Euler elasticae
(Lecture 8)

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7. The Ox Forgotten, Leaving the Man Alone:

Riding on the animal, he is at last back in his home,
Where lo! the ox is no more; the man alone sits serenely.
Though the red sun is high up in the sky, he is still quietly dreaming,
Under a straw-thatched roof are his whip and rope idly lying.

Pu-ming, "The Ten Oxherding Pictures"

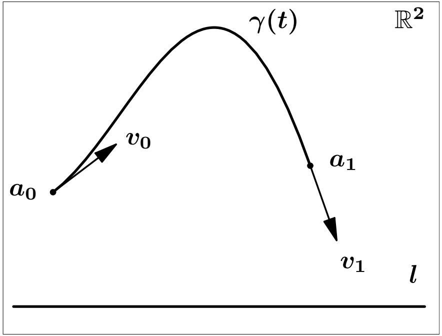
Reminder: Plan of the previous lecture

1. Proof of Pontryagin maximum principle for sub-Riemannian problems

Plan of this lecture

1. Statement and history of Euler's elastic problem
2. Optimal control problem
3. Attainable set and existence of optimal solutions
4. Extremal trajectories
5. Local and global optimality of extremal trajectories
6. Stability of elasticae
7. Global structure of exponential mapping
8. Movies
9. Conclusion

Problem statement: Stationary configurations of elastic rod



Given: $l > 0$, $a_0, a_1 \in \mathbb{R}^2$, $v_0 \in T_{a_0}\mathbb{R}^2$, $v_1 \in T_{a_1}\mathbb{R}^2$, $|v_0| = |v_1| = 1$.

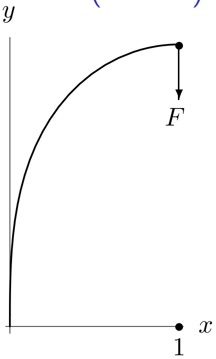
Find: $\gamma(t)$, $t \in [0, t_1]$:

$\gamma(0) = a_0$, $\gamma(t_1) = a_1$, $\dot{g}(0) = v_0$, $\dot{g}(t_1) = v_1$. $|\dot{g}(t)| \equiv 1 \quad \Rightarrow \quad t_1 = l$

Elastic energy $J = \frac{1}{2}\int_0^{t_1} k^2\, dt \to \min$, $k(t)$ — curvature of $\gamma(t)$.

1691: James (Jacob) Bernoulli

Rectangular elastica:



$$dy = \frac{x^2\,dx}{\sqrt{1-x^4}}, \qquad ds = \frac{dx}{\sqrt{1-x^4}}, \qquad x \in [0,1]$$

Integration in series

1742: Daniel Bernoulli

- Elastic energy

$$E = \text{const} \cdot \int \frac{ds}{R^2},$$

  R — radius of curvature,

- Letter to Leonhard Euler: proposal of the variational problem

$$E \to \min .$$

1744: Leonhard Euler

- "Methodus inveniendi lineas curvas maximi minimive proprietate gaudentes, sive Solutio problematis isoperimitrici latissimo sensu accepti", Lausanne, Geneva, 1744,
- Appendix "De curvis elasticis",
- *"That among all curves of the same length which not only pass through the points A and B, but are also tangent to given straight lines at these points, that curve be determined in which the value of* $\int_A^B \frac{ds}{R^2}$ *be a minimum."*

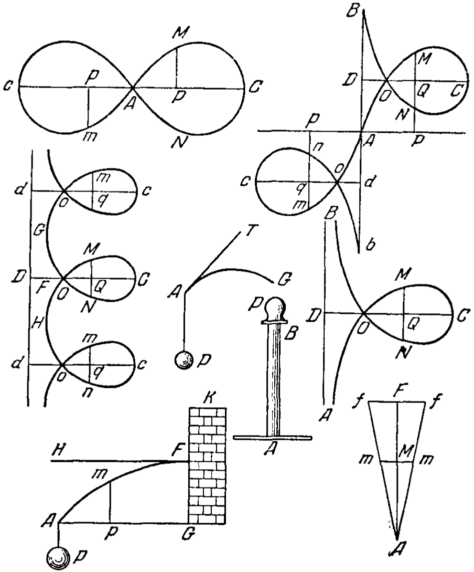
1744: Leonhard Euler

- Problem of calculus of variations,
- Euler-Lagrange equation,
- Reduction to quadratures

$$dy = \frac{(\alpha + \beta x + \gamma x^2)\, dx}{\sqrt{a^4 - (\alpha + \beta x + \gamma x^2)^2}}, \qquad ds = \frac{a^2\, dx}{\sqrt{a^4 - (\alpha + \beta x + \gamma x^2)^2}},$$

- Qualitative analysis of the integrals
- Types of solutions (elasticae)

Euler's sketches



1880: L.Saalchütz

Explicit parametrization of Euler elasticae by Jacobi's functions

1906: Max Born

- Ph.D. thesis "Stability of elastic lines in the plane and the space"
- Euler-Lagrange equation $\Rightarrow$

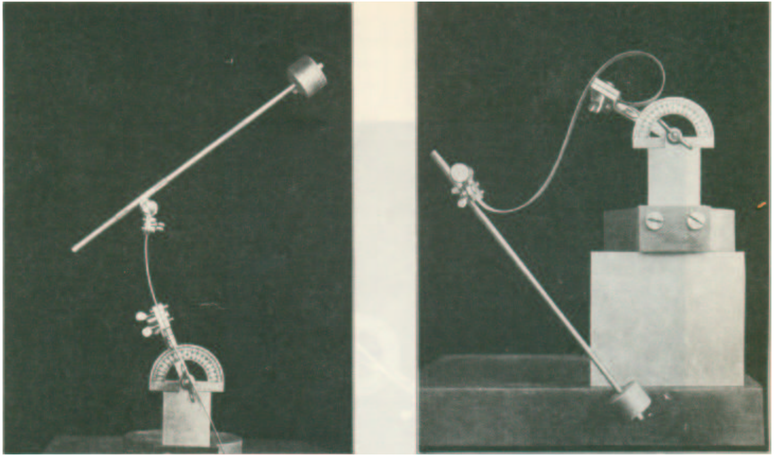
$$\dot{x} = \cos\theta, \qquad \dot{y} = \sin\theta,$$
$$A\ddot{\theta} + B\sin(\theta - \gamma) = 0, \qquad A,\ B,\ \gamma = \text{const},$$

equation of pendulum,

- elastic arc without inflection points $\Rightarrow$ stable,
- elastic arc with inflection points $\Rightarrow$ numerical investigation,
- numeric plots of elasticae.

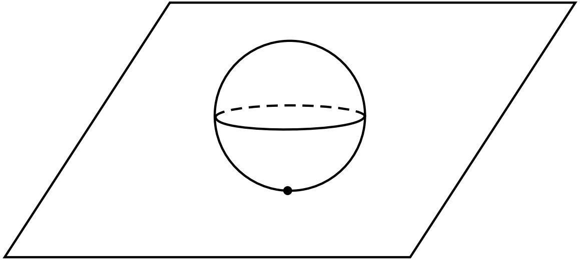
1906: Max Born

Experiments on elastic rods:



1993: Velimir Jurdjevic

Euler elasticae in the ball-plate problem

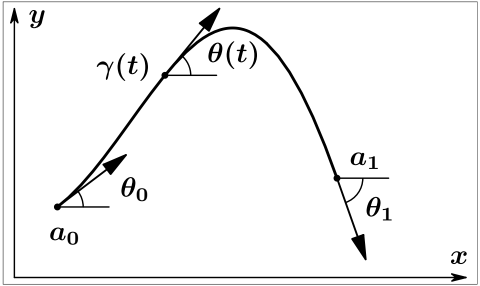


1993: Roger Brockett and L. Dai

Euler elasticae in the nilpotent sub-Riemannian problem with the growth vector (2,3,5):

$$
\begin{aligned}
&\dot q = u_1 X_1 + u_2 X_2, \qquad q \in \mathbb{R}^5, \quad u = (u_1, u_2) \in \mathbb{R}^2, \\
&q(0) = q_0, \qquad q(t_1) = q_1, \\
&I = \int_0^{t_1} \sqrt{u_1^2 + u_2^2}\, dt \to \min, \\
&[X_1, X_2] = X_3, \qquad [X_1, X_3] = X_4, \qquad [X_2, X_3] = X_5.
\end{aligned}
$$

Euler's problem: Coordinates in $\mathbb{R}^2 \times S^1$



- $(x, y) \in \mathbb{R}^2, \quad \theta \in S^1,$
- $\gamma(t) = (x(t), y(t)), \quad t \in [0, t_1],$
- $a_0 = (x_0, y_0), \quad a_1 = (x_1, y_1),$
- $v_0 = (\cos\theta_0, \sin\theta_0), \quad v_1 = (\cos\theta_1, \sin\theta_1).$

Optimal control problem

$$\begin{aligned}
\dot{x} &= \cos\theta, \\
\dot{y} &= \sin\theta, \\
\dot{\theta} &= u, \\
q &= (x, y, \theta) \in \mathbb{R}^2_{x,y} \times S^1_\theta, \qquad u \in \mathbb{R}, \\
q(0) &= q_0 = (x_0, y_0, \theta_0), \;\; q(t_1) = q_1 = (x_1, y_1, \theta_1), \;\; t_1 \text{ fixed}.
\end{aligned}$$

$$k^2 = \dot{\theta}^2 = u^2 \quad \Rightarrow \quad J = \frac{1}{2}\int_0^{t_1} u^2\, dt \to \min .$$

Admissible controls $u(t) \in L_2[0, t_1]$,
trajectories $q(t) \in AC[0, t_1]$

Left-invariant problem on the group of motions of a plane

$$\mathrm{SE}(2) = \mathbb{R}^2 \ltimes \mathrm{SO}(2) = \left\{ \begin{pmatrix} \cos\theta & -\sin\theta & x \\ \sin\theta & \cos\theta & y \\ 0 & 0 & 1 \end{pmatrix} \mid (x,y) \in \mathbb{R}^2,\ \theta \in S^1 \right\}$$

$$\dot{q} = X_1(q) + uX_2(q), \qquad q \in \mathrm{SE}(2), \quad u \in \mathbb{R}.$$

$$q(0) = q_0, \qquad q(t_1) = q_1, \qquad t_1 \text{ fixed},$$

$$J = \frac{1}{2}\int_0^{t_1} u^2\,dt \to \min,$$

Left-invariant frame on SE(2):

$$X_1(q) = qE_{13}, \quad X_2(q) = q(E_{21} - E_{12}), \quad X_3(q) = -qE_{23}$$

Continuous symmetries and normalization of conditions of the problem

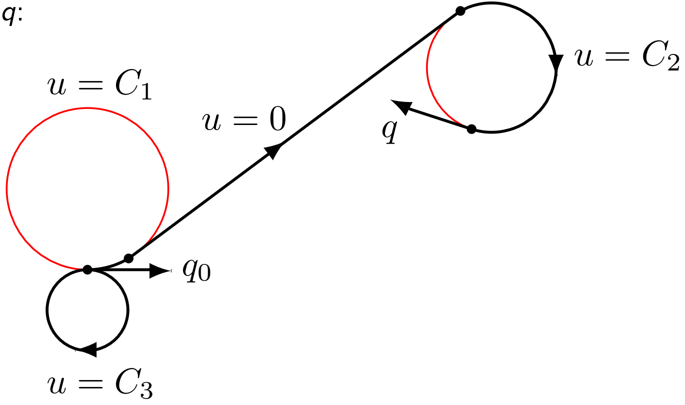
- Left translations on SE(2) $\Rightarrow$ $q_0 = \mathrm{Id} \in \mathrm{SE}(2)$:
  - Parallel translations in $\mathbb{R}^2$ $\Rightarrow$ $(x_0, y_0) = (0, 0)$
  - Rotations in $\mathbb{R}^2$ $\Rightarrow$ $\theta_0 = 0$
- Dilations in $\mathbb{R}^2$ $\Rightarrow$ $t_1 = 1$

Attainable set

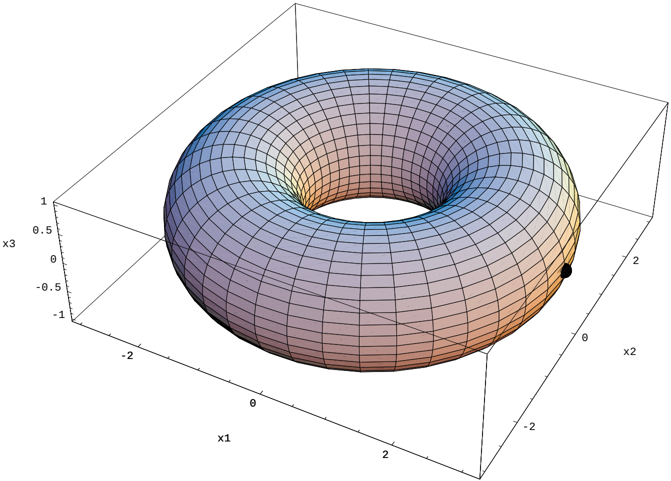
$q_0 = \mathrm{Id} = (0,0,0), \qquad t_1 = 1$

$$\mathscr{A}_{q_0}(1) = \{(x,y,\theta) \mid x^2 + y^2 < 1 \ \forall \theta \in S^1 \text{ or } (x,y,\theta) = (1,0,0)\}.$$

Steering q_0 to q:



Attainable set



In the sequel: $q_1 \in \mathscr{A}_{q_0}(t_1)$

Existence and regularity of optimal solutions

$$\dot{q} = X_1(q) + uX_2(q), \qquad q \in \mathbb{R}^2 \times S^1, \quad u \in \mathbb{R} \text{ unbounded}$$
$$q(0) = q_0, \quad q(t_1) = q_1,$$
$$J = \frac{1}{2}\int_0^{t_1} u^2\, dt \to \min,$$

- General existence theorem $\Rightarrow$ $\exists$ optimal $u(t) \in L_2$
- Compactification of the space of control parameters $\Rightarrow$ $\exists$ optimal $u(t) \in L_\infty$

$\Rightarrow$ Pontryagin Maximum Principle applicable

Pontryagin Maximum Principle in invariant form

$$\dot q = X_1(q) + uX_2(q),\ \ q \in M = \mathbb{R}^2 \times S^1,\ \ u \in \mathbb{R}, \quad J = \frac{1}{2}\int_0^{t_1} u^2\,dt \to \min$$

- $T_qM = \operatorname{span}(X_1(q), X_2(q), X_3(q)), \quad X_3 = [X_1, X_2]$
- $T_q^*M = \{(h_1, h_2, h_3)\}, \quad h_i(\lambda) = \langle \lambda, X_i \rangle,\ \lambda \in T^*M$
- Hamiltonian vector fields $\vec{h}_i \in \operatorname{Vec}(T^*M)$
- $h_u^\nu = \langle \lambda, X_1 + uX_2 \rangle + \frac{\nu}{2}u^2 = h_1(\lambda) + uh_2(\lambda) + \frac{\nu}{2}u^2$

Theorem 1 (Pontryagin Maximum Principle)

$u(t)$ and $q(t)$ optimal $\quad\Rightarrow\quad \exists\ \lambda_t \in T^*_{q(t)}M,\ \nu \le 0$:

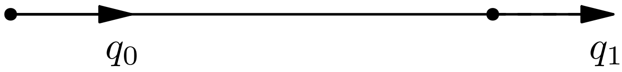
$$\dot\lambda_t = \vec{h}^\nu_{u(t)}(\lambda_t) = \vec{h}_1(\lambda_t) + u(t)\vec{h}_2(\lambda_t),$$

$$h^\nu_{u(t)}(\lambda_t) = \max_{u \in \mathbb{R}} h^\nu_u(\lambda_t),$$

$$(\nu, \lambda_t) \neq 0, \qquad t \in [0, t_1].$$

Abnormal extremal trajectories

$\nu = 0 \quad \Rightarrow \quad u(t) \equiv 0 \quad \Rightarrow \quad \theta \equiv 0, \qquad x = t, \qquad y \equiv 0$



$J = 0 = \min \quad \Rightarrow$

$\Rightarrow$ abnormal extremal trajectories optimal for $t \in [0, t_1]$

Unique trajectory from $q_0 = (0, 0, 0)$ to $(t_1, 0, 0) \in \partial \mathscr{A}_{q_0}(t_1)$.

Normal Hamiltonian system

$\nu = -1 \quad \Rightarrow \quad$ nonuniqueness of extremal trajectories

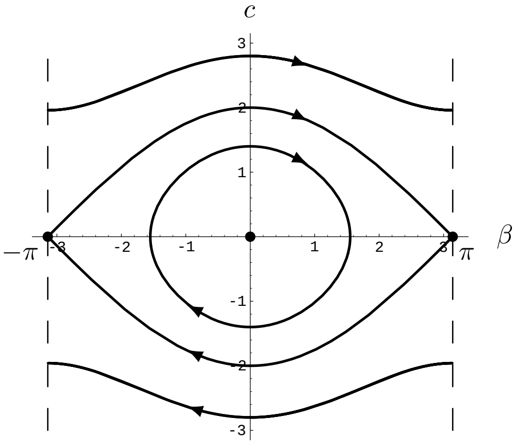
Hamiltonian system:

$$\dot h_1 = -h_2 h_3, \qquad \dot x = \cos\theta$$
$$\dot h_2 = h_3, \qquad \dot y = \sin\theta$$
$$\dot h_3 = h_1 h_2, \qquad \dot\theta = h_2$$

$$r^2 = h_1^2 + h_3^2 \equiv \text{const} \quad \Rightarrow \quad h_1 = -r\cos\beta, \; h_3 = -r\sin\beta$$

Equation of pendulum

$$\ddot{\beta} = -r\sin\beta \quad \Leftrightarrow \quad \begin{cases} \dot{\beta} = c, \\ \dot{c} = -r\sin\beta \end{cases}$$



Normal extremal trajectories

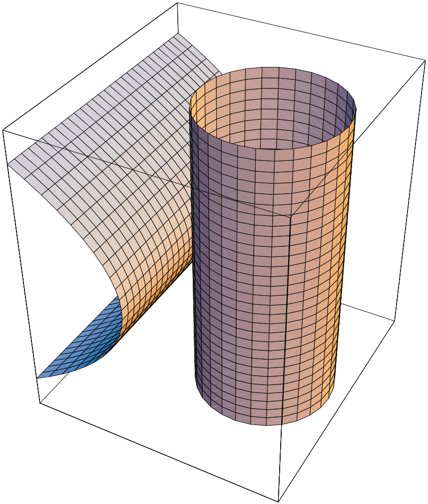
$$\ddot{\theta} = -r\sin(\theta - \gamma), \qquad r,\ \gamma = \text{const},$$
$$\dot{x} = \cos\theta,$$
$$\dot{y} = \sin\theta.$$

Integrable in Jacobi's functions.

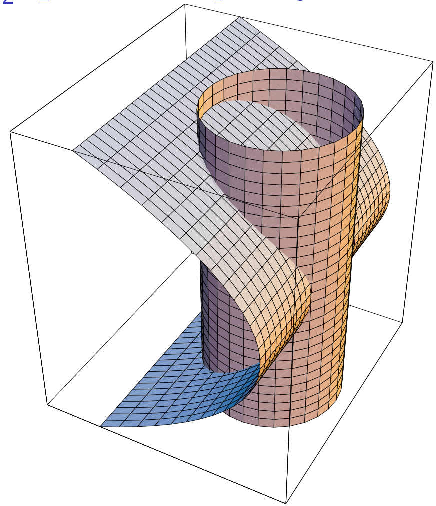
$\theta(t), x(t), y(t)$ parametrized by Jacobi's functions

$$\text{cn}(u,k), \quad \text{sn}(u,k), \quad \text{dn}(u,k), \quad \text{E}(u,k).$$

Geometry of integrals $H = h_1 + \frac{1}{2}h_2^2, \quad r^2 = h_1^2 + h_3^2$

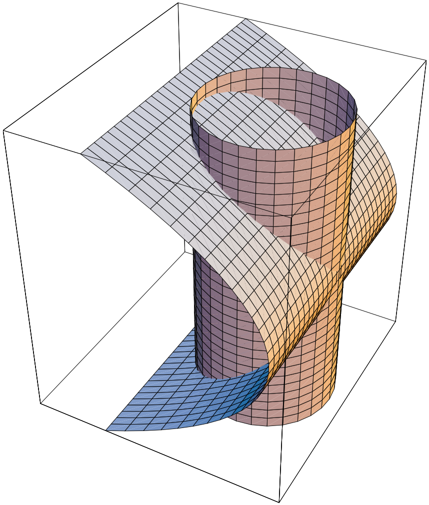


$H = -r \;,\; r > 0$

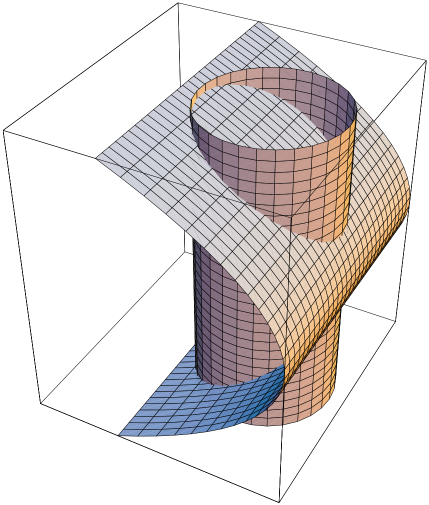


$H \in (-r, r),\; r > 0$

Geometry of integrals $H = h_1 + \frac{1}{2}h_2^2, \quad r^2 = h_1^2 + h_3^2$

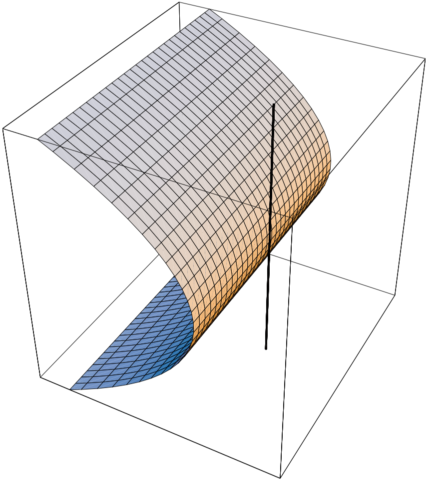


$H = r > 0$

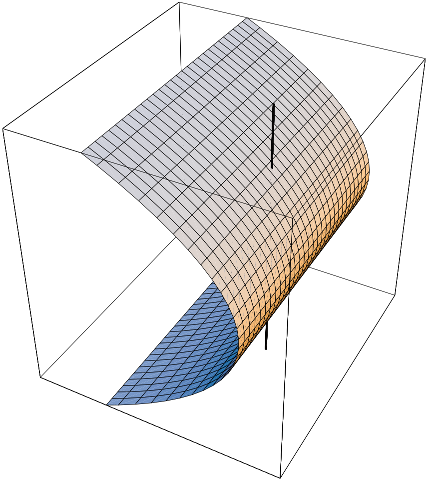


$H > r > 0$

Geometry of integrals $H = h_1 + \frac{1}{2}h_2^2, \quad r^2 = h_1^2 + h_3^2$



$H = r = 0$



$H > r = 0$

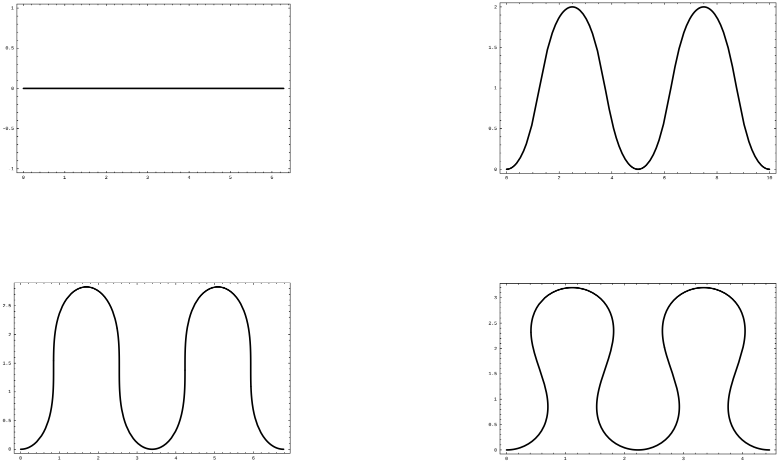
Euler elasticae

Energy of pendulum

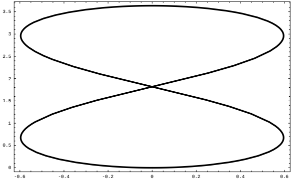
$$E = \frac{\dot{\theta}^2}{2} - r\cos(\theta - \gamma) \equiv \text{const} \in [-r, +\infty)$$

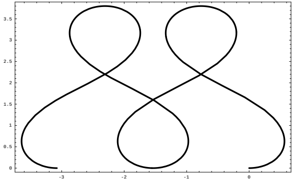
- $E = -r \neq 0 \quad \Rightarrow \quad$ straight lines
- $E \in (-r, r),\ r \neq 0 \quad \Rightarrow \quad$ inflectional elasticae
- $E = r \neq 0,\ \theta - \gamma = \pi \quad \Rightarrow \quad$ straight lines
- $E = r \neq 0,\ \theta - \gamma \neq \pi \quad \Rightarrow \quad$ critical elasticae
- $E > r \neq 0 \quad \Rightarrow \quad$ non-inflectional elasticae
- $r = 0 \quad \Rightarrow \quad$ straight lines and circles

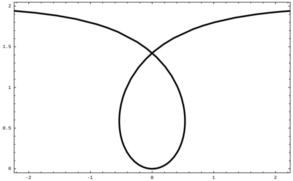
Euler elasticae

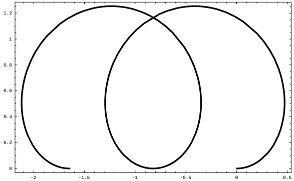


Euler elasticae

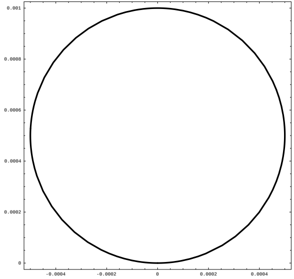








Euler elasticae



Optimality of normal extremal trajectories

$q(t)$ locally optimal:

$$\exists \varepsilon > 0 \quad \forall \widetilde{q} : \quad \|\widetilde{q} - q\|_C < \varepsilon,\ q(0) = \widetilde{q}(0),\ q(t_1) = \widetilde{q}(t_1) \quad \Rightarrow \quad J(q) \leq J(\widetilde{q})$$

Stable elastica $(x(t), y(t))$

$q(t)$ globally optimal:

$$\forall \widetilde{q} : \quad q(0) = \widetilde{q}(0),\ q(t_1) = \widetilde{q}(t_1) \quad \Rightarrow \quad J(q) \leq J(\widetilde{q})$$

Elastica $(x(t), y(t))$ of minimal energy.

Loss of optimality

Theorem 2 (Strong Legendre condition)

$$\left.\frac{\partial^2}{\partial u^2}\right|_{u(s)} h_u^{-1}(\lambda_s) < -\delta < 0 \quad \Rightarrow$$

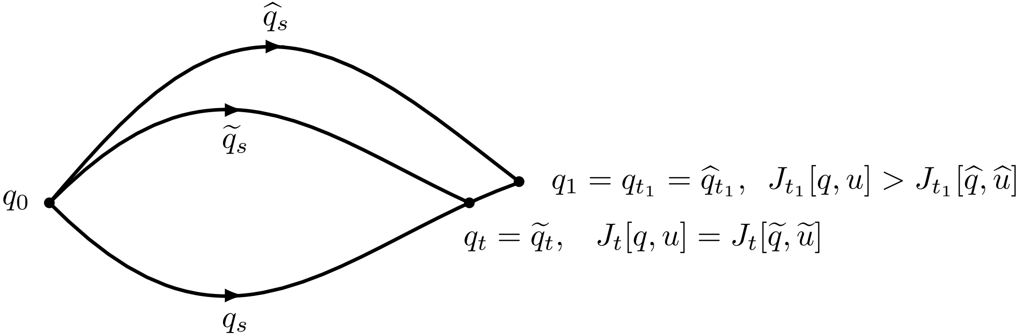
$$\Rightarrow \quad \textit{small arcs of normal extremal trajectories } q(s) \textit{ are optimal.}$$

Cut time along $q(s)$:

$$t_{\text{cut}}(q) = \sup\{t > 0 \mid q(s),\ s \in [0, t],\ \text{optimal}\ \}.$$

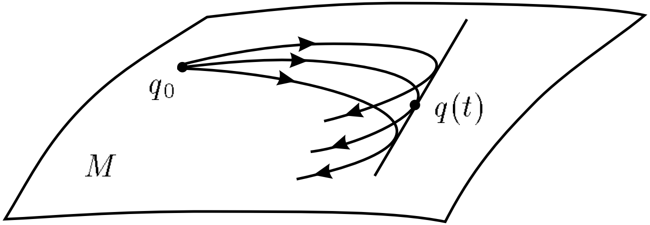
Reasons for loss of optimality: Maxwell point

Maxwell point q_t: $\exists\ \widetilde{q}_s \neq q_s : \quad q_t = \widetilde{q}_t,\ J_t[q,u] = J_t[\widetilde{q},\widetilde{u}]$



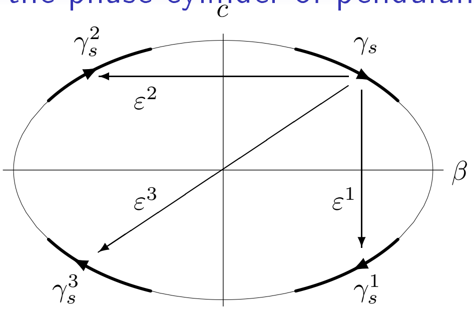
Reasons for loss of optimality: Conjugate point

Conjugate point: $q_t \in$ envelope of the family of extremal trajectories



$$t_{\text{cut}} \leq \min(t_{\text{Max}}, t_{\text{conj}})$$

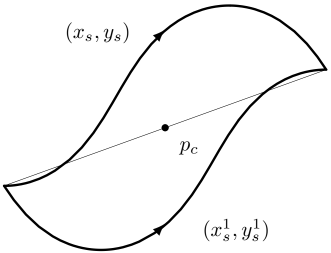
Reflections in the phase cylinder of pendulum $\ddot{\beta} = -r \sin \beta$

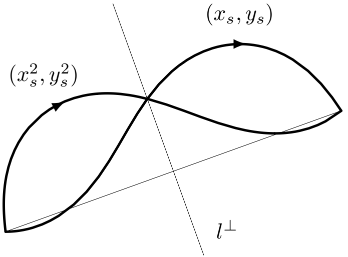


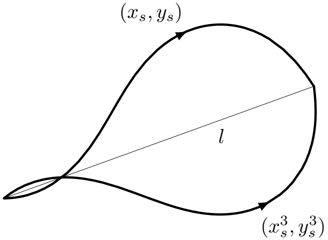
Dihedral group $D_2 = \{\mathrm{Id}, \varepsilon^1, \varepsilon^2, \varepsilon^3\}$

	ε^1	ε^2	ε^3
ε^1	Id	ε^3	ε^2
ε^2	ε^3	Id	ε^1
ε^3	ε^2	ε^1	Id

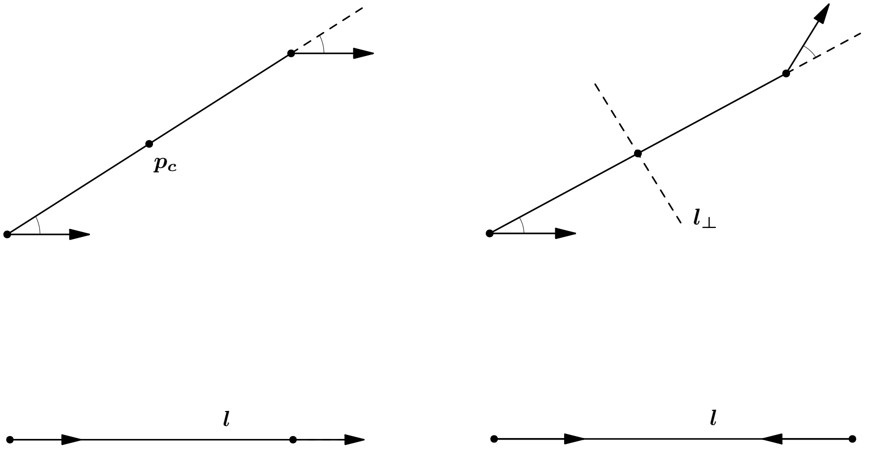
Action of reflections ε^1, ε^2, ε^3 on elasticae







Fixed points of reflections ε^1, ε^2, ε^3



Maxwell points corresponding to reflections

Fixed points of reflections $\varepsilon^i \quad \Rightarrow \quad$ Maxwell times:

$$t = t^n_{\varepsilon^i}, \qquad i = 1, 2, \quad n = 1, 2, \dots$$

T = period of pendulum $\quad \Rightarrow$

$$t^n_{\varepsilon^1} = nT, \qquad \left(n - \frac{1}{2}\right) T < t^n_{\varepsilon^2} < \left(n + \frac{1}{2}\right) T.$$

Upper bound of cut time:

$$t_{\text{cut}} \leq \min(t^1_{\varepsilon^1}, t^1_{\varepsilon^2}) \leq T.$$

Conjugate points

Exponential mapping

$$\mathrm{Exp}_t \; : \; T^*_{q_0} M \to M, \qquad \lambda_0 \mapsto q = q(t) = \pi \circ e^{t\vec{h}}(\lambda_0)$$

q — conjugate point $\quad \Leftrightarrow \quad$ q — critical value of Exp_t

$$\mathrm{Exp}_t(h_1, h_2, h_3) = (x, y, \theta)$$

$$\frac{\partial(x, y, \theta)}{\partial(h_1, h_2, h_3)} = 0$$

Local optimality of normal extremal trajectories

$q(t) = (x(t), y(t), \theta(t))$ normal extremal trajectory

Theorem 3 (Jacobi condition)

- *no conjugate points at* $(0, t_1]$ $\Rightarrow$ $q(t)$ *is locally optimal;*
- $(0, t_1)$ *contains conjugate points* $\Rightarrow$ $q(t)$ *is not locally optimal.*

Local optimality is lost at the first conjugate point $t^1_{\text{conj}} \in (0, +\infty]$

- No inflection points $\Rightarrow$ no conjugate points
- Inflectional case $\Rightarrow$ $t^1_{\text{conj}} \in [t^1_{\varepsilon^1}, t^1_{\varepsilon^2}] \subset [\frac{1}{2}T, \frac{3}{2}T]$

Stability of Euler elasticae

$$(x(s), y(s)) \text{ stable} \quad \Leftrightarrow \quad q(s) = (x(s), y(s), \theta(s)) \text{ locally optimal}$$

- $t_1 < t^1_{\text{conj}} \quad \Rightarrow \quad$ stability
- $t_1 > t^1_{\text{conj}} \quad \Rightarrow \quad$ instability
- straight lines, circles, non-inflectional elasticae are stable

Stability of inflectional elasticae

Loss of stability at the first conjugate point

- $t_1 \leq \frac{1}{2}T \quad \Rightarrow \quad$ stability
- $t_1 \geq \frac{3}{2}T \quad \Rightarrow \quad$ instability

In particular:

- no inflection points $\quad \Rightarrow \quad$ stability
- 1 or 2 inflection points $\quad \Rightarrow \quad$ stability or instability
- 3 inflection points $\quad \Rightarrow \quad$ instability

Global optimality of elasticae

$$q_1 \in \mathscr{A}_{q_0}(t_1), \qquad \text{optimal } q(t) = ?$$

$$q(t) = \operatorname{Exp}_t(\lambda) \text{ optimal for } t \in [0, t_1] \quad \Rightarrow \quad t_1 \leq \min(t^1_{\varepsilon_1}(\lambda), t^1_{\varepsilon_2}(\lambda))$$

$$N' = \{\lambda \in T^*_{q_0}M \mid t_1 \leq \min(t^1_{\varepsilon_1}(\lambda), t^1_{\varepsilon_2}(\lambda))\}$$

$$\operatorname{Exp}_{t_1} \; : \; N' \to \mathscr{A}_{q_0}(t_1) \text{ surjective, with singularities and multiple points}$$

$\exists$ open dense $\widetilde{N} \subset N'$, $\widetilde{M} \subset \mathscr{A}_{q_0}(t_1)$ such that

$$\operatorname{Exp}_{t_1} \; : \; \widetilde{N} \to \widetilde{M} \text{ double covering}$$

Global structure of exponential mapping

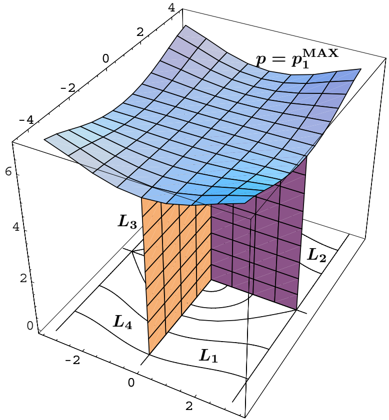


Figure: $\widetilde{N} = \cup_{i=1}^4 L_i$

$\xrightarrow{\mathrm{Exp}_{t_1}}$

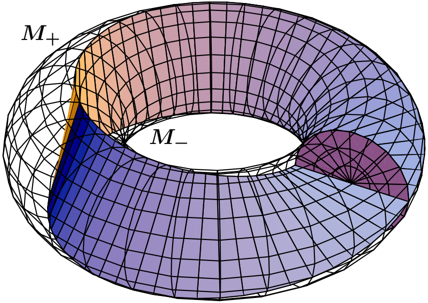
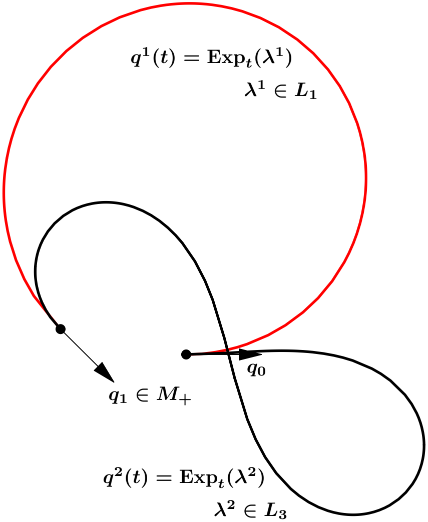


Figure: $\widetilde{M} = M_+ \cup M_-$

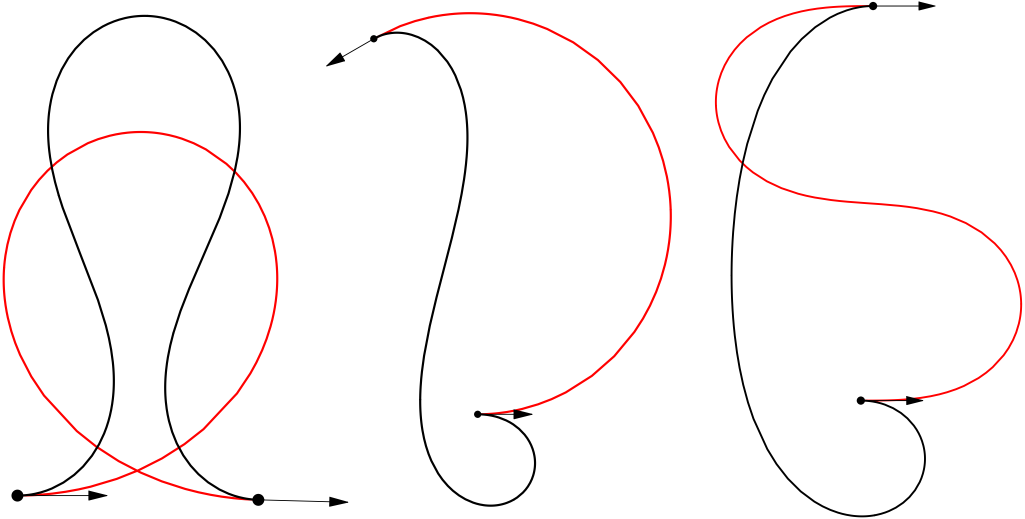
$\mathrm{Exp}_{t_1} \; : \; L_1, \; L_3 \to M_+$ diffeo, $\qquad \mathrm{Exp}_{t_1} \; : \; L_2, \; L_4 \to M_-$ diffeo

Competing elasticae



$$? \; : \; J[q^1] \lesseqgtr J[q^2]$$

Competing elasticae



Questions and perspectives

- Cut time $t_{\text{cut}} = ?$
- Optimal synthesis in Euler's elastic problem
- Nilpotent $(2, 3, 5)$ sub-Riemannian problem
- The ball-plate problem

Movies

Papers on Euler's elastic problem

[1] Yu. L. Sachkov, Maxwell strata in Euler's elastic problem, *Journal of Dynamical and Control Systems*, Vol. 14 (2008), No. 2 (April), 169–234.
[2] Yu. L. Sachkov, Conjugate points in Euler's elastic problem, *Journal of Dynamical and Control Systems*, 2008 Vol. 14 (2008), No. 3 (July), 409–439.
[3] Yu. L. Sachkov, Optimality of Euler's elasticae (in Russian), *Doklady Mathematics*, Vol. 76 (2007), No. 3, 817–819.
[4] A.A. Ardentov, Yu. L. Sachkov, Solution of Euler's elastic problem (in Russian), *Avtomatika i Telemekhanika*, 2009, No. 4, 78–88. (English translation in *Automation and remote control.*)
[5] Yu. L. Sachkov, S. Levyakov, Stability of Euler elasticae centered at vertices or inflection points, *Proceedings of the Steklov Institute of Mathematics*, V. 271 (2010), 187–203.
[6] Yu. L. Sachkov, Closed Euler Elasticae, *Proceedings of the Steklov Institute of Mathematics*, V. 278 (2012), 218–232.
[7] Yu. L. Sachkov, E.F. Sachkova, Exponential mapping in Euler's elastic problem, *Journal of Dynamical and Control Systems*, Vol. 20 (2014), No. 4, 443–464.
[8] A. Mashtakov, A. Ardentov, Yu. L. Sachkov, Relation between Euler's Elasticae and Sub-Riemannian Geodesics on SE(2), *Regular and Chaotic Dynamics*, December 2016, Volume 21, Issue 7, pp 832–839.

Conclusion: Euler's elastic problem

- Optimal control problem
- Extremal trajectories
- Local and global optimality of extremal trajectories
- Stability of Euler elasticae