Pontryagin maximum principle for various optimal control problems *(Lecture 11)*

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«Elements of Control Theory»

Lecture course in Program Systems Institute, Pereslavl-Zalessky

27 June 2023

Reminder: Plan of previous lecture

1. Proof of the geometric statement of PMP with fixed terminal time

Plan of this lecture

1. Geometric statement of PMP for free time
2. PMP for optimal control problems
3. Statement of PMP with transversality conditions

Geometric statement of PMP for fixed time

Theorem 1 (PMP)

Let $\widetilde{u}(t)$, $t \in [0, t_1]$, be an admissible control and $\widetilde{q}(t) = q_{\widetilde{u}}(t)$ the corresponding trajectory. If $\widetilde{q}(t_1) \in \partial \mathcal{A}_{q_0}(t_1)$, then there exists a Lipschitzian curve in the cotangent bundle

$$\lambda_t \in T^*_{\widetilde{q}(t)}M, \qquad 0 \leq t \leq t_1,$$

such that

$$\lambda_t \neq 0, \tag{1}$$
$$\dot{\lambda}_t = \vec{h}_{\widetilde{u}(t)}(\lambda_t), \tag{2}$$
$$h_{\widetilde{u}(t)}(\lambda_t) = \max_{u \in U} h_u(\lambda_t) \tag{3}$$

for almost all $t \in [0, t_1]$.

Geometric statement of PMP for free time

Theorem 2

Let $\widetilde{u}(\cdot)$ be an admissible control such that $\widetilde{q}(t_1) \in \partial \left(\cup_{|t-t_1|<\varepsilon} \mathcal{A}_{q_0}(t) \right)$ for some $t_1 > 0$ and $\varepsilon \in (0, t_1)$. Then there exists a Lipschitzian curve

$$\lambda_t \in T^*_{\widetilde{q}(t)} M, \quad \lambda_t \neq 0, \qquad 0 \leq t \leq t_1,$$

such that

$$\begin{aligned} \dot{\lambda}_t &= \vec{h}_{\widetilde{u}(t)}(\lambda_t), \\ h_{\widetilde{u}(t)}(\lambda_t) &= \max_{u \in U} h_u(\lambda_t), \\ h_{\widetilde{u}(t)}(\lambda_t) &= 0 \end{aligned} \tag{4}$$

for almost all $t \in [0, t_1]$.

Remark 1

In problems with free time, there appears one more variable, the terminal time t_1. In order to eliminate it, we have one additional condition — equality (4). This condition is indeed scalar since the previous two equalities imply that $h_{\widetilde{u}(t)}(\lambda_t) = \text{const}$.

Proof of Theorem 3.

- We reduce the case of free time to the case of fixed time by extension of the control system via substitution of time. Admissible trajectories of the extended system are reparametrized admissible trajectories of the initial system (the positive direction of time on trajectories is preserved).
- Let a new time be a smooth function

$$\varphi : \mathbb{R} \to \mathbb{R}, \quad \dot{\varphi} > 0.$$

- We find an ODE for a reparametrized trajectory:

$$\frac{d}{d\,t} q_u(\varphi(t)) = \dot{\varphi}(t) f_{u(\varphi(t))}(q_u(\varphi(t))),$$

  so the required equation is

$$\dot{q} = \dot{\varphi}(t) f_{u(\varphi(t))}(q).$$

- Now consider along with the initial control system

$$\dot{q} = f_u(q), \qquad u \in U,$$

an extended system of the form

$$\dot{q} = v f_u(q), \qquad u \in U, \ |v - 1| < \delta, \tag{5}$$

where $\delta = \varepsilon / t_1 \in (0, 1)$.
- Admissible controls of the new system are

$$w(t) = (v(t), u(t)),$$

and the reference control corresponding to the control $\widetilde{u}(\cdot)$ of the initial system is

$$\tilde{w}(t) = (1, \widetilde{u}(t)).$$

- It is easy to see that since $\widetilde{q}(t_1) \in \partial \left(\cup_{|t - t_1| < \varepsilon} \mathcal{A}_{q_0}(t) \right)$, then the trajectory of the new system through the point q_0 corresponding to the control $\tilde{w}(\cdot)$ comes at the moment t_1 to the boundary of the attainable set of the new system for time t_1.
- Thus $\tilde{w}(t)$ satisfies PMP with fixed time.

- We apply the geometric statement of PMP for fixed time to the new system (5).
- The Hamiltonian for the new system is $vh_u(\lambda)$.
- Then the maximality condition reads

$$1 \cdot h_{\widetilde{u}(t)}(\lambda_t) = \max_{u \in U,\ |v-1|<\delta} vh_u(\lambda_t).$$

- We take $u = \widetilde{u}(t)$ under the maximum and obtain

$$h_{\widetilde{u}(t)}(\lambda_t) = 0,$$

  then we restrict the maximum to the set $v = 1$ and come to

$$h_{\widetilde{u}(t)}(\lambda_t) = \max_{u \in U} h_u(\lambda_t).$$

- The Hamiltonian systems along $\tilde{w}(\cdot)$ and $\widetilde{u}(\cdot)$ coincide one with another, thus the proposition follows.

□

PMP for optimal control problems

- Now we apply PMP in geometric form to optimal control problems, starting from problems with fixed time.
- For a control system

$$\dot{q} = f_u(q), \qquad q \in M, \quad u \in U, \tag{6}$$

with the boundary conditions

$$q(0) = q_0, \quad q(t_1) = q_1, \quad q_0, q_1 \in M \text{ fixed}, \tag{7}$$
$$t_1 > 0 \text{ fixed}, \tag{8}$$

and the cost functional

$$J(u) = \int_0^{t_1} \varphi(q_u(t), u(t))\, dt \tag{9}$$

we consider the optimal control problem

$$J(u) \to \min. \tag{10}$$

- We transform the problem to a geometric one.

- We extend the state space:

$$\widehat{q} = \begin{pmatrix} y \\ q \end{pmatrix} \in \mathbb{R} \times M,$$

define the extended vector field $\widehat{f_u} \in \mathrm{Vec}(\mathbb{R} \times M)$:

$$\widehat{f_u}(q) = \begin{pmatrix} \varphi(q,u) \\ f_u(q) \end{pmatrix},$$

and come to the new control system:

$$\frac{d\,\widehat{q}}{d\,t} = \widehat{f_u}(q) \quad \Leftrightarrow \quad \begin{cases} \dot{y} = \varphi(q,u), \\ \dot{q} = f_u(q) \end{cases} \tag{11}$$

with the boundary conditions

$$\widehat{q}(0) = \widehat{q}_0 = \begin{pmatrix} 0 \\ q_0 \end{pmatrix}, \qquad \widehat{q}(t_1) = \begin{pmatrix} J(u) \\ q_1 \end{pmatrix}.$$

- If a control $\widetilde{u}(\cdot)$ is optimal for problem (6)–(10), then the trajectory $\widehat{q}_{\widetilde{u}}(t)$ of the extended system (11) starting from $\widehat{q}_0$ satisfies the condition

$$\widehat{q}_{\widetilde{u}}(t_1) \in \partial \widehat{\mathcal{A}}_{\widehat{q}_0}(t_1),$$

  where $\widehat{\mathcal{A}}_{\widehat{q}_0}(t_1)$ is the attainable set of system (11) from the point $\widehat{q}_0$ for time t_1.
- So we can apply the geometric statement of PMP.
- But the geometric statement of PMP applied to the extended system (11) does not distinguish minimum and maximum of the cost $J(u)$.
- In order to have conditions valid only for minimum, we introduce a new control parameter v and consider a new system of the form

$$\begin{cases} \dot{y} = \varphi(q, u) + v, \\ \dot{q} = f_u(q), \end{cases} \qquad v \geq 0, \quad u \in U. \tag{12}$$

- Now the trajectory of system (12) corresponding to the controls $\tilde{v}(t) \equiv 0$, $\widetilde{u}(t)$, comes to the boundary of the attainable set of this system at time t_1.

- We apply the geometric statement of PMP to system (12).
- We have

$$T_{(y,q)}(\mathbb{R} \times M) = \mathbb{R} \oplus T_q M,$$
$$T^*_{(y,q)}(\mathbb{R} \times M) = \mathbb{R} \oplus T^*_q M = \{(\nu, \lambda)\}.$$

- The Hamiltonian function for system (12) has the form

$$\widehat{h}_{(v,u)}(\nu, \lambda) = \langle \lambda, f_u \rangle + \nu(\varphi + v),$$

and the Hamiltonian system of PMP is

$$\begin{cases} \dot{\nu} = \frac{\partial \widehat{h}}{\partial y} = 0, \\ \dot{y} = \varphi(q, u) + v, \\ \dot{\lambda} = \vec{h}_{\widetilde{u}(t)}(\nu, \lambda). \end{cases} \tag{13}$$

- Here $\vec{h}_u(\nu, \lambda)$ is the Hamiltonian vector field with the Hamiltonian function $h_u(\nu, \lambda) = \langle \lambda, f_u \rangle + \nu\varphi$.

- The first of equations (13) means that

$$\nu = \text{const}$$

along the reference trajectory.

- The maximality condition has the form

$$\langle \lambda_t, f_{\widetilde{u}(t)} \rangle + \nu\varphi(\widetilde{q}(t), \widetilde{u}(t)) = \max_{u \in U,\ v \geq 0} \left(\langle \lambda_t, f_u \rangle + \nu\varphi(\widetilde{q}(t), u) + \nu v \right).$$

- Since the previous maximum is attained, we have

$$\nu \leq 0,$$

thus we can set $v = 0$ in the right-hand side of the maximality condition:

$$\langle \lambda_t, f_{\widetilde{u}(t)} \rangle + \nu\varphi(\widetilde{q}(t), \widetilde{u}(t)) = \max_{u \in U} \left(\langle \lambda_t, f_u \rangle + \nu\varphi(\widetilde{q}(t), u) \right).$$

- So we proved the PMP for optimal control problems with fixed terminal time.

Theorem 3

Let $\widetilde{u}(t)$*,* $t \in [0, t_1]$*, be an optimal control for problem* (6)–(10)*:*

$$J(\widetilde{u}) = \min\{J(u) \mid q_u(t_1) = q_1\}.$$

Define a Hamiltonian function

$$h_u^\nu(\lambda) = \langle \lambda, f_u \rangle + \nu\varphi(q, u), \qquad \lambda \in T_q^* M, \quad u \in U, \quad \nu \in \mathbb{R}.$$

Then there exists a nontrivial pair:

$$(\nu, \lambda_t) \neq 0, \qquad \nu \in \mathbb{R}, \quad \lambda_t \in T_{\widetilde{q}(t)}^* M,$$

such that the following conditions hold:

$$\begin{aligned} &\dot{\lambda}_t = \vec{h}_{\widetilde{u}(t)}^\nu(\lambda_t), \\ &h_{\widetilde{u}(t)}^\nu(\lambda_t) = \max_{u \in U} h_u^\nu(\lambda_t) \quad \forall \text{ a.e. } t \in [0, t_1], \\ &\nu \leq 0. \end{aligned}$$

Remarks

(1) If we have a maximization problem instead of minimization problem (10), then the preceding inequality for ν should be reversed:

$$\nu \geq 0.$$

(2) For the problem with free time t_1: (6), (7), (9), (10), necessary optimality conditions of PMP are the same as in Theorem 3 plus one additional scalar equality $h^{\nu}_{\widetilde{u}(t)}(\lambda_t) \equiv 0$ (exercise).

- There are two distinct possibilities for the constant parameter ν in Theorem 3:

  (a) if $\nu \neq 0$, then the curve λ_t is called a *normal extremal*. Since the pair (ν, λ_t) can be multiplied by any positive number, we can normalize $\nu < 0$ and assume that $\nu = -1$ in the normal case;

  (b) if $\nu = 0$, then λ_t is an *abnormal extremal*.
- So we can always assume that $\nu = -1$ or 0.

Time-optimal problem

$$\dot{q} = f_u(q), \qquad q \in M, \quad u \in U,$$

$$q(0) = q_0, \quad q(t_1) = q_1, \quad q_0, q_1 \text{ fixed}, \qquad t_1 = \int_0^{t_1} 1\, dt \to \min .$$

Corollary 4

Let an admissible control $\widetilde{u}(t)$, $t \in [0, t_1]$, *be time-optimal. Define a Hamiltonian function* $h_u(\lambda) = \langle \lambda, f_u \rangle$, $\qquad \lambda \in T_q^* M$, $\quad u \in U$. *Then there exists a Lipschitzian curve* $\lambda_t \in T^* M$, $\quad \lambda_t \neq 0$, $\qquad t \in [0, t_1]$, *such that the following conditions hold for almost all* $t \in [0, t_1]$*:*

$$\dot{\lambda}_t = \vec{h}_{\widetilde{u}(t)}(\lambda_t),$$
$$h_{\widetilde{u}(t)}(\lambda_t) = \max_{u \in U} h_u(\lambda_t),$$
$$h_{\widetilde{u}(t)}(\lambda_t) \geq 0. \tag{14}$$

Proof of Corollary 1.

- Apply PMP for optimal control problems with free terminal time, taking $\varphi \equiv 1$.
- Then the Hamiltonian system and the maximality condition follow.
- Inequality (14) is equivalent to conditions $h_{\widetilde{u}(t)}(\lambda_t) + \nu = 0$ and $\nu \leq 0$.
- The inequality $\lambda_t \neq 0$ is obtained as follows: if $\lambda_t = 0$, then $h_{\widetilde{u}(t)}(\lambda_t) = 0$, thus $\nu = 0$.
- But the pair (ν, λ_t) must be nontrivial, consequently, $\lambda_t \neq 0$.

□

PMP with general boundary conditions

- We prove versions of Pontryagin Maximum Principle for optimal control problems in which boundary points of trajectories belong to prescribed manifolds.
- First consider the following problem:

$$\dot{q} = f_u(q), \qquad q \in M, \quad u \in U \subset \mathbb{R}^m, \tag{15}$$

$$q(0) \in N_0, \qquad q(t_1) \in N_1, \tag{16}$$

$$t_1 > 0 \text{ fixed}, \tag{17}$$

$$J(u) = \int_0^{t_1} \varphi(q(t), u(t))\, dt \to \min . \tag{18}$$

- Here N_0 and N_1 are given immersed submanifolds of the state space M.
- So the boundary points $q(0)$ and $q(t_1)$ are not fixed as before, but should belong to N_0 and N_1 respectively.

- If a trajectory $\tilde{q}(t)$ is optimal for this problem, then it is optimal as well for the problem with the fixed boundary points $\tilde{q}(0)$, $\tilde{q}(t_1)$ considered before.
- Consequently, the statement of Theorem 3 should be satisfied for $\tilde{q}(t)$.
- But now we need additional conditions that select boundary points $\tilde{q}(0) \in N_0$ and $\tilde{q}(t_1) \in N_1$.
- It is reasonable to expect that they should be determined by $(\dim N_0 + \dim N_1)$ scalar equalities.
- Such conditions can easily be formulated in the Hamiltonian framework, they are called *transversality conditions*, see (23) below.

Theorem 5

Let $\widetilde{u}(t)$, $t \in [0, t_1]$, *be an optimal control in problem* (15)–(18). *Define a family of Hamiltonians:*

$$h_u^\nu(\lambda) = \langle \lambda, f_u(q) \rangle + \nu \varphi(q, u), \qquad \lambda \in T_q^* M, \ q \in M, \quad \nu \in \mathbb{R}, \ u \in U.$$

Then there exists a Lipschitzian curve $\lambda_t \in T^*_{\widetilde{q}(t)} M$, $t \in [0, t_1]$, *and a number* $\nu \in \mathbb{R}$ *such that:*

$$\dot{\lambda}_t = \overrightarrow{h^\nu_{\widetilde{u}(t)}}(\lambda_t), \tag{19}$$

$$h^\nu_{\widetilde{u}(t)}(\lambda_t) = \max_{u \in U} h_u^\nu(\lambda_t), \tag{20}$$

$$(\lambda_t, \nu) \not\equiv (0, 0), \qquad t \in [0, t_1], \tag{21}$$

$$\nu \leq 0, \tag{22}$$

$$\lambda_0 \perp T_{\widetilde{q}(0)} N_0, \qquad \lambda_{t_1} \perp T_{\widetilde{q}(t_1)} N_1. \tag{23}$$

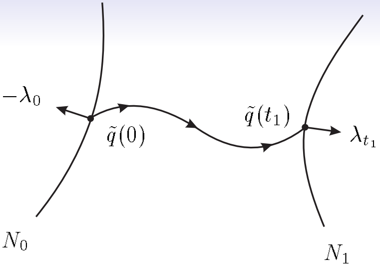

Figure: Transversality conditions (23)

- Any linear functional on a linear space acts naturally on a subspace by restriction, so transversality conditions (23) read respectively as follows:

$$\langle \lambda_0, v \rangle = 0, \quad v \in T_{\tilde{q}(0)} N_0, \qquad \langle \lambda_{t_1}, w \rangle = 0, \quad w \in T_{\tilde{q}(t_1)} N_1 .$$

- The problem with free time: (15), (16), (18), is reduced to the case of fixed t_1 as before, so for this problem holds the previous theorem with the additional condition $h^{\nu}_{\tilde{u}(t)}(\lambda_t) \equiv 0$.