t} = \mathrm{Id}, \qquad t \in (-\varepsilon, \varepsilon), \\ & P^{-t} = \left(P^t\right)^{-1}, \qquad t \in (-\varepsilon, \varepsilon). \end{aligned}$$

In particular, all P^t are diffeomorphisms.

Now we extend the mappings P^t for all $t \in \mathbb{R}$. Any $t \in \mathbb{R}$ can be represented as

$$t = \frac{\varepsilon}{2} K + \tau, \qquad 0 \le \tau < \frac{\varepsilon}{2}, \quad K = 0, \pm 1, \pm 2, \dots .$$

We set

$$P^t \stackrel{\text{def}}{=} P^\tau \circ \underbrace{P^{\pm\varepsilon/2} \circ \dots \circ P^{\pm\varepsilon/2}}_{|K| \text{ times}}, \qquad \pm = \operatorname{sgn} t.$$

Then the curve

$$t \mapsto P^t(q_0), \qquad t \in \mathbb{R},$$

is a solution to Cauchy problem (6). □

Definition 16

For a complete vector field $V \in \operatorname{Vec} M$, the mapping

$$t \mapsto P^t, \qquad t \in \mathbb{R},$$

is called the *flow* generated by V.

Example 17

The linear vector field $V(x) = Ax$, $x \in \mathbb{R}^n$, has the flow $P^t = e^{tA} = \sum_{k=0}^{\infty} \frac{(tA)^k}{k!}$.
By this reason the flow of any complete vector field $V \in \operatorname{Vec} M$ is denoted as $P^t = e^{tV}$.

Remark 4

It is useful to imagine a vector field $V \in \operatorname{Vec} M$ as a field of velocity vectors of a moving liquid in M. Then the flow P^t takes any particle of the liquid from a position $q \in M$ and transfers it for a time $t \in \mathbb{R}$ to the position $P^t(q) \in M$.

Proposition 2

Let $K \subset M$ be a compact subset, and let $V \in \operatorname{Vec} M$. Then there exists $\varepsilon_K > 0$ such that for all $q_0 \in K$ the solution $q(t, q_0)$ to Cauchy problem (6) is defined for all $t \in (-\varepsilon_K, \varepsilon_K)$.

Proof.

By Theorem 12, domain of the solution $q(t, q_0)$ can be chosen continuously depending on q_0. The diameter of this domain has a positive infimum $2\varepsilon_K$ for q_0 in the compact set K. □

Corollary 18

If a smooth manifold M is compact, then any vector field $V \in \operatorname{Vec} M$ is complete.

Corollary 19

Suppose that a vector field $V \in \operatorname{Vec} M$ *has a compact support:*

$$\operatorname{supp} V \stackrel{\text{def}}{=} \overline{\{ q \in M \mid V(q) \neq 0 \}} \text{ is compact.}$$

Then V *is complete.*

Proof.

Indeed, by Proposition 2, there exists $\varepsilon > 0$ such that all trajectories of V starting in $\operatorname{supp} V$ are defined for $t \in (-\varepsilon, \varepsilon)$. But $V|_{M \setminus \operatorname{supp} V} = 0$, and all trajectories of V starting outside of $\operatorname{supp} V$ are constant, thus defined for all $t \in \mathbb{R}$. By Proposition 1, the vector field V is complete. □

Remark 5

If we are interested in the behaviour of (trajectories of) a vector field $V \in \operatorname{Vec} M$ in a compact subset $K \subset M$, we can suppose that V is complete. Indeed, take an open neighbourhood O_K of K with the compact closure $\overline{O_K}$. We can find a function $a \in C^\infty(M)$ such that

$$a(q) = \begin{cases} 1, & q \in K, \\ 0, & q \in M \setminus O_K. \end{cases}$$

Then the vector field $a(q)V(q) \in \operatorname{Vec} M$ is complete since it has a compact support. On the other hand, in K the vector fields $a(q)V(q)$ and $V(q)$ coincide, thus have the same trajectories.