Proof of Pontryagin maximum principle
(Lecture 10)

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Reminder: Plan of previous lecture

1. Linear on fibers Hamiltonians
2. Geometric statement of PMP and discussion

Plan of this lecture

1. Proof of the geometric statement of PMP with fixed terminal time

Proof of the geometric statement of PMP with fixed terminal time

- We start from two auxiliary lemmas.
- Denote the positive orthant in $\mathbb{R}^m$ as

$$\mathbb{R}^m_+ = \{(x_1, \ldots, x_m) \in \mathbb{R}^m \mid x_i \geq 0, \; i = 1, \ldots, m\}.$$

Lemma 1

Let a vector-function $F : \mathbb{R}^m \to \mathbb{R}^n$ be Lipschitzian, $F(0) = 0$, and differentiable at 0:

$$\exists \; F'_0 = \left. \frac{d\,F}{d\,x} \right|_0 .$$

Assume that

$$F'_0(\mathbb{R}^m_+) = \mathbb{R}^n.$$

Then for any neighborhood of the origin $O_0 \subset \mathbb{R}^m$

$$0 \in \operatorname{int} F(O_0 \cap \mathbb{R}^m_+).$$

Remark 1

The statement of this lemma holds if the orthant $\mathbb{R}^m_+$ is replaced by an arbitrary convex cone $C \subset \mathbb{R}^m$. In this case the proof given below works without any changes.

Proof of Lemma 1.

- Choose points $y_0, \ldots, y_n \in \mathbb{R}^n$ that generate an n-dimensional simplex centered at the origin: $\quad \frac{1}{n+1} \sum_{i=0}^{n} y_i = 0.$
- Since the mapping $F_0' : \mathbb{R}^m_+ \to \mathbb{R}^n$ is surjective and the positive orthant $\mathbb{R}^m_+$ is a convex cone, it is easy to show that restriction to the interior $F_0'|_{\operatorname{int} \mathbb{R}^m_+}$ is also surjective:

$$\exists\ v_i \in \operatorname{int} \mathbb{R}^m_+ \quad \text{such that} \quad F_0' v_i = y_i, \quad i = 0, \ldots, n.$$

- The points $y_0, \ldots, y_n$ are affinely independent in $\mathbb{R}^n$, thus their preimages $v_0, \ldots, v_n$ are also affinely independent in $\mathbb{R}^m$.

- The mean

$$v = \frac{1}{n+1} \sum_{i=0}^{n} v_i$$

  belongs to $\operatorname{int} \mathbb{R}^m_+$ and satisfies the equality

$$F_0' v = 0.$$

- Further, the subspace

$$W = \operatorname{span}\{v_i - v \mid i = 0, \ldots, n\} \subset \mathbb{R}^m$$

  is n-dimensional.
- Since $v \in \operatorname{int} \mathbb{R}^m_+$, we can find an n-dimensional ball $B_\delta \subset W$ of a sufficiently small radius δ centered at the origin such that

$$v + B_\delta \subset \operatorname{int} \mathbb{R}^m_+.$$

- Since $F_0'(v_i - v) = F_0' v_i$, then $F_0' W = \mathbb{R}^n$, i.e., the linear mapping $F_0' : W \to \mathbb{R}^n$ is invertible.

- Consider the following family of mappings:

$$G_\alpha : B_\delta \to \mathbb{R}^n, \qquad \alpha \in [0, \alpha_0),$$
$$G_\alpha(w) = \frac{1}{\alpha} F(\alpha(v+w)), \qquad \alpha > 0,$$
$$G_0(w) = F_0' w.$$

- By the hypotheses of this lemma,

$$F(x) = F_0' x + o(x), \qquad x \in \mathbb{R}^m,\ x \to 0,$$

thus

$$G_\alpha(w) = \frac{1}{\alpha}(F_0'(\alpha(v+w)) + o(\alpha(v+w))) = F_0' w + o(1),\ \alpha \to 0,\ w \in B_\delta. \quad (1)$$

- Since the mapping F is Lipschitzian, all mappings G_α are Lipschitzian with a common constant.
- Thus the family G_α is equicontinuous. Equality (1) means that uniformly in $w \in B_\delta$

$$G_\alpha \to G_0, \qquad \alpha \to 0.$$

- So the continuous mapping $G_\alpha \circ G_0^{-1} : G_0(B_\delta) \to \mathbb{R}^n$ is uniformly close to the identity mapping, hence the difference $\mathrm{Id} - G_\alpha \circ G_0^{-1}$ is uniformly close to the zero mapping.
- For any $\tilde{x} \in \mathbb{R}^n$ sufficiently close to the origin, the continuous mapping

$$\mathrm{Id} - G_\alpha \circ G_0^{-1} + \tilde{x}$$

  transforms the set $G_0(B_\delta)$ into itself.
- By Brower's fixed point theorem, this mapping has a fixed point $x \in G_0(B_\delta)$:

$$x - G_\alpha \circ G_0^{-1}(x) + \tilde{x} = x,$$

  i.e.,

$$G_\alpha \circ G_0^{-1}(x) = \tilde{x}.$$

- It follows that $\mathrm{int}\, G_\alpha(B_\delta) \ni 0$, consequently, $\mathrm{int}\, F(\alpha(v + B_\delta)) \ni 0$ for small $\alpha > 0$. Thus $\mathrm{int}\, F(O_0 \cap \mathbb{R}^m_+) \ni 0$ for a small neighborhood $O_0 \in \mathbb{R}^m$. $\square$

- Now we start to compute a convex approximation of the attainable set $\mathcal{A}_{q_0}(t_1)$ at the point $q_1 = \widetilde{q}(t_1)$ corresponding to a reference control $\widetilde{u}(\cdot)$.
- Take any admissible control $u(t)$ and express the endpoint of a trajectory via Variations Formula:

$$\begin{aligned} q_u(t_1) &= q_0 \circ \overrightarrow{\exp} \int_0^{t_1} f_{u(\tau)}\, d\tau = q_0 \circ \overrightarrow{\exp} \int_0^{t_1} f_{\widetilde{u}(\tau)} + (f_{u(\tau)} - f_{\widetilde{u}(\tau)})\, d\tau \\ &= q_0 \circ \overrightarrow{\exp} \int_0^{t_1} f_{\widetilde{u}(\tau)}\, d\tau \circ \overrightarrow{\exp} \int_0^{t_1} \left(P_\tau^{t_1}\right)_* (f_{u(\tau)} - f_{\widetilde{u}(\tau)})\, d\tau \\ &= q_1 \circ \overrightarrow{\exp} \int_0^{t_1} \left(P_\tau^{t_1}\right)_* (f_{u(\tau)} - f_{\widetilde{u}(\tau)})\, d\tau. \end{aligned}$$

- Introduce the following vector field depending on two parameters:

$$g_{\tau,u} = \left(P_\tau^{t_1}\right)_* (f_u - f_{\widetilde{u}(\tau)}), \qquad \tau \in [0, t_1], \quad u \in U. \tag{2}$$

- We showed that

$$q_u(t_1) = q_1 \circ \overrightarrow{\exp} \int_0^{t_1} g_{\tau,u(\tau)}\, d\tau. \tag{3}$$

- Notice that $g_{\tau,\widetilde{u}(\tau)} \equiv 0, \qquad \tau \in [0, t_1]$.

Lemma 2

Let $\mathcal{T} \subset [0, t_1]$ be the set of Lebesgue points of the control $\widetilde{u}(\cdot)$. If

$$\text{cone}\{g_{\tau,u}(q_1) \mid \tau \in \mathcal{T},\ u \in U\} = T_{q_1} M,$$

then $q_1 \in \text{int}\, \mathcal{A}_{q_0}(t_1)$.

Remark 2

The set $\text{cone}\{g_{\tau,u}(q_1) \mid \tau \in \mathcal{T},\ u \in U\} \subset T_{q_1} M$ is a local convex approximation of the attainable set $\mathcal{A}_{q_0}(t_1)$ at the point q_1 corresponding to a reference control $\widetilde{u}(\cdot)$.

- Recall that a point $\tau \in [0, t_1]$ is called a *Lebesgue point* of a function $u \in L^1[0, t_1]$ if $\lim\limits_{t \to \tau} \dfrac{1}{|t - \tau|} \displaystyle\int_\tau^t |u(\theta) - u(\tau)|\, d\theta = 0$.
- At Lebesgue points of u, the integral $\displaystyle\int_0^t u(\theta)\, d\theta$ is differentiable and $\dfrac{d}{d\,t}\left(\displaystyle\int_0^t u(\theta)\, d\theta\right) = u(t)$.

- The set of Lebesgue points has the full measure in the domain $[0, t_1]$.

Proof of Lemma 2.

- We can choose vectors

$$g_{\tau_i,u_i}(q_1) \in T_{q_1}M, \qquad \tau_i \in \mathcal{T}, \quad u_i \in U, \quad i = 1, \dots, k,$$

  that generate the whole tangent space as a positive convex cone:

$$\text{cone}\,\{g_{\tau_i,u_i}(q_1) \mid i = 1, \dots, k\} = T_{q_1}M,$$

  moreover, we can choose points τ_i distinct: $\tau_i \neq \tau_j$, $i \neq j$.
- Indeed, if $\tau_i = \tau_j$ for some $i \neq j$, we can find a sufficiently close Lebesgue point $\tau_j' \neq \tau_j$ such that the difference $g_{\tau_j',u_j}(q_1) - g_{\tau_j,u_j}(q_1)$ is as small as we wish.
- This is possible since for any $\tau \in \mathcal{T}$ and any $\varepsilon > 0$

$$\frac{1}{|t-\tau|}\,\text{meas}\{t' \in [\tau, t] \mid |u(t') - u(\tau)| \leq \varepsilon\} \to 1 \text{ as } t \to \tau.$$

- We suppose that $\tau_1 < \tau_2 < \cdots < \tau_k$.

- We define a family of variations of controls that follow the reference control $\widetilde{u}(\cdot)$ everywhere except neighborhoods of τ_i, and follow u_i near τ_i (such variations are called *needle-like*).
- More precisely, for any $s = (s_1, \ldots, s_k) \in \mathbb{R}_+^k$ consider a control of the form

$$u_s(t) = \begin{cases} u_i, & t \in [\tau_i, \tau_i + s_i], \\ \widetilde{u}(t), & t \notin \cup_{i=1}^k [\tau_i, \tau_i + s_i]. \end{cases} \tag{4}$$

- For small s, the segments $[\tau_i, \tau_i + s_i]$ do not overlap since $\tau_i \neq \tau_j$, $i \neq j$.
- In view of formula (3), the endpoint of the trajectory corresponding to the control constructed is expressed as follows:

$$\begin{aligned} q_{u_s}(t_1) &= q_0 \circ \overrightarrow{\exp} \int_0^{t_1} f_{u_s(t)}\, dt \\ &= q_1 \circ \overrightarrow{\exp} \int_{\tau_1}^{\tau_1+s_1} g_{t,u_1}\, dt \circ \overrightarrow{\exp} \int_{\tau_2}^{\tau_2+s_2} g_{t,u_2}\, dt \circ \cdots \\ &\quad \circ \overrightarrow{\exp} \int_{\tau_k}^{\tau_k+s_k} g_{t,u_k}\, dt. \end{aligned}$$

- The mapping
$$F \; : \; s = (s_1, \ldots, s_k) \mapsto q_{u_s}(t_1)$$
is Lipschitzian, differentiable at $s = 0$, and
$$\left. \frac{\partial F}{\partial s_i} \right|_{s=0} = g_{\tau_i, u_i}(q_1).$$
- By Lemma 1,
$$F(0) = q_1 \in \operatorname{int} F(O_0 \cap \mathbb{R}^k_+)$$
for any neighborhood $O_0 \subset \mathbb{R}^k$.
- But the curve $q_{u_s}(t)$, $t \in [0, t_1]$, is an admissible trajectory for small $s \in \mathbb{R}^k_+$, thus $F(O_0 \cap \mathbb{R}^k_+) \subset \mathcal{A}_{q_0}(t_1)$ and $q_1 \in \operatorname{int} \mathcal{A}_{q_0}(t_1)$.

□

Now we can prove the geometric statement of Pontryagin Maximum Principle:

Theorem 3 (PMP)

Let $\widetilde{u}(t)$, $t \in [0, t_1]$, be an admissible control and $\widetilde{q}(t) = q_{\widetilde{u}}(t)$ the corresponding trajectory of the control system. If $\widetilde{q}(t_1) \in \partial \mathcal{A}_{q_0}(t_1)$, then there exists a Lipschitzian curve in the cotangent bundle

$$\lambda_t \in T^*_{\widetilde{q}(t)} M, \qquad 0 \leq t \leq t_1,$$

such that

$$\lambda_t \neq 0, \tag{5}$$

$$\dot{\lambda}_t = \vec{h}_{\widetilde{u}(t)}(\lambda_t), \tag{6}$$

$$h_{\widetilde{u}(t)}(\lambda_t) = \max_{u \in U} h_u(\lambda_t) \tag{7}$$

for almost all $t \in [0, t_1]$.

Proof.

- Let the endpoint of the reference trajectory $q_1 = \widetilde{q}(t_1) \in \partial \mathcal{A}_{q_0}(t_1)$.
- By Lemma 2, the origin $0 \in T_{q_1} M$ belongs to the boundary of the convex set $\operatorname{cone}\{g_{t,u}(q_1) \mid t \in \mathcal{T},\ u \in U\}$, so this set has a hyperplane of support at the origin:

$$\exists\ \lambda_{t_1} \in T^*_{q_1} M, \quad \lambda_{t_1} \neq 0,$$

such that

$$\langle \lambda_{t_1}, g_{t,u}(q_1) \rangle \leq 0 \qquad \forall \text{ a.e. } t \in [0, t_1], \quad u \in U.$$

- Taking into account definition (2) of the field $g_{t,u}$, we rewrite this inequality as follows:

$$\langle \lambda_{t_1}, \left(P^{t_1}_{t*} f_u\right)(q_1) \rangle \leq \langle \lambda_{t_1}, \left(P^{t_1}_{t*} f_{\widetilde{u}(t)}\right)(q_1) \rangle,$$

i.e.,

$$\langle \left(P^{t_1}_{t}\right)^* \lambda_{t_1}, f_u(\widetilde{q}(t)) \rangle \leq \langle \left(P^{t_1}_{t}\right)^* \lambda_{t_1}, f_{\widetilde{u}(t)}(\widetilde{q}(t)) \rangle.$$

- The action of the flow $P_t^{t_1}$ on covectors defines the curve in the cotangent bundle:

$$\lambda_t \stackrel{\text{def}}{=} \left(P_t^{t_1}\right)^* \lambda_{t_1} \in T^*_{\widetilde{q}(t)} M, \qquad t \in [0, t_1].$$

- In terms of this covector curve, the inequality above reads

$$\langle \lambda_t, f_u(\widetilde{q}(t)) \rangle \leq \langle \lambda_t, f_{\widetilde{u}(t)}(\widetilde{q}(t)) \rangle.$$

- Thus the maximality condition of PMP holds along the reference trajectory:

$$h_u(\lambda_t) \leq h_{\widetilde{u}(t)}(\lambda_t) \qquad \forall\ u \in U \quad \forall \text{ a.e. } t \in [0, t_1].$$

- The curve λ_t is a trajectory of the nonautonomous Hamiltonian flow with the Hamiltonian function $f^*_{\widetilde{u}(t)} = h_{\widetilde{u}(t)}$:

$$\lambda_t = \lambda_{t_1} \circ \left(\overrightarrow{\exp} \int_t^{t_1} f_{\widetilde{u}(\theta)} \, d\theta \right)^* = \lambda_{t_1} \circ \overrightarrow{\exp} \int_{t_1}^{t} \vec{h}_{\widetilde{u}(\theta)} \, d\theta,$$

thus it satisfies the Hamiltonian equation of PMP

$$\dot{\lambda}_t = \vec{h}_{\widetilde{u}(t)}(\lambda_t).$$

□