Optimal Control Problem Statement. Lebesgue measure and integral

(Lecture 1)

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Plan of lecture

1. Optimal Control Problem Statement
2. Lebesgue measurable sets and functions
3. Lebesgue integral
4. Carathéodory ODEs

Optimal Control Problem Statement

Control system:

$$\dot{q} = f_u(q), \qquad q \in M, \quad u \in U \subset \mathbb{R}^m. \tag{1}$$

- M a smooth manifold
- U an arbitrary subset of $\mathbb{R}^m$
- right-hand side of (1):

$$q \mapsto f_u(q) \text{ is a smooth vector field on } M \text{ for any fixed } u \in U, \tag{2}$$

$$(q, u) \mapsto f_u(q) \text{ is a continuous mapping for } q \in M,\ u \in \overline{U}, \tag{3}$$

  and moreover, in any local coordinates on M

$$(q, u) \mapsto \frac{\partial f_u}{\partial q}(q) \text{ is a continuous mapping for } q \in M,\ u \in \overline{U}. \tag{4}$$

- *Admissible controls* are measurable locally bounded mappings

$$u \,:\, t \mapsto u(t) \in U,$$

  i.e., $u \in L_\infty([0, t_1], U)$.

- Substitute such a control $u = u(t)$ for control parameter into system (1)
- $\Rightarrow$ nonautonomous ODE $\dot{q} = f_u(q)$
- By Carathéodory's Theorem, for any point $q_0 \in M$, the Cauchy problem

$$\dot{q} = f_u(q), \qquad q(0) = q_0, \tag{5}$$

  has a unique solution $q_u(t)$.
- In order to compare admissible controls one with another on a segment $[0, t_1]$, introduce a *cost functional*:

$$J(u) = \int_0^{t_1} \varphi(q_u(t), u(t))\, dt \tag{6}$$

  with an integrand

$$\varphi : M \times U \to \mathbb{R}$$

  satisfying the same regularity assumptions as the right-hand side f, see (2)–(4).

- Take any pair of points $q_0, q_1 \in M$.
- Consider the following *optimal control problem*:

Problem

Minimize the functional J among all admissible controls $u = u(t)$, $t \in [0, t_1]$, for which the corresponding solution $q_u(t)$ of Cauchy problem (5) satisfies the boundary condition

$$q_u(t_1) = q_1 . \tag{7}$$

- This problem can also be written as follows:

$$\dot{q} = f_u(q), \qquad q \in M, \quad u \in U \subset \mathbb{R}^m, \tag{8}$$

$$q(0) = q_0, \qquad q(t_1) = q_1, \tag{9}$$

$$J(u) = \int_0^{t_1} \varphi(q(t), u(t))\, dt \to \min . \tag{10}$$

- Two types of problems: with fixed terminal time t_1 and free t_1.
- A solution u of this problem is called an *optimal control*, and the corresponding curve $q_u(t)$ is an *optimal trajectory*.

Definition of Lebesgue measure in $I = [0,1]$: H. Lebesgue, 1902 [1]

- Measure of intervals:

$$m(\emptyset) := 0, \qquad m(|a,b|) := b-a, \quad b \geq a, \quad | = [\text{ or }].$$

- Measure of elementary sets: $m'(\sqcup_{i=1}^{\infty} |a_i, b_i|) := \sum_{i=1}^{\infty} m(|a_i, b_i|)$
- Outer measure: $\mu^*(A) := \inf \{\sum_{i=1}^{\infty} m(P_i) \mid A \subset \cup_{i=1}^{\infty} P_i,\ P_i \text{ intervals}\}$.
- Lebesgue measure:
  - $A \subset I$ is called *measurable* if

$$\forall\, \varepsilon > 0\ \exists \text{ elementary set } B \subset I:\ \mu^*(A \triangle B) < \varepsilon, \qquad A \triangle B := (A \setminus B) \cup (B \setminus A).$$

  - A measurable $\Rightarrow$ *Lebesgue measure* $\mu(A) := \mu^*(A)$.

[1] A.N. Kolmogorov, S.V. Fomin, "Elements of theory of functions and functional analysis"

Properties of Lebesgue measure

1. System of measurable sets is closed w.r.t. $\cup_{i=1}^{\infty}$, $\cap_{i=1}^{\infty}$, $\setminus$, $\triangle$
2. σ-additivity: A_i measurable $\Rightarrow$ $\mu(\sqcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i)$.
3. Continuity: $A_1 \supset A_2 \supset \cdots$ measurable $\Rightarrow$ $\mu(\cap_{i=1}^{\infty} A_i) = \lim_{i\to\infty} \mu(A_i)$.
4. Open, closed sets are measurable.
5. There exist non-measurable sets (G. Vitali, 1905)
6. $A \subset \mathbb{R}$ is measurable if $\forall$ $A \cap I_n$ is measurable, $I_n = (n, n+1]$, $n \in \mathbb{Z}$,
7. $\mu(A) := \sum_{n=-\infty}^{+\infty} \mu(A \cap I_n) \in [0, +\infty]$.
8. $\mu(A) = 0$ $\Leftrightarrow$ $\forall \varepsilon > 0$ $\exists$ intervals: $\cup_{i=1}^{\infty} P_i \supset A$, $\sum_{i=1}^{\infty} m(P_i) < \varepsilon$.
9. A property P holds *almost everywhere* (a.e.) on a set X if $\exists$ $A \subset X$, $\mu(A) = 0$, s.t. P holds on $X \setminus A$.
10. $f : \mathbb{R} \to \mathbb{R}^m$ is *measurable* if $f^{-1}(O)$ is measurable for any open $O \subset \mathbb{R}^m$.

Lebesgue integral: Definition

- Let $\mu(X) < +\infty$. A function $f : X \to \mathbb{R}$ is simple if it is measurable and takes not more than countable number of values.
- Th.: A function $f(x)$ taking not more than countable number of values y_1, y_2, ... is measurable iff al sets $f^{-1}(y_n)$ are measurable.
- Th.: A function $f(x)$ is measurable iff it is a uniform limit of simple measurable functions.
- Let f be a simple measurable function taking values y_1, y_2, ... Let $A \subset X$ be measurable. Then

$$\int_A f(x)d\mu := \sum_n y_n \mu(f^{-1}(y_n)).$$

  A function f is called integrable on A if this series absolutely converges.
- A measurable function f is called *integrable* on $A \subset X$ if there exist a sequence of simple integrable on A functions $\{f_n\}$ that converges uniformly to f. Then

$$\int_A f(x)d\mu := \lim_{n\to\infty} \int_A f_n(x)d\mu.$$

Lebesgue integral: Properties

1. $\int_A 1 d\mu = \mu(A)$.
2. Linearity: $\int_A (af(x) + bg(x))d\mu = a\int_A f(x)d\mu + b\int_A g(x)d\mu$.
3. $f(x)$ bounded on $A \quad \Rightarrow \quad f(x)$ integrable on A.
4. Monotonicity: $f(x) \le g(x) \quad \Rightarrow \quad \int_A f(x)d\mu \le \int_A g(x)d\mu$.
5. $\mu(A) = 0 \quad \Rightarrow \quad \int_A f(x)d\mu = 0$.
6. $f(x) = g(x)$ a.e. $\quad \Rightarrow \quad \int_A f(x)d\mu = \int_A g(x)d\mu$.
7. $g(x)$ integrable on A and $|f(x)| \le g(x)$ a.e. $\quad \Rightarrow \quad f(x)$ integrable on A.
8. Functions f and $|f|$ are integrable or non-integrable simultaneously.
9. σ-additivity: if $A = \sqcup_n A_n$ then $\int_A f(x)d\mu = \sum_n \int_{A_n} f(x)d\mu$.
10. Absolute continuity: f in integrable on $A \quad \Rightarrow \quad \forall \varepsilon > 0\ \exists \delta > 0$ s.t. $\left|\int_E f(x)d\mu\right| < \varepsilon$ for any measurable $E \subset A$, $\mu(E) < \varepsilon$.
11. $\mu(X) = \infty$, $X = \cup_n X_n$, $X_n \subset X_{n+1}$, $\mu(X_n) < \infty \quad \Rightarrow$ $\int_X f(x)d\mu := \lim_{n\to\infty} \int_{X_n} f(x)d\mu$.

Spaces of integrable functions

$f : X \to \mathbb{R}$ measurable, $p \in [1, +\infty)$.

1. $L_p(X,\mu) = \{f \mid \|f\|_p < \infty\}$, $\|f\|_p = \left(\int_X |f(x)|^p d\mu\right)^{1/p}$.
2. $L_\infty(X,\mu) = \{f \mid \|f\|_\infty < \infty\}$, $\|f\|_\infty = \sup_{x\in X} |f(x)|$.
3. $1 \le p_1 < p_2 \le \infty \quad \Rightarrow \quad L_{p_1} \supsetneq L_{p_2}$.
4. L_p, $p \in [1, +\infty]$, are Banach spaces (complete normed spaces).
5. L_2 is a Hilbert space (complete Euclidean infinite-dimensional space), $(f,g) = \int_X f(x)g(x)d\mu$.

Carathéodory ODEs: C. Carathéodory, 1873–1950 [2]

- Carathéodory conditions: let for a domain $D \subset \mathbb{R}^{1+n}_{t,x}$
  1. $f(t,x)$ is defined and continuous in x for almost all t
  2. $f(t,x)$ is measurable in t for any x
  3. $|f(t,x)| \leq m(t)$, where $m(t)$ is Lebesgue integrable on any segment
- Carathéodory ODE: $\dot{x} = f(t,x)$, where $f : D \to \mathbb{R}^n$ satisfies conditions 1–3.
- Solution to Carathéodory ODE: $x : |a,b| \to \mathbb{R}^n$, $x(t) = x(t_0) + \int_{t_0}^{t} f(s,x(s))ds$, $t_0 \in |a,b|$.
- Existence: Solutions exist on sufficiently small segments $[t_0, t_0 + \varepsilon]$, $\varepsilon > 0$.
- Uniqueness: If $|f(t,x) - f(t,y)| \leq l(t)|x-y|$, $l(t)$ Lebesgue integrable, then a solution is unique.
- Extension: Any solution in compact D can be extended in both sides up to ∂D.

[2] A.F. Filippov, "Differential equations with discontinuous right-hand side"