Existence of optimal control

(Lecture 2)

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«Elements of Optimal Control»

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Reminder: Plan of previous lecture

1. Optimal control problem statement
2. Lebesgue measurable sets and functions
3. Lebesgue integral
4. Carathéodory ODEs

Plan of this lecture

1. Banach-Tarski Paradox
2. Reduction of Optimal Control Problem to Study of Attainable Sets
3. Filippov's theorem: Compactness of Attainable Sets
4. Time-Optimal Problem

Banach-Tarski Paradox

Theorem

Let $B, B' \subset \mathbb{R}^3$ be balls of different radii. Then there exist decompositions

$$B = X_1 \sqcup \cdots \sqcup X_n, \qquad B' = X'_1 \sqcup \cdots \sqcup X'_n$$

such that

$$\exists f_i \in \mathrm{SE}(3) \ : \ f_i(X_i) = X'_i, \qquad i = 1, \ldots, n.$$

- Sets X_i, X'_i are not measurable.
- $n \geq 5$.
- X, X' can be raplaced by any bounded subsets in $\mathbb{R}^3$ with nonempty interior.
- Similar theorem for $\mathbb{R}^2$ instead of $\mathbb{R}^3$ fails.
  Reason: SE(2) is solvable, while SE(3) is not:
  $[\mathfrak{se}(3), \mathfrak{se}(3)] = \mathfrak{so}(3)$, $[\mathfrak{so}(3), \mathfrak{so}(3)] = \mathfrak{so}(3) \neq \{0\}$.

Optimal Control Problem Statement

$$\dot{q} = f_u(q), \qquad q \in M, \quad u \in U \subset \mathbb{R}^m, \tag{1}$$

$$q(0) = q_0\,, \tag{2}$$

$$q(t_1) = q_1\,, \tag{3}$$

$$J(u) = \int_0^{t_1} \varphi(q, u)dt \to \min\,. \tag{4}$$

$q = q_u(\cdot)$ — solution to Cauchy problem (1), (2) corresponding to an admissible control $u(\cdot)$.

Attainable sets

- Fix an initial point $q_0 \in M$.
- *Attainable set* of control system (1) for time $t \geq 0$ from q_0 with measurable locally bounded controls is defined as follows:

$$\mathcal{A}_{q_0}(t) = \{q_u(t) \mid u \in L^\infty([0,t],U)\}\,.$$

- Similarly, one can consider the attainable sets for time not greater than t:

$$\mathcal{A}^t_{q_0} = \bigcup_{0 \leq \tau \leq t} \mathcal{A}_{q_0}(\tau)$$

and for arbitrary nonnegative time:

$$\mathcal{A}_{q_0} = \bigcup_{0 \leq \tau < \infty} \mathcal{A}_{q_0}(\tau).$$

Extended system

- Optimal control problems on M can be reduced to the study of attainable sets of some auxiliary control systems on the extended state space

$$\widehat{M} = \mathbb{R} \times M = \{\widehat{q} = (y, q) \mid y \in \mathbb{R},\ q \in M\}.$$

- Consider the following extended control system on $\widehat{M}$:

$$\frac{d\,\widehat{q}}{d\,t} = \widehat{f}_u(\widehat{q}), \qquad \widehat{q} \in \widehat{M},\ u \in U, \tag{5}$$

with the right-hand side

$$\widehat{f}_u(\widehat{q}) = \begin{pmatrix} \varphi(q,u) \\ f_u(q) \end{pmatrix}, \qquad q \in M, \quad u \in U,$$

where φ is the integrand of the cost functional J, see (4).

- Denote by $\widehat{q}_u(t)$ the solution of the extended system (5) with the initial conditions

$$\widehat{q}_u(0) = \begin{pmatrix} y(0) \\ q(0) \end{pmatrix} = \begin{pmatrix} 0 \\ q_0 \end{pmatrix}.$$

Reduction to Study of Attainable Sets

Proposition

Let $q_{\widetilde{u}}(t)$, $t \in [0, t_1]$, be an optimal trajectory in the problem (1)–(4) with the fixed terminal time t_1. Then $\widehat{q}_{\widetilde{u}}(t_1) \in \partial \widehat{A}_{(0,q_0)}(t_1)$.

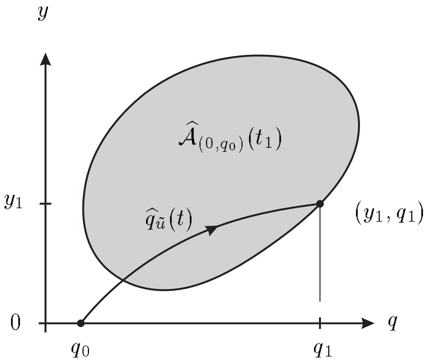

Figure: $q_{\widetilde{u}}(t)$ optimal

Proof.

- Solutions $\widehat{q}_u(t)$ of the extended system are expressed through solutions $q_u(t)$ of the original system (1) as

$$\widehat{q}_u(t) = \begin{pmatrix} J_t(u) \\ q_u(t) \end{pmatrix}, \qquad J_t(u) = \int_0^t \varphi(q_u(\tau), u(\tau))\, d\tau.$$

- Thus attainable sets of the extended system (5) have the form

$$\widehat{\mathcal{A}}_{(0,q_0)}(t) = \{(J_t(u), q_u(t)) \mid u \in L^\infty([0,t], U)\}.$$

- The set $\widehat{\mathcal{A}}_{(0,q_0)}(t_1)$ should not intersect the ray $\left\{(y, q_1) \in \widehat{M} \mid y < J_{t_1}(\widetilde{u})\right\}$.
- Indeed, suppose that there exists a point $(y, q_1) \in \widehat{\mathcal{A}}_{(0,q_0)}(t_1), \quad y < J_{t_1}(\widetilde{u})$.
- Then the trajectory of the extended system $\widehat{q}_u(t)$ that steers $(0, q_0)$ to (y, q_1):

$$\widehat{q}_u(0) = \begin{pmatrix} 0 \\ q_0 \end{pmatrix}, \qquad \widehat{q}_u(t_1) = \begin{pmatrix} y \\ q_1 \end{pmatrix},$$

gives a trajectory $q_u(t)$, $q_u(0) = q_0$, $q_u(t_1) = q_1$, with $J_{t_1}(u) = y < J_{t_1}(\widetilde{u})$, a contradiction to optimality of $\widetilde{u}$. □

Existence of optimal trajectories for problems with fixed t_1

Proposition

Let $q_1 \in \mathcal{A}_{q_0}(t_1)$. If $\widehat{\mathcal{A}}_{(0,q_0)}(t_1)$ is compact, then there exists an optimal trajectory in the problem (1)–(4) with the fixed terminal time t_1.

Proof.

- The intersection $\widehat{\mathcal{A}}_{(0,q_0)}(t_1) \cap \{(y, q_1) \in \widehat{M}\}$ is nonempty and compact.
- Denote $\widetilde{J} = \min\{y \in \mathbb{R} \mid (y, q_1) \in \widehat{\mathcal{A}}_{(0,q_0)}(t_1)\}$.
- $(\widetilde{J}, q_1) \in \widehat{\mathcal{A}}_{(0,q_0)}(t_1)$.
- There exists an admissible control $\widetilde{u}$ such that $q_{\widetilde{u}}$ steers q_0 to q_1 for time t_1 with the cost $\widetilde{J}$.
- The trajectory $q_{\widetilde{u}}$ is optimal.

□

Existence of optimal trajectories for problems with free t_1

Proposition

Let $q_1 \in \mathcal{A}_{q_0}$. Let $\widehat{\mathcal{A}}^t_{(0,q_0)}$, $t > 0$, be compact. Let there extist $\overline{u} \in L^\infty[0, \overline{t}_1]$ that steers q_0 to q_1 such that for any $u \in L^\infty[0, t_1]$ that steers q_0 to q_1:

$$t_1 > \overline{t}_1 \quad \Rightarrow \quad J(u) > J(\overline{u}).$$

Then there exists an optimal trajectory in the problem (1)–(4) with the free t_1.

Proof.

- Denote $I^t = \left\{ y \in \mathbb{R} \mid (y, q_1) \in \widehat{A}^t_{(0,q_0)} \right\}$, $J^t = \min I^t$.
- Since $q_1 \in \mathcal{A}_{q_0}(t_1)$ for some $t_1 > 0$, then $I^{t_1} \neq \emptyset$.
- Let $T = \max(t_1, \overline{t}_1)$. We have $I^T \neq \emptyset$. Denote $\widetilde{J} = J^T$.
- There exists $\widetilde{u} \in L^\infty[0, \widetilde{t}_1]$ that steers q_0 to q_1 with the cost $\widetilde{J} = J(\widetilde{u})$.
- The control $\widetilde{u}$ is optimal in the problem with the free t_1.

Compactness of attainable sets

Theorem (Filippov)

Let the space of control parameters $U \Subset \mathbb{R}^m$ be compact. Let there exist a compact $K \Subset M$ such that $f_u(q) = 0$ for $q \notin K$, $u \in U$. Moreover, let the velocity sets

$$f_U(q) = \{f_u(q) \mid u \in U\} \subset T_q M, \qquad q \in M,$$

be convex. Then the attainable sets $\mathcal{A}_{q_0}(t)$ and $\mathcal{A}^t_{q_0}$ are compact for all $q_0 \in M$, $t > 0$.

Remark

The condition of convexity of the velocity sets $f_U(q)$ is natural: the flow of the ODE

$$\dot{q} = \alpha(t) f_{u_1}(q) + (1 - \alpha(t)) f_{u_2}(q), \qquad 0 \le \alpha(t) \le 1,$$

can be approximated by flows of the systems of the form

$$\dot{q} = f_v(q), \quad \text{where} \quad v(t) \in \{u_1(t), u_2(t)\}.$$

Sketch of the proof of Filippov's Theorem: 1/5

- All nonautonomous vector fields $f_u(q)$ with admissible controls u have a common compact support, thus are complete.
- Under hypotheses of the theorem, velocities $f_u(q)$, $q \in M$, $u \in U$, are uniformly bounded, thus all trajectories $q(t)$ of control system (1) starting at q_0 are Lipschitzian with the same Lipschitz constant.
- Embed the manifold M into a Euclidean space $\mathbb{R}^N$, then the space of continuous curves $q(t)$ becomes endowed with the uniform topology of continuous mappings from $[0, t_1]$ to $\mathbb{R}^N$.
- The set of trajectories $q(t)$ of control system (1) starting at q_0 is uniformly bounded:

$$\|q(t)\| \leq C$$

and equicontinous:

$$\forall \varepsilon > 0 \ \exists \delta > 0 \ \forall q(\cdot) \ \forall |t_1 - t_2| < \delta \quad \|q(t_1) - q(t_2)\| < \varepsilon.$$

Sketch of the proof of Filippov's Theorem: 2/5

Theorem (Arzelà–Ascoli)

Consider a family of mappins $\mathcal{F} \subset C([0, t_1], M)$, where M is a complete metric space. If $\mathcal{F}$ is uniformly bounded and equicontinuous, then it is precompact:

$$\forall \{q_n\} \subset \mathcal{F} \ \exists \text{ a converging subsequence } q_{n_k} \to q \in C([0, t_1], M).$$

- Thus the set of admissible trajectories is precompact in the topology of uniform convergence.
- For any sequence $q_n(t)$ of admissible trajectories:

$$\dot{q}_n(t) = f_{u_n}(q_n(t)), \qquad 0 \leq t \leq t_1, \quad q_n(0) = q_0,$$

  there exists a uniformly converging subsequence, we denote it again by $q_n(t)$:

$$q_n(\cdot) \to q(\cdot) \text{ in } C([0, t_1], M) \text{ as } n \to \infty.$$

- Now we show that $q(t)$ is an admissible trajectory of control system (1).

Sketch of the proof of Filippov's Theorem: 3/5

- Fix a sufficiently small $\varepsilon > 0$.
- Then in local coordinates

$$\frac{1}{\varepsilon}(q_n(t+\varepsilon) - q_n(t)) = \frac{1}{\varepsilon}\int_t^{t+\varepsilon} f_{u_n}(q_n(\tau))\, d\tau$$

$$\in \operatorname{conv} \bigcup_{\tau \in [t, t+\varepsilon]} f_U(q_n(\tau)) \subset \operatorname{conv} \bigcup_{q \in O_{q(t)}(c\varepsilon)} f_U(q),$$

where c is the doubled Lipschitz constant of admissible trajectories.
- We pass to the limit $n \to \infty$ and obtain

$$\frac{1}{\varepsilon}(q(t+\varepsilon) - q(t)) \in \operatorname{conv} \bigcup_{q \in O_{q(t)}(c\varepsilon)} f_U(q).$$

- Now let $\varepsilon \to 0$. If t is a point of differentiability of $q(t)$, then

$$\dot{q}(t) \in f_U(q)$$

since $f_U(q)$ is convex.

Sketch of the proof of Filippov's Theorem: 4/5

- In order to show that $q(t)$ is an admissible trajectory of control system (1), we should find a measurable selection $u(t) \in U$ that generates $q(t)$.
- We do this via the lexicographic order on the set $U = \{(u_1, \dots, u_m)\} \subset \mathbb{R}^m$.
- The set

$$V_t = \{v \in U \mid \dot{q}(t) = f_v(q(t))\}$$

  is a compact subset of U, thus of $\mathbb{R}^m$.
- There exists a vector $v^{\min}(t) \in V_t$ minimal in the sense of lexicographic order. To find $v^{\min}(t)$, we minimize the first coordinate on V_t:

$$v_1^{\min} = \min\{ v_1 \mid v = (v_1, \dots, v_m) \in V_t \},$$

  then minimize the second coordinate on the compact set found at the first step:

$$v_2^{\min} = \min\{ v_2 \mid v = (v_1^{\min}, v_2, \dots, v_m) \in V_t \}, \quad \dots,$$
$$v_m^{\min} = \min\{ v_m \mid v = (v_1^{\min}, \dots, v_{m-1}^{\min}, v_m) \in V_t \}.$$

Sketch of the proof of Filippov's Theorem: 5/5

- The control $v^{\min}(t) = (v_1^{\min}(t), \ldots, v_m^{\min}(t))$ is measurable, thus $q(t)$ is an admissible trajectory of system (1) generated by this control.
- The proof of compactness of the attainable set $\mathcal{A}_{q_0}(t)$ is complete.
- Compactness of $\mathcal{A}_{q_0}^t$ is proved similarly. $\square$

Discussion on completeness

- In Filippov's theorem, the hypothesis of common compact support of the vector fields in the right-hand side is essential to ensure the uniform boundedness of velocities and completeness of vector fields.
- On a manifold, sufficient conditions for completeness of a vector field cannot be given in terms of boundedness of the vector field and its derivatives: a constant vector field is not complete on a bounded domain in $\mathbb{R}^n$.
- Nevertheless, one can prove compactness of attainable sets for many systems without the assumption of common compact support. If for such a system we have a priori bounds on solutions, then we can multiply its right-hand side by a cut-off function, and obtain a system with vector fields having compact support.
- We can apply Filippov's theorem to the new system. Since trajectories of the initial and new systems coincide in a domain of interest for us, we obtain a conclusion on compactness of attainable sets for the initial system.

A priori bound in $\mathbb{R}^n$

- For control systems on $M = \mathbb{R}^n$, there exist well-known sufficient conditions for completeness of vector fields.
- If the right-hand side grows at infinity not faster than a linear field, i.e.,

$$|f_u(x)| \le C(1 + |x|), \qquad x \in \mathbb{R}^n, \quad u \in U, \tag{6}$$

  for some constant C, then the nonautonomous vector fields $f_u(x)$ are complete (here $|x| = \sqrt{x_1^2 + \cdots + x_n^2}$ is the norm of a point $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$).
- These conditions provide an a priori bound for solutions: any solution $x(t)$ of the control system

$$\dot{x} = f_u(x), \qquad x \in \mathbb{R}^n, \quad u \in U, \tag{7}$$

  with the right-hand side satisfying (6) admits the bound

$$|x(t)| \le e^{2Ct} \left(|x(0)| + 1 \right), \qquad t \ge 0.$$

Compactness of attainable sets in $\mathbb{R}^n$

- Filippov's theorem plus the previous remark imply the following sufficient condition for compactness of attainable sets for systems in $\mathbb{R}^n$.

Corollary

Let system (7) *have a compact space of control parameters* $U \Subset \mathbb{R}^m$ *and convex velocity sets* $f_U(x)$, $x \in \mathbb{R}^n$.
Suppose moreover that the right-hand side of the system satisfies a sublinear bound of the form (6).
Then the attainable sets $\mathcal{A}_{x_0}(t)$ *and* $\mathcal{A}_{x_0}^t$ *are compact for all* $x_0 \in \mathbb{R}^n$, $t > 0$.

Time-optimal problem

- Given a pair of points $q_0 \in M$ and $q_1 \in \mathcal{A}_{q_0}$, the *time-optimal problem* consists in minimizing the time of motion from q_0 to q_1 via admissible controls of control system (1):

$$\min_{u} \{t_1 \mid q_u(t_1) = q_1\}. \tag{8}$$

- That is, we consider the optimal control problem with the integrand $\varphi(q, u) \equiv 1$ and free terminal time t_1.
- Reduction of optimal control problems to the study of attainable sets and Filippov's Theorem yield the following existence result.

Corollary

Under the hypotheses of Filippov's Theorem 2*, time-optimal problem* (1)*,* (8) *has a solution for any points* $q_0 \in M$*,* $q_1 \in \mathcal{A}_{q_0}$*.*