Proof of Pontryagin maximum principle for sub-Riemannian problems
(Lecture 7)

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6. Coming Home on the Ox's Back:

Riding on the animal, he leisurely wends his way home:
Enveloped in the evening mist, how tunefully the flute vanishes away!
Singing a ditty, beating time, his heart is filled with a joy indescribable!
That he is now one of those who know, need it be told?

Pu-ming, "The Ten Oxherding Pictures"

Reminder: Plan of the previous lecture

1. Sub-Riemannian problems
2. The Lie algebra rank condition for SR problems
3. The Filippov theorem for SR problems
4. The Pontryagin maximum principle for SR problems
5. Optimality of SR extremal trajectories
6. A symmetry method for construction of optimal synthesis
7. The sub-Riemannian problem on the Heisenberg group.

Optimal control problem

At this lecture we prove Pontryagin maximum principle for the sub-Riemannian optimal control problem:

$$\dot{q} = \sum_{i=1}^{k} u_i f_i(q) =: f_u(q), \qquad q \in M, \quad u = (u_1, \ldots, u_k) \in \mathbb{R}^k,$$

$$q(0) = q_0, \qquad q(t_1) = q_1,$$

$$I = \int_0^{t_1} \left(\sum_{i=1}^{k} u_i^2 \right)^{1/2} dt \to \min .$$

Statement of PMP for SR problem

Theorem 1 (PMP for SR problems)

Let $\overline{q} \in \mathrm{Lip}([0,t_1],M)$ *be a SR minimizer for which the corresponding control* $\overline{u}(t)$ *satisfies the condition* $\sum_{i=1}^{k} \overline{u}_i^2(t) \equiv \mathrm{const}$. *Then there exists a curve* $\lambda_t \in \mathrm{Lip}([0,t_1],T^*M)$, $\pi(\lambda_t) = \overline{q}(t)$, *such that for almost all* $t \in [0,t_1]$

$$\dot{\lambda}_t = \sum_{i=1}^{k} \overline{u}_i(t)\vec{h}_i(\lambda_t), \qquad (1)$$

and one of the conditions hold:

(N) $\quad h_i(\lambda_t) \equiv \overline{u}_i(t), \quad i = 1,\dots,k,$ *or*

(A) $\quad h_i(\lambda_t) \equiv 0, \quad i = 1,\dots,k, \qquad \lambda_t \neq 0 \qquad \forall t \in [0,t_1].$

- In conditions (N), (A) corresponding to the normal and abnormal cases, as always, $h_i(\lambda) = \langle \lambda, X_i \rangle$, $i = 1,\dots,k$

Reduction to Theorems 2, 3

Theorem 1 follows from the next two theorems.

Theorem 2

Let the hypotheses of Theorem 1 *hold. For any* $t \in [0, t_1]$, *let* $P_t : M \to M$ *denote the flow of the nonautonomous vector field* $f_{\overline{u}(t)} = \sum_{i=1}^{k} \overline{u}_i(t) f_i$ *from the time* 0 *to the time* t. *Then there exists* $\lambda_0 \in T^*_{q_0}M$ *such that the curve*

$$\lambda_t = (P_t^{-1})^*(\lambda_0) \in T^*_{\overline{q}(t)}M \qquad (2)$$

satisfies one of conditions (N), (A) *of Theorem* 1.

Theorem 3

Let the hypotheses of Theorems 1 *and* 2 *hold. Then ODE* (1) *follows from identity* (2).

Flow of nonautonomous vector field

- In Theorem 2, the flow $P_t : M \to M$ of the nonautonomous field $f_{\overline{u}(t)}$ from the time 0 to the time t is given as follows:

$$P_t(q) = \overline{q}(t), \qquad q \in M, \quad t \in [0, t_1],$$

$$\frac{d}{dt}\overline{q}(t) = \sum_{i=1}^{k} \overline{u}_i(t) f_i(\overline{q}(t)), \qquad \overline{q}(0) = q.$$

- Further, in Theorem 2 we use the mapping $(P_t^{-1})^* : T^*_{q_0}M \to T^*_{\overline{q}(t)}M$, recall the necessary definition. If $F : M \to N$ is a smooth mapping between smooth manifolds and $q \in M$, then there is defined the differential

$$F_{*q} : T_q M \to T_{F(q)} N,$$

and the dual mapping of cotangent spaces:

$$F_q^* = (F_{*q})^* : T^*_{F(q)} N \to T^*_q M,$$

$$\langle F_q^*(\lambda), v \rangle = \langle \lambda, F_{q*}(v) \rangle, \qquad v \in T_q M, \quad \lambda \in T^*_{F(q)} N.$$

Reduction to the study of attainable sets

- Replace the length $l = \int_0^{t_1} (\sum_{i=1}^{k} u_i^2)^{1/2}\, dt$ by the energy $J = \int_0^{t_1} \sum_{i=1}^{k} u_i^2\, dt$.
- In order to include the functional J into dynamics of the system, introduce a new variable equal to the running value of the cost functional along a trajectory $q_u(t)$: $y(t) = \int_0^{t} \sum_{i=1}^{k} u_i^2\, dt$.
- Respectively, we introduce an extended state $\widehat{q} = \begin{pmatrix} y \\ q \end{pmatrix} \in \mathbb{R} \times M$ that satisfies an *extended control system*

$$\frac{d\widehat{q}}{dt} = \begin{pmatrix} \dot{y} \\ \dot{q} \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^{k} u_i^2 \\ f(q, u) \end{pmatrix} =: \widehat{f}(\widehat{q}, u).$$

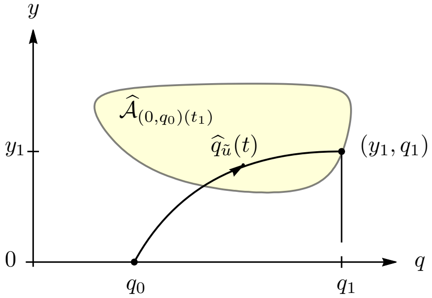
- The boundary conditions for this system are

$$\widehat{q}(0) = \begin{pmatrix} 0 \\ q_0 \end{pmatrix}, \qquad \widehat{q}(t_1) = \begin{pmatrix} J \\ q_1 \end{pmatrix}.$$

Reduction to the study of attainable sets

- A trajectory $q_{\tilde{u}}(t)$ is optimal for the optimal control problem with fixed time t_1 if and only if the corresponding trajectory $\widehat{q}_{\tilde{u}}(t)$ of the extended system comes to a point (y_1, q_1) of the attainable set $\widehat{\mathscr{A}_{(0,q_0)}}(t_1)$ such that

$$\widehat{\mathscr{A}_{(0,q_0)}}(t_1) \cap \{(y, q_1) \mid y < y_1\} = \emptyset.$$



Proof of Theorem 2: 1/11

- The curve $\overline{q}(t)$ is a minimizer of the length functional $l = \int_0^{t_1} \left(\sum_{i=1}^{k} u_i^2 \right)^{1/2} dt$ of constant velocity, thus it is a minimizer of the energy functional $J(u) = \frac{1}{2} \int_0^{t_1} \sum_{i=1}^{k} u_i^2(t)\, dt$ for a fixed t_1.
- Take any control $u(\,\cdot\,) = \overline{u}(\,\cdot\,) + v(\,\cdot\,) \in L^\infty([0, t_1], \mathbb{R}^k)$ and consider the corresponding Cauchy problem

$$\dot{q}(t) = f_{u(t)}(q(t)) = \sum_{i=1}^{k} u_i(t) f_i(q(t)), \qquad q(0) = q_0.$$

- Recall that $P_t : M \to M$ is the flow of the nonautonomous vector field $f_{\overline{u}(t)}$ from the time 0 to the time t.
- Consider the curve $x(t) = P_t^{-1}(q(t))$ and derive an ODE for $x(t)$.

Proof of Theorem 2: 2/11

- We differentiate the identity $q(t) = P_t(x(t))$ and get

$$\dot{q}(t) = f_{\overline{u}(t)}(P_t(x(t))) + (P_t)_* \dot{x}(t),$$

  whence

$$\begin{aligned}
\dot{x}(t) &= (P_t^{-1})_*[\dot{q}(t) - f_{\overline{u}(t)}(P_t(x(t)))] \\
&= (P_t^{-1})_*[(f_{u(t)} - f_{\overline{u}(t)})(P_t(x(t)))] \\
&= [(P_t^{-1})_*(f_{u(t)-\overline{u}(t)})](x(t)) \\
&= [(P_t^{-1})_* f_{v(t)}](x(t)).
\end{aligned}$$

- We denote the nonautonomous vector field $g_v^t = (P_t^{-1})_* f_v$ and get the required ODE

$$\dot{x}(t) = g_{v(t)}^t(x(t)), \qquad x(0) = P_0^{-1}(q_0) = q_0. \tag{3}$$

- Notice that f_v is linear in v, thus g_v^t is linear in v.

Proof of Theorem 2: 3/11

- For any $v \in L^\infty([0, t_1], \mathbb{R}^k)$, consider a mapping

$$\mathbb{R} \ni s \mapsto \begin{pmatrix} x(t_1; \overline{u} + sv) \\ J(\overline{u} + sv) \end{pmatrix} \in M \times \mathbb{R},$$

  where $x(t_1; \overline{u} + sv)$ is the solution to Cauchy problem (3) corresponding to the control $\overline{u} + sv$, and $J(\overline{u} + sv)$ is the corresponding energy.

Lemma 4

There exists a covector $\overline{\lambda} \in (T_{q_0}M \oplus \mathbb{R})^*$, $\overline{\lambda} \neq 0$, such that for any $v \in L^\infty([0, t_1], \mathbb{R}^k)$ there holds the equality

$$\left\langle \overline{\lambda}, \left(\left.\frac{\partial x(t_1; \overline{u} + sv)}{\partial s}\right|_{s=0}, \left.\frac{\partial J(\overline{u} + sv)}{\partial s}\right|_{s=0} \right) \right\rangle = 0. \tag{4}$$

Proof of Theorem 2: 4/11, Proof of Lemma 4

- Denote

$$\Phi(v) = \left(\frac{\partial x(t_1; \overline{u} + sv)}{\partial s}\bigg|_{s=0}, \frac{\partial J(\overline{u} + sv)}{\partial s}\bigg|_{s=0} \right),$$

$$\Phi : L^\infty([0, t_1], \mathbb{R}^k) \to T_{q_0} M \oplus \mathbb{R}.$$

- We compute the derivatives in the definition of the mapping Φ. It is easy to see that

$$\frac{\partial J(\overline{u} + sv)}{\partial s}\bigg|_{s=0} = \int_0^{t_1} \sum_{i=1}^{k} \overline{u}_i(t) v_i(t)\, dt. \tag{5}$$

Indeed, this follows from the expansion

$$J(\overline{u} + sv) = \frac{1}{2} \int_0^{t_1} |\overline{u} + sv|^2\, dt$$

$$= \frac{1}{2} \int_0^{t_1} \left(|\overline{u}|^2 + 2s \sum_{i=1}^{k} \overline{u}_i(t) v_i(t) + s^2 |v|^2 \right) dt.$$

Proof of Theorem 2: 5/11, Proof of Lemma 4

- Further, we show that

$$\left.\frac{\partial x(t_1;\overline{u}+sv)}{\partial s}\right|_{s=0} = \int_0^{t_1} g^t_{v(t)}(q_0)\,dt = \int_0^{t_1} \sum_{i=1}^{k} ((P_t^{-1})_* f_i)(q_0) v_i(t)\,dt. \qquad (6)$$

- The ODE $\dot{x}(t;\overline{u}+sv) = g^t_{sv}(x(t;\overline{u}+sv))$ implies in local coordinates that

$$\begin{aligned} x(t_1;\overline{u}+sv) &= q_0 + \int_0^{t_1} g^t_{sv(t)}(x(t;\overline{u}+sv))\,dt \\ &= q_0 + s\int_0^{t_1} g^t_{v(t)}(x(t;\overline{u}+sv))\,dt, \end{aligned}$$

whence

$$\begin{aligned} \left.\frac{\partial x(t_1;\overline{u}+sv)}{\partial s}\right|_{s=0} &= \int_0^{t_1} g^t_{v(t)}(x(t;\overline{u}))\,dt \\ &= \int_0^{t_1} g^t_{v(t)}(q_0)\,dt = \int_0^{t_1} \sum_{i=1}^{k} ((P_t^{-1})_* f_i)(q_0) v_i(t)\,dt. \end{aligned}$$

Proof of Theorem 2: 6/11, Proof of Lemma 4

- One can see from (5), (6) that the mapping Φ is linear. We show that it is not surjective.
- By contradiction, let $\operatorname{Im}\Phi = T_{q_0}M \oplus \mathbb{R}$, then there exist $v^0, \ldots, v^n \in L^\infty([0,t_1],\mathbb{R}^k)$ such that $\Phi(v^0), \ldots, \Phi(v^n)$ are linearly independent, i.e., the vectors

$$\begin{pmatrix} \left.\frac{\partial x(t_1;\overline{u}+sv^0)}{\partial s}\right|_{s=0} \\ \left.\frac{\partial J(\overline{u}+sv^0)}{\partial s}\right|_{s=0} \end{pmatrix}, \quad \ldots, \quad \begin{pmatrix} \left.\frac{\partial x(t_1;\overline{u}+sv^n)}{\partial s}\right|_{s=0} \\ \left.\frac{\partial J(\overline{u}+sv^n)}{\partial s}\right|_{s=0} \end{pmatrix}$$

  are linearly independent.
- Consider the mapping

$$F \ : \ (s_0, \ldots, s_n) \mapsto \begin{pmatrix} x\left(t_1; \overline{u} + \sum_{i=0}^{n} s_i v^i\right) \\ J\left(\overline{u} + \sum_{i=0}^{n} s_i v^i\right) \end{pmatrix}, \qquad \mathbb{R}^{n+1} \to M \times \mathbb{R}.$$

Proof of Theorem 2: 7/11, Proof of Lemma 4

- The mapping F is smooth near the point $0 \in \mathbb{R}^{n+1}$ and has a nondegenerate Jacobian at this point.
- Thus there exists a neighbourhood $O_0 \subset \mathbb{R}^{n+1}$ such that the restriction $F|_{O_0}$ is a diffeomorphism.
- Consequently,

$$F(0) = \begin{pmatrix} x(t_1; \overline{u}) \\ J(\overline{u}) \end{pmatrix} = \begin{pmatrix} q_0 \\ J(\overline{u}) \end{pmatrix} \in \operatorname{int} F(O_0).$$

- Thus there exists a control $v(\cdot) = \sum_{i=0}^{n} s_i v^i(\cdot)$ for which

$$x(t_1; \overline{u} + v) = q_0, \qquad J(\overline{u} + v) < J(\overline{u}).$$

Proof of Theorem 2: 8/11, Proof of Lemma 4

- Consider the corresponding trajectory $t \mapsto q(t; \overline{u} + v)$. We have

$$q(0; \overline{u} + v) = q_0,$$
$$q(t_1; \overline{u} + v) = P_{t_1}(x(t_1; \overline{u} + v)) = P_{t_1}(q_0) = q_1.$$

- So the curve $q(t; \overline{u} + v)$ connects the points q_0 and q_1 with a lesser value of the functional J than the optimal trajectory $\overline{q}(t) = q(t; \overline{u})$.
- The contradiction obtained completes the proof of Lemma 4.

Proof of Theorem 2: 9/11

- We continue the proof of Theorem 2.
- By the previous lemma, there exists a covector $0 \neq \overline{\lambda} \in (T_{q_0}M \oplus \mathbb{R})^*$ such that for any $v \in L^\infty([0, t_1], \mathbb{R}^k)$ we have

$$\left\langle \overline{\lambda}, \left(\left. \frac{\partial x(t_1; \overline{u} + sv)}{\partial s} \right|_{s=0}, \left. \frac{\partial J(\overline{u} + sv)}{\partial s} \right|_{s=0} \right) \right\rangle = 0.$$

- It is obvious that if this condition holds for some covector $\overline{\lambda}$, then it also holds for any covector $\alpha\overline{\lambda}$, $\alpha \neq 0$.
- Consequently, we can choose a covector $\overline{\lambda}$ of the form

$$\overline{\lambda} = (\lambda_0, -1) \qquad \text{or} \qquad \overline{\lambda} = (\lambda_0, 0), \quad \lambda_0 \neq 0.$$

Proof of Theorem 2: 10/11

- Thus there exists a covector $\lambda_0 \in T^*_{q_0}M$ such that for any $v \in L^\infty([0,t_1],\mathbb{R}^k)$

$$\left.\frac{\partial J(\overline{u}+sv)}{\partial s}\right|_{s=0} - \left\langle \lambda_0, \left.\frac{\partial x(t_1;\overline{u}+sv)}{\partial s}\right|_{s=0} \right\rangle = 0 \tag{7}$$

or

$$0 = \left\langle \lambda_0, \left.\frac{\partial x(t_1;\overline{u}+sv)}{\partial s}\right|_{s=0} \right\rangle, \qquad \lambda_0 \neq 0. \tag{8}$$

- Consider the case (7).
- Equalities (5) and (6) imply that for any $v \in L^\infty([0,t_1],\mathbb{R}^k)$

$$\int_0^{t_1} \sum_{i=1}^k \overline{u}_i(t)v_i(t)\,dt = \int_0^{t_1} \sum_{i=1}^k \left\langle \lambda_0, ((P_t^{-1})_* f_i)(q_0)\right\rangle v_i(t)\,dt$$
$$= \int_0^{t_1} \sum_{i=1}^k \langle \lambda_t, f_i(\overline{q}(t))\rangle v_i(t)\,dt = \int_0^{t_1} \sum_{i=1}^k h_i(\lambda_t)v_i(t)\,dt.$$

Proof of Theorem 2: 11/11

- Since the functions $v_i \in L^\infty[0, t_1]$ are arbitrary, we get in case (7)
  $(N) \quad \overline{u}_i(t) = h_i(\lambda_t), \quad i = 1, \ldots, k.$
- Similarly, in case (8) we get the condition
  $(A) \quad 0 = h_i(\lambda_t), \quad i = 1, \ldots, k; \qquad \lambda_0 \neq 0.$
- Theorem 2 is proved.

Proof of Theorem 3: 1/7

- Now we prove Theorem 3.
- Recall: we should show that the curve $\lambda_t = (P_t^{-1})^* \lambda_0 \in T^*_{\overline{q}(t)} M$ satisfies the ODE
  $$\dot{\lambda}_t = \sum_{i=1}^{k} \overline{u}_i(t) \vec{h}_i(\lambda_t).$$
- Now we prove this for the flow of an autonomous vector field.

Proof of Theorem 3: 2/7, Proof of Lemma 5

Lemma 5

Let $X \in \mathrm{Vec}(M)$, $P_t = e^{tX}$. *Then the curve* $\lambda_t = (P_t^{-1})^* \lambda_0$ *satisfies the ODE* $\dot{\lambda}_t = \vec{h}_X(\lambda_t)$.

- We set $\varphi(t) = (P_t^{-1})^*(\lambda_0)$, then we have to prove that

$$\dot{\varphi}(t) = \vec{h}_X(\varphi(t)) \in T_{\varphi(t)}(T^*M).$$

- A function $a \in C^\infty(T^*M)$ is called *constant on fibers of T^*M* if it has the form $a = \alpha \circ \pi$ for some function $\alpha \in C^\infty(M)$. Notation: $a \in C^\infty_{\mathrm{cst}}(T^*M)$.
- A function $h_Y \in C^\infty(T^*M)$ is called *linear on fibers of T^*M* if

$$h_Y(\lambda) = \langle \lambda, Y(q) \rangle, \qquad q = \pi(\lambda), \quad \lambda \in T^*M,$$

  for some vector field $Y \in \mathrm{Vec}(M)$. Notation: $h_Y \in C^\infty_{\mathrm{lin}}(T^*M)$.
- An *affine on fibers of T^*M function* is a sum of a constant on fibers and a linear on fibers functions:

$$C^\infty_{\mathrm{aff}}(T^*M) = C^\infty_{\mathrm{cst}}(T^*M) + C^\infty_{\mathrm{lin}}(T^*M).$$

Proof of Theorem 3: 3/7, Proof of Lemma 5

- Remark: Let $v, \omega \in T_\lambda(T^*M)$. The equality $v = \omega$ holds if and only if

$$vg = \omega g \qquad \forall g \in C^\infty_{\mathrm{aff}}(T^*M).$$

  Indeed, the value $vg = \langle d_\lambda g, v\rangle$ depends only on the first order Taylor polynomial of the function g.
- So we check the required equality $\dot\varphi(t) = \vec{h}_X(\varphi(t))$ for affine on fibers of T^*M functions.
- Let $a = \alpha \circ \pi \in C^\infty_{\mathrm{cst}}(T^*M)$, we check the equality $\dot\varphi(t)a = \vec{h}_X a$. We have

$$\vec{h}_X a = \{h_X, a\} = \sum_{i=1}^n \frac{\partial h_X}{\partial p_i}\frac{\partial \alpha}{\partial q_i} = \sum_{i=1}^n X_i \frac{\partial \alpha}{\partial q_i} = X\alpha,$$

$$\dot\varphi(t)a = \frac{d}{dt}a(\varphi(t)) = \frac{d}{dt}\alpha \circ e^{tX}(q_0) = (X\alpha)(\varphi(t)),$$

  and the required equality is proved for functions $a \in C^\infty_{\mathrm{cst}}(T^*M)$.

Proof of Theorem 3: 4/7, Proof of Lemma 5

- Now let $h_Y \in C^\infty_{\mathrm{lin}}(T^*M)$, we check the equality $\dot\varphi(t)h_Y = \vec{h}_X h_Y$. We have

$$\vec{h}_X h_Y = \{h_X, h_Y\} = h_{[X,Y]}.$$

- On the other hand,

$$\begin{aligned}
\dot\varphi(t)h_Y &= \frac{d}{dt} h_Y \circ \varphi(t) = \left.\frac{d}{d\tau}\right|_{\tau=0} h_Y \circ \varphi(t+\tau) \\
&= \left.\frac{d}{d\tau}\right|_{\tau=0} h_Y \circ (e^{-\tau X})^* \circ (e^{-tX})^*(\lambda_0) \\
&= \left.\frac{d}{d\tau}\right|_{\tau=0} \left\langle (e^{-\tau X})^* \circ (e^{-tX})^*(\lambda_0),\ Y(e^{(t+\tau)X}(q_0)) \right\rangle \\
&= \left\langle \varphi(t), \left.\frac{d}{d\tau}\right|_{\tau=0} e_*^{-\tau X} Y(e^{\tau X} \circ e^{tX}(q_0)) \right\rangle \\
&= \left\langle \varphi(t), [X,Y](e^{tX}(q_0)) \right\rangle = h_{[X,Y]}(\varphi(t)).
\end{aligned}$$

Proof of Theorem 3: 5/7, Proof of Lemma 5

- In the penultimate transition we used the equality

$$\frac{d}{d\tau}\bigg|_{\tau=0} e_*^{-\tau X} Y(e^{\tau X}(q)) = [X, Y](q), \tag{9}$$

which we prove now.

- We have

$$\frac{d}{d\tau}\bigg|_{\tau=0} e_*^{-\tau X} Y(e^{\tau X}(q)) = \frac{\partial^2}{\partial\tau\partial s}\bigg|_{\tau=0,s=0} e^{-\tau X} \circ e^{sY} \circ e^{\tau X}(q).$$

- We compute Taylor expansions of the compositions in the right-hand side:

$$e^{\tau X}(q) = q + \tau X(q) + o(\tau),$$

$$\begin{aligned} e^{sY} \circ e^{\tau X} &= e^{sY}(q + \tau X(q) + o(\tau)) \\ &= q + \tau X(q) + o(\tau) + sY(q + \tau X(q) + o(\tau)) + o(s) \\ &= q + \tau X(q) + sY(q) + s\tau \frac{\partial Y}{\partial q} X(q) + \dots, \end{aligned}$$

Proof of Theorem 3: 6/7, Proof of Lemma 5

- Consequently,

$$\begin{aligned} e^{-\tau X} \circ e^{sY} \circ e^{\tau X}(q) &= q + \tau X(q) + sY(q) + s\tau \frac{\partial Y}{\partial q} X(q) \\ &\quad - \tau X(q) - \tau s \frac{\partial X}{\partial q} Y(q) + \dots \\ &= q + sY(q) + s\tau [X, Y](q) + \dots, \end{aligned}$$

  thus

$$\left.\frac{\partial^2}{\partial \tau \partial s}\right|_{\tau=0, s=0} e^{-\tau X} \circ e^{sY} \circ e^{\tau X}(q) = [X, Y](q),$$

  and equality (9) follows.
- Lemma 5 is proved.

Proof of Theorem 3: 7/7

- Similarly to Lemma 5 for an autonomous vector field X, one proves the equality $\dot{\lambda}_t = \sum_{i=1}^{k} \overline{u}_i(t)\vec{h}_i(\lambda_t)$ for a curve $\lambda_t = (P_t^{-1})^* \lambda_0$ in the case of a nonautonomous vector field $f_{\overline{u}(t)}$.
- This completes the proof of Theorem 3.
- As we noticed above, Theorem 1 follows from Theorems 2 and 3.
- The Pontryagin maximum principle for sub-Riemannian problems is proved.