Orbit theorem
(Lecture 3)

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2. Seeing the Traces:

By the stream and under the trees, scattered are the traces of the lost;
The sweet-scented grasses are growing thick — did he find the way?
However remote over the hills and far away the beast may wander,
His nose reaches the heavens and none can conceal it.

Pu-ming, "The Ten Oxherding Pictures"

Reminder: Plan of the previous lecture

1. Lie groups, Lie algebras, and left-invariant optimal control problems
2. Controllability of linear systems
3. Local controllability of nonlinear systems
4. Statement of the Orbit theorem
5. Corollary of the Orbit theorem: Orbit and Lie algebra of the system.

Plan of this lecture

1. Proof of the Orbit theorem.
2. Corollaries of the Orbit theorem:
    - Rashevskii–Chow theorem,
    - Lie algebra rank condition,
    - Frobenius theorem.
3. Krener's theorem.

Theorem (*Orbit theorem*, Nagano–Sussmann)

Let $\mathcal{F} \subset \mathrm{Vec}(M)$, and let $q_0 \in M$.

(1) *The orbit $\mathcal{O}_{q_0}$ is a connected immersed submanifold of M.*

(2) *For any $q \in \mathcal{O}_{q_0}$*

$$T_q\mathcal{O}_{q_0} = \mathrm{span}(\mathcal{P}_*\mathcal{F})(q) = \mathrm{span}\{(P_*V)(q) \mid P \in \mathcal{P}, \quad V \in \mathcal{F}\},$$
$$\mathcal{P} = \{e^{t_N f_N} \circ \cdots \circ e^{t_1 f_1} \mid t_i \in \mathbb{R}, \quad f_i \in \mathcal{F}, \quad N \in \mathbb{N}\}.$$

Proof of the Orbit theorem: 1/7

Proof.

- Introduce a vector space important in the sequel

$$\Pi_q = \mathrm{span}(\mathcal{P}_* \mathcal{F})(q) \subset T_q M, \qquad q \in M,$$

  this is a candidate tangent space to the orbit $\mathcal{O}_{q_0}$.
- *1)* We prove that for all $q \in \mathcal{O}_{q_0}$ we have $\dim \Pi_q = \dim \Pi_{q_0}$.
- Choose any point $q \in \mathcal{O}_{q_0}$, then $q = Q(q_0)$, $Q \in \mathcal{P}$. Let us show that $Q_*^{-1}(\Pi_q) \subset \Pi_{q_0}$.
- Choose any element $(P_* f)(q) \in \Pi_q$, $P \in \mathcal{P}$, $f \in \mathcal{F}$. Then

$$\begin{aligned} Q_*^{-1}[(P_* f)(q)] &= (Q_*^{-1} \circ P_* f)(Q^{-1}(q)) \\ &= [(Q^{-1} \circ P)_* f](q_0) \in (\mathcal{P}_* \mathcal{F})(q_0) \subset \Pi_{q_0}. \end{aligned}$$

  Thus $Q_*^{-1}(\Pi_q) \subset \Pi_{q_0}$, whence $\dim \Pi_q \leq \dim \Pi_{q_0}$. Interchanging in this arguments q and q_0, we get $\dim \Pi_{q_0} \leq \dim \Pi_q$.
- Finally we have $\dim \Pi_q = \dim \Pi_{q_0}$, $q \in \mathcal{O}_{q_0}$.

Proof of the Orbit theorem: 2/7

- *2)* For any point $q \in M$ denote $m = \dim \Pi_q$, and choose such vector fields $V_1, \ldots, V_m \in \mathcal{P}_* \mathcal{F}$ that $\Pi_q = \operatorname{span}(V_1(q), \ldots, V_m(q))$.
- Further, define a mapping

$$G_q : (t_1, \ldots, t_m) \mapsto e^{t_m V_m} \circ \cdots \circ e^{t_1 V_1}(q), \qquad \mathbb{R}^m \to M.$$

- We have $\frac{\partial G_q}{\partial t_i}(0) = V_i(q)$, thus the vectors $\frac{\partial G_q}{\partial t_1}(0), \ldots, \frac{\partial G_q}{\partial t_m}(0)$ are linearly independent.
- Consequently, the restriction of G_q to a sufficiently small neighbourhood W_0 of the origin in $\mathbb{R}^m$ is a submersion.
- *3)* The image $G_q(W_0)$ is an (embedded) submanifold of M, may be, for a smaller neighbourhood W_0.

Proof of the Orbit theorem: 3/7

- *4)* We show that $G_q(W_0) \subset \mathcal{O}_q$.
- We have $G_q(W_0) = \{e^{t_m V_m} \circ \cdots \circ e^{t_1 V_1}(q) \mid t \in W_0\}$.
- Since $V_1 = P_* f, P \in \mathcal{P}$, $f \in \mathcal{F}$, we get

$$e^{t_1 V_1}(q) = e^{t_1 P_* f}(q) = P \circ e^{t_1 f} \circ P^{-1}(q) \in \mathcal{O}_q.$$

- We conclude similarly that $e^{t_2 V_2} \circ e^{t_1 V_1}(q) \in \mathcal{O}_q$ etc. Finally we have $G_q(t) \in \mathcal{O}_q$, $t \in W_0$.

Proof of the Orbit theorem: 4/7

- 5) We show that $G_{q*}(T_t\mathbb{R}^m) = \Pi_{G_q(t)}$, $t \in W_0$. We have $\dim G_{q*}(T_t\mathbb{R}^m) = m = \dim \Pi_{G_q(t)}$, thus it suffices to prove the inclusion $\frac{\partial G_q}{\partial t_i}(t) \in \Pi_{G_q(t)}, \qquad t \in W_0.$
- Let us compute this partial derivative:

$$\frac{\partial G_q}{\partial t_i} = \frac{\partial}{\partial t_i} e^{t_m V_m} \circ \cdots \circ e^{t_i V_i} \circ \cdots \circ e^{t_1 V_1}(q)$$

denote $R = e^{t_m V_m} \circ \cdots \circ e^{t_{i+1} V_{i+1}}$, $q' = e^{t_{i-1} V_{i-1}} \circ \cdots \circ e^{t_1 V_1}(q)$,

$$\begin{aligned} &= \frac{\partial}{\partial t_i} R \circ e^{t_i V_i}(q') = R_* V_i(e^{t_i V_i}(q')) \\ &= (R_* V_i)[R \circ e^{t_i V_i} \circ \cdots \circ e^{t_1 V_1}(q)] \\ &= (R_* V_i)(G_q(t)) \in (\mathcal{P}_* \mathcal{F})(G_q(t)) \subset \Pi_{G_q(t)}. \end{aligned}$$

- Thus $G_{q*}(T_t\mathbb{R}^m) = \Pi_{G_q(t)}$, i.e., the space $\Pi_{G_q(t)}$ is a tangent space to the smooth manifold $G_q(W_0)$ at the point $G_q(t)$.

Proof of the Orbit theorem: 5/7

- *6)* We prove that the sets $G_q(W_0)$ form a base of a ("strong") topology on M.
- *6a)* It is obvious that any point $q \in M$ is contained in the set $G_q(W_0)$.
- *6b)* Let us show that for any point $\widehat{q} \in G_q(W_0) \cap G_{\widetilde{q}}(\widetilde{W_0})$ there exists a set $G_{\widehat{q}}(\widehat{W_0}) \subset G_q(W_0) \cap G_{\widetilde{q}}(\widetilde{W_0})$.
- Take any point $\widehat{q} \in G_q(W_0) \cap G_{\widetilde{q}}(\widetilde{W_0})$ and consider $G_{\widehat{q}}(t) = e^{t_m \widehat{V}_m} \circ \cdots \circ e^{t_1 \widehat{V}_1}(\widehat{q})$.
- For any point $q' \in G_q(W_0)$ we have $\widehat{V}_1(q') \in (\mathcal{P}_* \mathcal{F})(q') \subset \Pi_{q'}$. But $G_q(W_0)$ is a submanifold with the tangent space $T_{q'} G_q(W_0) = \Pi_{q'}$. The vector field $\widehat{V}_1$ is tangent to this submanifold, thus $e^{t_1 \widehat{V}_1}(\widehat{q}) \in G_q(W_0)$ for small $|t_1|$. We conclude similarly that $e^{t_2 \widehat{V}_2} \circ e^{t_1 \widehat{V}_1}(\widehat{q}) \in G_q(W_0)$ for small $|t_1|$, $|t_2|$ etc. Finally we get

$$G_{\widehat{q}}(t) \in G_q(W_0) \text{ for small } |t|.$$

- Similarly $G_{\widehat{q}}(t) \in G_{\widetilde{q}}(\widetilde{W_0})$ for small $|t|$. Thus $G_{\widehat{q}}(\widehat{W_0}) \subset G_q(W_0) \cap G_{\widetilde{q}}(\widetilde{W_0})$ for some neighbourhood $\widehat{W_0}$, and property 6b) is proved.

Proof of the Orbit theorem: 6/7

- It follows from properties 6a) and 6b) that the sets $G_q(W_0)$ form a base of topology on the set M. Denote the corresponding topological space as $M^{\mathcal{F}}$.
- *7)* We show that for any $q_0 \in M$ the orbit $\mathcal{O}_{q_0}$ is connected, open and closed in the space $M^{\mathcal{F}}$.
- The mappings $t_i \mapsto e^{t_i f_i}(q)$ are continuous in $M^{\mathcal{F}}$, thus $\mathcal{O}_{q_0}$ is connected.
- Any point $q \in \mathcal{O}_{q_0}$ is contained in the neighbourhood $G_q(W_0) \subset \mathcal{O}_q = \mathcal{O}_{q_0}$, thus the orbit is open in $M^{\mathcal{F}}$.
- Finally, any orbit is a complement in M to orbits with which it does not intersect. Thus any orbit is closed in $M^{\mathcal{F}}$.
- So any orbit $\mathcal{O}_{q_0}$ is a connected component of the topological space $M^{\mathcal{F}}$.

Proof of the Orbit theorem: 7/7

- *8)* Introduce a smooth structure on $\mathcal{O}_{q_0}$ as follows:
  - the sets $G_q(W_0)$ are called coordinate neighbourhoods
  - the mappings $G_q^{-1} : G_q(W_0) \to W_0$ are called coordinate mappings.
- It is easy to see that these coordinate neighbourhoods and mappings agree: for any intersecting neighbourhoods $G_q(W_0)$ and $G_{\widetilde{q}}(\widetilde{W_0})$ the composition

$$G_{\widetilde{q}} \circ G_q \ : \ G_q^{-1}(G_q(W_0) \cap G_{\widetilde{q}}(\widetilde{W_0})) \to G_{\widetilde{q}}^{-1}(G_q(W_0) \cap G_{\widetilde{q}}(\widetilde{W_0}))$$

  is a diffeomorphism.
- Thus the orbit $\mathcal{O}_{q_0}$ is a smooth manifold.
- Moreover, $\mathcal{O}_{q_0} \subset M$ is an immersed submanifold of dimension $m = \dim \Pi_{q_0}$.
- *9)* It follows from item 5) above that the smooth manifold $\mathcal{O}_{q_0}$ has a tangent space

$$T_q \mathcal{O}_{q_0} = \Pi_q = \operatorname{span}(\mathcal{P}_* \mathcal{F})(q), \qquad q \in \mathcal{O}_{q_0}.$$

- The Orbit theorem is proved.

Rashevskii–Chow theorem

- A system $\mathcal{F} \subset \mathrm{Vec}(M)$ is called *completely nonholonomic* (*full-rank*, *bracket-generating*) if $\mathrm{Lie}_q(\mathcal{F}) = T_q M \quad \forall q \in M$.

Theorem (Rashevskii–Chow)

If $\mathcal{F} \subset \mathrm{Vec}(M)$ is full-rank and M is connected, then $\mathcal{O}_q = M \; \forall \; q \in M$.

Proof.

- Take any $q \in M$ and any $q_1 \in \mathcal{O}_q$.
- We have $T_{q_1}\mathcal{O}_q \supset \mathrm{Lie}_{q_1}(\mathcal{F}) = T_{q_1}M$, thus $\dim \mathcal{O}_q = \dim M$, i.e., $\mathcal{O}_q$ is open in M.
- On the other hand, any orbit is closed as a complement to the union of all other orbits.
- Thus any orbit is a connected component of M. Since M is connected, each orbit coincides with M.

□

Lie algebra rank condition

Corollary (Lie algebra rank condition, LARC)

If a manifold M is connected, and a system $\mathcal{F} \subset \mathrm{Vec}(M)$ is symmetric and completely nonholonomic, then it is controllable on M.

Distributions

- A *distribution* on a smooth manifold M is a smooth mapping

$$\Delta\colon q \mapsto \Delta_q \subset T_qM, \quad q \in M,$$

  where the vector subspaces Δ_q have the same dimension called the *rank* of Δ.
- An immersed submanifold $N \subset M$ is called an *integral manifold* of a distribution Δ if

$$\forall q \in N \quad T_qN = \Delta_q.$$

- A distribution Δ on M is called *integrable* if for any point $q \in M$ there exists an integral manifold $N_q \ni q$.
- Denote by

$$\bar{\Delta} = \{f \in \mathrm{Vec}(M) \mid f(q) \in \Delta_q \quad \forall q \in M\}$$

  the set of vector fields tangent to Δ.
- A distribution Δ is called *holonomic* if $[\bar{\Delta}, \bar{\Delta}] \subset \bar{\Delta}$.

Theorem (Frobenius)

A distribution is integrable iff it is holonomic.

Proof:

- *Necessity*. Take any $f, g \in \bar{\Delta}$. Let $q \in M$, and let $N_q \ni q$ be the integral manifold of Δ through q.
- Then
$$\varphi(t) = e^{-tg} \circ e^{-tf} \circ e^{tg} \circ e^{tf}(q) \in N_q,$$
thus
$$\left.\frac{d}{dt}\right|_{t=0} \varphi(\sqrt{t}) = [f, g](q) \in T_q N_q = \Delta_q.$$
- So $[f, g] \in \bar{\Delta}$, and the inclusion $[\bar{\Delta}, \bar{\Delta}] \subset \bar{\Delta}$ follows.

- *Sufficiency*. We consider only the analytic case.
- We have

$$[\bar{\Delta}, \bar{\Delta}] \subset \bar{\Delta}, \qquad [[\bar{\Delta}, \bar{\Delta}], \bar{\Delta}] \subset [\bar{\Delta}, \bar{\Delta}] \subset \bar{\Delta}.$$

- Inductively $\operatorname{Lie}_q(\bar{\Delta}) \subset \bar{\Delta}_q = \Delta_q$.
- The reverse inclusion is obvious, thus $\operatorname{Lie}_q(\bar{\Delta}) = \Delta_q$, $q \in M$.
  Denote $N_q = \mathcal{O}_q(\bar{\Delta})$ and prove that N_q is an integral manifold of Δ:

$$T_{q'} N_q = T_{q'}(\mathcal{O}_q(\bar{\Delta})) = \operatorname{Lie}_{q'}(\bar{\Delta}) = \Delta_{q'}, \quad q' \in N_q.$$

- So $N_q \ni q$ is the integral manifold of Δ, and Δ is integrable. □

Frobenius condition

- Consider a *local frame* of Δ:

$$\Delta_q = \mathrm{span}(f_1(q), \ldots, f_k(q)), \quad q \in S \subset M, \quad f_1, \ldots, f_k \in \mathrm{Vec}(S), \quad k = \dim \Delta_q,$$

where S is an open subset of M.

- Then the inclusion $[\bar{\Delta}, \bar{\Delta}] \subset \bar{\Delta}$ takes the form

$$[f_i, f_j](q) = \sum_{l=1}^{k} c_{ij}^l(q) f_l(q), \qquad q \in S, \quad c_{ij}^l \in C^\infty(S).$$

- This equality is called the *Frobenius condition*.

Example:
The sub-Riemannian problem on the group of motions of the plane

- The control system has the following form:

$$\mathcal{F} = \{u_1 f_1 + u_2 f_2 \mid (u_1, u_2) \in \mathbb{R}^2\} \subset \mathrm{Vec}(\mathbb{R}^2 \times S^1),$$
$$f_1 = \cos\theta \frac{\partial}{\partial x} + \sin\theta \frac{\partial}{\partial y}, \qquad f_2 = \frac{\partial}{\partial \theta}.$$

- The system is symmetric: $\mathcal{F} = -\mathcal{F}$.
- Compute its Lie algebra:

$$[f_1, f_2] = \sin\theta \frac{\partial}{\partial x} - \cos\theta \frac{\partial}{\partial y} =: f_3,$$
$$\mathrm{Lie}_q(\mathcal{F}) = \mathrm{span}(f_1(q), f_2(q), f_3(q)) = T_q(\mathbb{R}^2 \times S^1).$$

- The system $\mathcal{F}$ is completely nonholonomic, thus controllable.

Example: Orbits of different dimensions

- Let

$$M = \mathbb{R}_x, \qquad \mathcal{F} = \left\{ x \frac{\partial}{\partial x} \right\} \subset \mathrm{Vec}(M).$$

- We have:

$$\begin{aligned} x_0 > 0 \quad &\Rightarrow \quad \mathcal{O}_{x_0} = \{x > 0\}, \\ x_0 = 0 \quad &\Rightarrow \quad \mathcal{O}_{x_0} = \{x = 0\}, \\ x_0 < 0 \quad &\Rightarrow \quad \mathcal{O}_{x_0} = \{x < 0\}, \end{aligned}$$

- Thus the system has two one-dimensional orbits and one zero-dimensional orbit.

Example:
More orbits of different dimensions

- Let

$$M = \mathbb{R}^3_{x,y,z}, \qquad \mathcal{F} = \left\{ x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x},\ y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y},\ z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z} \right\} \subset \mathrm{Vec}(M).$$

- Then for any point $q \in \mathbb{R}^3$

$$\mathcal{O}_q = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = |q|^2\},$$

- This is a sphere for $q \neq 0$ and a point for $q = 0$.
- An orbit of a control system is a generalisation of a trajectory of a vector field to the case of more than one vector field.

Attainable sets of full-rank systems

- Let $\mathcal{F} \subset \mathrm{Vec}(M)$ be a full-rank system. The assumption of full rank is not very strong in the analytic case: if it is violated, we can consider the restriction of $\mathcal{F}$ to its orbit, and this restriction is full-rank.
- What is the possible structure of *attainable sets* of $\mathcal{F}$?
- It is easy to construct systems in the two-dimensional plane that have the following attainable sets:
  - a smooth full-dimensional manifold without boundary;
  - a full-dimensional manifold with smooth boundary;
  - a full-dimensional manifold with non-smooth boundary, with corner or cusp singularity.

Possible attainable sets of full-rank systems

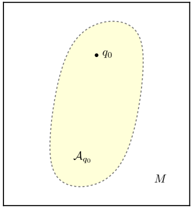

Figure: Attainable set — smooth manifold without boundary

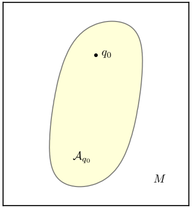

Figure: Attainable set — manifold with smooth boundary

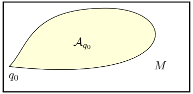

Figure: Attainable set — manifold with nonsmooth boundary

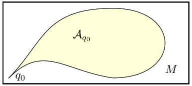

Figure: Attainable set — manifold with nonsmooth boundary

Impossible attainable sets of full-rank systems

- But it is impossible to construct an attainable set that is:
  - a lower-dimensional submanifold;
  - a set whose boundary points are isolated from its interior points.

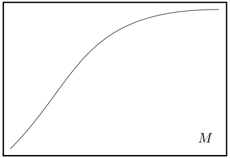

Figure: Forbidden attainable set: subset of lower dimension

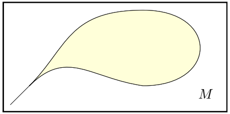

Figure: Forbidden attainable set: subset with isolated boundary points

- These possibilities are forbidden respectively by the following theorem.

Krener's theorem

Theorem (Krener)

Let $\mathcal{F} \subset \mathrm{Vec}(M)$, *and let* $\mathrm{Lie}_q \mathcal{F} = T_q M$ *for any* $q \in M$. *Then:*

(1) $\mathrm{int}\, \mathcal{A}_q \neq \varnothing$ *for any* $q \in M$

(2) $\mathrm{cl}(\mathrm{int}\, \mathcal{A}_q) \supset \mathcal{A}_q$ *for any* $q \in M$.

Proof of Krener's theorem: 1/2

- Since item (2) implies item (1), we prove item (2): $\mathrm{cl}(\operatorname{int}\mathcal{A}_q) \supset \mathcal{A}_q$.
- We argue by induction on dimension of M. If $\dim M = 0$, the statement is obvious. Let $\dim M > 0$.
- Take any $q_1 \in \mathcal{A}_q$, and fix any neighbourhood $q_1 \in W(q_1) \subset M$. We show that $\operatorname{int}\mathcal{A}_q \cap W(q_1) \neq \varnothing$.
- There exists $f_1 \in \mathcal{F}$ such that $f_1(q_1) \neq 0$, otherwise $\mathcal{F}(q_1) = \{0\} = \mathrm{Lie}_{q_1}(\mathcal{F}) = T_{q_1}M$, a contradiction. Consider the following set for a small $\varepsilon_1 > 0$:

$$N_1 = \{e^{t_1 f_1}(q_1) \mid 0 < t_1 < \varepsilon_1\} \subset W(q_1) \cap \mathcal{A}_q.$$

- N_1 is a smooth 1-dimensional manifold. If $\dim M = 1$, then N_1 is open, thus $N_1 \subset \operatorname{int}\mathcal{A}_q$, so $\operatorname{int}\mathcal{A}_q \cap W(q_1) \neq \varnothing$. Since the neighbourhood $W(q_1)$ is arbitrary, $q_1 \in \mathrm{cl}(\operatorname{int}\mathcal{A}_q)$.

Proof of Krener's theorem: 2/2

- Let $\dim M > 1$. There exist $q_2 = e^{t_1^1 f_1}(q_1) \in N_1 \cap W(q_1)$ and $f_2 \in \mathcal{F}$ such that $f_2(q_2) \notin T_{q_2} N_1$. Otherwise $\dim \mathcal{F}(q_2) = \dim \mathrm{Lie}_{q_2}(\mathcal{F}) = \dim T_{q_2} M = 1$ for any $q_2 \in N_2 \cap W$, and $\dim M = 1$.
- Consider the following set for a small ε_2:

$$N_2 = \{e^{t_2 f_2} \circ e^{t_1 f_1}(q_2) \mid t_1^1 < t_1 < t_1^1 + \varepsilon_2,\ 0 < t_2 < \varepsilon_2\} \subset W(q_1) \cap \mathcal{A}_q.$$

- N_2 is a smooth 2-dimensional manifold.
- If $\dim M = 2$, then N_2 is open, thus $N_2 \subset \operatorname{int} \mathcal{A}_q \cap W(q_1) \neq \varnothing$ and $q_1 \in \mathrm{cl}(\operatorname{int} \mathcal{A}_q)$.
- If $\dim M > 2$, we proceed by induction. □

A control system $\mathcal{F} \subset \mathrm{Vec}(M)$ is called *accessible* at a point $q \in M$ if $\operatorname{int} \mathcal{A}_q \neq \emptyset$. In the analytic case the accessibility property is equivalent to the full-rank condition.

Example: Stopping a train

- The control system has the form

$$\dot{x} = f_1(x) + u f_2(x), \qquad x = (x_1, x_2) \in \mathbb{R}^2, \quad |u| \le 1,$$

$$f_1 = x_2 \frac{\partial}{\partial x_1}, \qquad f_2 = \frac{\partial}{\partial x_2}.$$

- We have $[f_1, f_2] = -\frac{\partial}{\partial x_1}$, whence the system $\mathcal{F} = \{f_1 + u f_2 \mid u \in [-1, 1]\}$ is full-rank: $\mathrm{Lie}_x(\mathcal{F}) = \mathrm{span}\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right)(x) = T_x\mathbb{R}^2 \qquad \forall x \in \mathbb{R}^2$.
- Thus

$$\mathcal{O}_x = \mathbb{R}^2 \qquad \forall x \in \mathbb{R}^2.$$

- In order to find the attainable sets, we compute trajectories of the system with a constant control $u \neq 0$: they are the parabolas

$$\frac{x_2^2}{2} = u x_1 + C.$$

- Now it is visually obvious that the system is controllable.

Example: Markov–Dubins car (1/2)

- The control system has the form

$$\dot{q} = f_1(q) + u f_2(q), \qquad q = (x, y, \theta) \in M = \mathbb{R}^2 \times S^1, \quad |u| \le 1,$$

$$f_1 = \cos\theta \frac{\partial}{\partial x} + \sin\theta \frac{\partial}{\partial y}, \qquad f_2 = \frac{\partial}{\partial \theta}.$$

- We have

$$[f_1, f_2] = \sin\theta \frac{\partial}{\partial x} - \cos\theta \frac{\partial}{\partial y} =: f_3.$$

- Thus the system $\mathcal{F} = \{f_1 + u f_2 \mid u \in [-1, 1]\}$ is full-rank:

$$\mathrm{Lie}_q(\mathcal{F}) = \mathrm{span}(f_1(q), f_2(q), f_3(q)) = T_q M \qquad \forall q \in M,$$

  consequently,

$$\mathcal{O}_q = M \qquad \forall q \in M.$$

- In order to describe the attainable sets, we replace the initial system $\mathcal{F}$ by a restricted system $\mathcal{F}_1 = \{f_1 \pm f_2\} \subset \mathcal{F}$ and prove that $\mathcal{F}_1$ is controllable (then $\mathcal{F}$ is controllable as well).

Example: Markov–Dubins car (2/2)

- Trajectories of the restricted system $\dot{x} = \cos\theta$, $\dot{y} = \sin\theta$, $\dot{\theta} = \pm 1$, have the form

$$\theta = \theta_0 \pm t, \qquad x = x_0 \pm (\sin(\theta_0 \pm t) - \sin\theta_0), \qquad y = y_0 \pm (\cos\theta_0 - \cos(\theta_0 \pm t)).$$

- These trajectories are periodic: $e^{(t+2\pi n)(f_1 \pm f_2)} = e^{t(f_1 \pm f_2)}, \qquad t \in \mathbb{R}, \quad n \in \mathbb{Z}$. So a shift along the fields $f_1 \pm f_2$ in the negative time can be obtained as a shift in the positive time.
- Consequently, if we introduce the system $\mathcal{F}_2 = \{f_1 \pm f_2, \ -f_1 \pm f_2\}$, then we get

$$\mathcal{A}_q(\mathcal{F}_2) = \mathcal{A}_q(\mathcal{F}_1), \qquad q \in M.$$

- But the system $\mathcal{F}_2$ is symmetric and full-rank, thus $\mathcal{A}_q(\mathcal{F}_2) = \mathcal{O}_q(\mathcal{F}_2) = M$, whence

$$\mathcal{A}_q(\mathcal{F}) = \mathcal{A}_q(\mathcal{F}_1) = M \text{ for all } q \in M.$$

  That is, the Markov–Dubins car is completely controllable in the space $\mathbb{R}^2 \times S^1$.