Examples and statements of control problems
(Lecture 1)

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Plan of lecture

1. Examples of optimal control problems
2. Statements of the main problems of this course:
   2.1 controllability problem,
   2.2 optimal control problem.
3. Smooth manifolds and vector fields.

Examples of optimal control problems:
1. Stopping a train

Given:

- material point of mass $m > 0$ with coordinate $x \in \mathbb{R}$
- force F bounded by the absolute value by $F_{\max} > 0$
- initial position x_0 and initial velocity $\dot{x}_0$ of the material point

Find:

- force F that steers the point to the origin with zero velocity, for a minimal time.

$$
\begin{aligned}
&\dot{x}_1 = x_2, \qquad (x_1, x_2) \in \mathbb{R}^2,\\
&\dot{x}_2 = u, \qquad |u| \leq 1,\\
&(x_1, x_2)(0) = (x_0, \dot{x}_0), \qquad (x_1, x_2)(t_1) = (0, 0),\\
&t_1 \to \min .
\end{aligned}
$$

2. Control of linear oscillator

Given:

- pendulum that performs small oscillations under the action of a force bounded by the absolute value

Find:

- force that steers the pendulum from an arbitrary position and velocity to the stable equilibrium for a minimum time.

$$\begin{aligned}
&\dot{x}_1 = x_2, \qquad x = (x_1, x_2) \in \mathbb{R}^2, \\
&\dot{x}_2 = -x_1 + u, \qquad |u| \leq 1, \\
&x(0) = x^0, \qquad x(t_1) = 0, \\
&t_1 \to \min .
\end{aligned}$$

3. The Markov–Dubins car

Given:

- model of a car given by a unit vector attached at a point $(x, y) \in \mathbb{R}^2$, with orientation $\theta \in S^1$
- The car moves forward with the unit velocity and can simultaneously rotate with an angular velocity $|\dot{\theta}| \leq 1$
- an initial and a terminal state of the car

Find:

- angular velocity in such a way that the time of motion is as minimum as possible.

$$
\begin{aligned}
&\dot{x} = \cos\theta, \qquad && q = (x, y, \theta) \in \mathbb{R}^2_{x,y} \times S^1_\theta = M,\\
&\dot{y} = \sin\theta, \qquad && |u| \leq 1,\\
&\dot{\theta} = u, && \\
&q(0) = q_0, \qquad && q(t_1) = q_1,\\
&t_1 \to \min.
\end{aligned}
$$

4. The sub-Riemannian problem on the group of motions of the plane

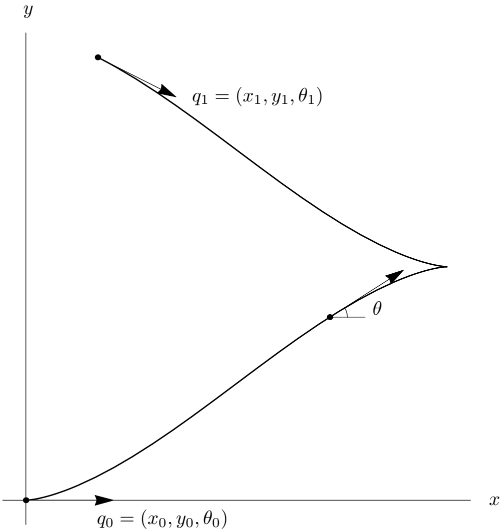
Given:

- model of a car in the plane that can move forward or backward with an arbitrary linear velocity and simultaneously rotate it with an arbitrary angular velocity
- state of the car is given by its position in the plane and orientation angle
- an initial and a terminal state of the car

Find:

- motion of the car from a given initial state to a given terminal state, so that the length of the path in the space of positions and orientations is as minimum as possible.

4. The sub-Riemannian problem on the group of motions of the plane



$$\dot{x} = u \cos \theta, \qquad q = (x, y, \theta) \in \mathbb{R}^2 \times S^1,$$
$$\dot{y} = u \sin \theta, \qquad (u, v) \in \mathbb{R}^2,$$
$$\dot{\theta} = v,$$
$$q(0) = q_0, \qquad q(t_1) = q_1,$$
$$l = \int_0^{t_1} \sqrt{u^2 + v^2}\, dt \to \min .$$

5. Euler elasticae

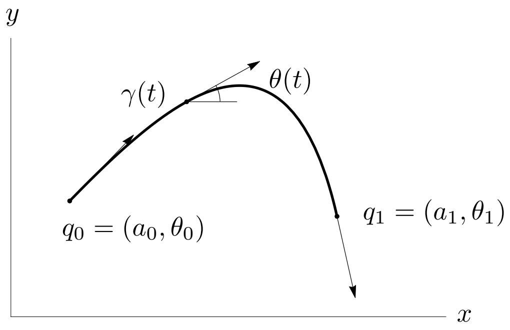
Given:

- uniform elastic rod of length l in the plane
- the rod has fixed endpoints and tangents at endpoints

Find:

- the profile of the rod.

5. Euler elasticae



$$\dot{x} = \cos\theta, \qquad q = (x, y, \theta) \in \mathbb{R}^2 \times S^1,$$
$$\dot{y} = \sin\theta, \qquad u \in \mathbb{R},$$
$$\dot{\theta} = u,$$
$$q(0) = q_0, \qquad q(t_1) = q_1,$$
$t_1 = l$ is the length of the rod,
$$J = \frac{1}{2}\int_0^{t_1} u^2\, dt \to \min .$$

6. The plate-ball problem

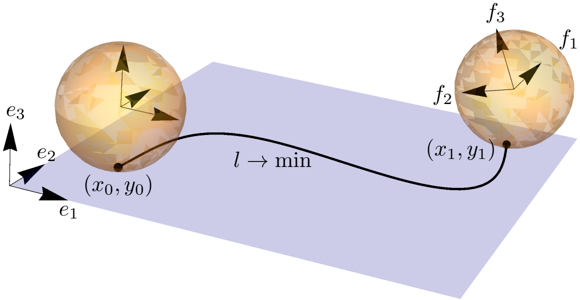
Given:

- uniform sphere roll without slipping or twisting on a horizontal plane
- imagine: the sphere rolls between two horizontal planes, a fixed lower one and a moving upper one
- absence of slipping: the contact point of the sphere with the plane has zero instantaneous velocity
- absence of twisting means that the angular velocity vector of the sphere is horizontal
- admissible motions are obtained by horizontal motions of the upper plane
- initial and terminal states of the sphere.

Find:

- roll the sphere so that the length of the curve in the plane traced by the contact point was the minimum possible.

6. The plate-ball problem



$$\dot{x} = u, \quad \dot{y} = v, \qquad (u, v) \in \mathbb{R}^2,$$

$$\dot{R} = R \begin{pmatrix} 0 & 0 & -u \\ 0 & 0 & -v \\ u & v & 0 \end{pmatrix},$$

$$q = (x, y, R) \in \mathbb{R}^2 \times \mathrm{SO}(3),$$

$$q(0) = q_0, \qquad q(t_1) = q_1,$$

$$l = \int_0^{t_1} \sqrt{u^2 + v^2}\, dt \to \min .$$

7. Anthropomorphic curve reconstruction

Given:

- greyscale image as a set of isophotes (level lines of brightness)
- image corrupted in some domain.

Find:

- anthropomorphic reconstruction of the image.

7. Anthropomorphic curve reconstruction

- D. Hubel and T. Wiesel (1981 Nobel Prize): a human brain stores curves not as sequences of planar points (x_i, y_i), but as sequences of positions and orientations (x_i, y_i, θ_i)
- model of the primary visual cortex $V1$ of the human brain by J. Petitot, G. Citti and A. Sarti: corrupted curves of images are reconstructed according to a variational principle
- human brain lifts images $(x(t), y(t))$ from the plane to the space of positions and orientations $(x(t), y(t), \theta(t))$.

$$\dot{x} = u \cos\theta, \quad \dot{y} = u \sin\theta, \quad \dot{\theta} = v,$$

$$J = \int_0^{t_1} (u^2 + v^2)\, dt \to \min \quad \Leftrightarrow \quad l = \int_0^{t_1} \sqrt{u^2 + v^2}\, dt \to \min,$$

- Optimal trajectories for the sub-Riemannian problem on the group of motions of the plane solve the problem of anthropomorphic curve reconstruction!

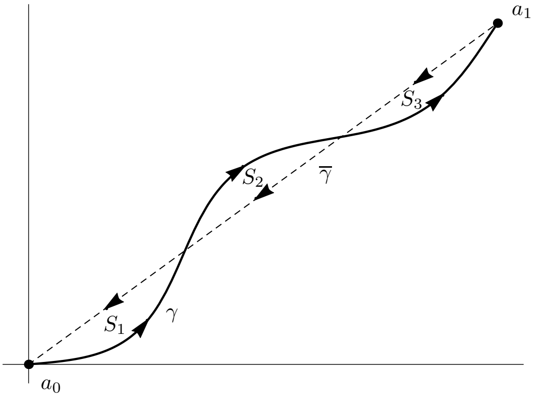
8. Dido's problem

Given:

- points $a_0, a_1 \in \mathbb{R}^2$
- Lipschitzian curve $\overline{\gamma} \subset \mathbb{R}^2$ connecting a_1 with a_0
- number $S \in \mathbb{R}$.

Find:

- the shortest Lipschitzian curve $\gamma \subset \mathbb{R}^2$ connecting a_0 with a_1 for which the closed curve $\gamma \cup \overline{\gamma}$ bounds a domain in $\mathbb{R}^2$ of the algebraic area S.



8. Dido's problem

- coordinates x, y in the plane $\mathbb{R}^2$ with the origin a_0. Then $a_0 = (0,0)$, $a_1 = (x_1, y_1)$, $\gamma(t) = (x(t), y(t))$, $t \in [0, t_1]$, $\overline{\gamma}(t) = (\overline{x}(t), \overline{y}(t))$, $t \in [0, \overline{t}_1]$.
- closed curve $\widehat{\gamma} = \gamma \cup \overline{\gamma}$ and a domain bounded by it: $D \subset \mathbb{R}^2$, $\partial D = \widehat{\gamma}$

$$S(D) = \frac{1}{2} \oint_{\widehat{\gamma}} x\,dy - y\,dx = \frac{1}{2} \int_0^{t_1} (x\dot{y} - y\dot{x})\,dt - \overline{I},$$

$$\dot{x}(t) =: u_1(t), \qquad \dot{y}(t) =: u_2(t), \qquad \dot{z}(t) = \frac{1}{2}(xu_2 - yu_1).$$

$$\dot{q} = u_1 X_1(q) + u_2 X_2(q), \qquad u = (u_1, u_2) \in \mathbb{R}^2, \quad q = (x, y, z) \in \mathbb{R}^3,$$

$$X_1 = \frac{\partial}{\partial x} - \frac{y}{2}\frac{\partial}{\partial z}, \qquad X_2 = \frac{\partial}{\partial y} + \frac{x}{2}\frac{\partial}{\partial z},$$

$$q(0) = q_0 = (0,0,0), \qquad q(t_1) = q_1 = (x_1, y_1, z_1),$$

$$l(\gamma) = \int_0^{t_1} \sqrt{\dot{x}^2 + \dot{y}^2}\,dt = \int_0^{t_1} \sqrt{u_1^2 + u_2^2}\,dt \to \min.$$

Dynamical systems and control systems

- *smooth dynamical system*, or an *ordinary differential equation*:

$$\dot{q} = f(q), \qquad q \in M$$

- deterministic: given an initial condition $q(0) = q_0$ and a time $t > 0$, there exists a unique solution $q(t)$
- *control system*:

$$\dot{q} = f(q, u), \qquad q \in M, \quad u \in U. \tag{1}$$

- *control* function $u = u(t) \in U \quad \Rightarrow \quad$ a nonautonomous ODE

$$\dot{q} = f(q, u(t)). \tag{2}$$

- Together with an initial condition $q(0) = q_0$, ODE (2) determines a unique solution — a *trajectory* $q_u(t)$.
- Regularity assumptions for controls $u(\cdot)$: piecewise constant, L^∞, ...

Attainable sets

- *Attainable set* of control system (1) from a point q_0 for arbitrary times:

$$\mathcal{A}_{q_0} = \{q_u(t) \mid q_u(0) = q_0, \quad u \in L^\infty([0,t],U), \quad t \geq 0\}.$$

- attainable set from the point q_0 for a time $t_1 \geq 0$:

$$\mathcal{A}_{q_0}(t_1) = \{q_u(t_1) \mid q_u(0) = q_0, \quad u \in L^\infty([0,t_1],U)\},$$

- attainable set from the point q_0 for times not greater than $t_1 \geq 0$:

$$\mathcal{A}_{q_0}(\leq t_1) = \bigcup_{t=0}^{t_1} \mathcal{A}_{q_0}(t).$$

Controllability problem

A control system (1) is called:

- *globally (completely) controllable* if $\mathcal{A}_{q_0} = M$ for any $q_0 \in M$
- *globally controllable from a point* $q_0 \in M$ if $\mathcal{A}_{q_0} = M$
- *locally controllable at* q_0 if $q_0 \in \operatorname{int} \mathcal{A}_{q_0}$
- *small time locally controllable (STLC) at* q_0 if $q_0 \in \operatorname{int} \mathcal{A}_{q_0}(\leq t_1)$ for any $t_1 > 0$

- Local controllability problem: necessary conditions and sufficient conditions of STLC for arbitrary dimension of the state space M, but local controllability tests are available only for the case $\dim M = 2$.
- Global controllability problem: global controllability conditions only for very symmetric systems (linear systems, left-invariant systems on Lie groups).

Optimal control problem

- Let $q_1 \in \mathcal{A}_{q_0}$. Then typically q_0, q_1 are connected by continuum of trajectories
- *Cost functional* to be minimized:

$$J = \int_0^{t_1} \varphi(q, u)\, dt.$$

- *Optimal control problem*:

$$\begin{aligned} &\dot{q} = f(q, u), \qquad q \in M, \quad u \in U, \\ &q(0) = q_0, \qquad q(t_1) = q_1, \\ &J = \int_0^{t_1} \varphi(q, u)\, dt \to \min. \end{aligned}$$

- Other important mathematical control problems: equivalence, stabilization, observability, etc., which we do not touch upon.

Smooth manifolds

A smooth k-dimensional *submanifold* $M \subset \mathbb{R}^n$ is usually defined by one of the following equivalences:

(a) implicitly by a system of regular equations:

$$f_1(x) = \cdots = f_{n-k}(x) = 0, \qquad x \in \mathbb{R}^n,$$
$$\operatorname{rank}\left(\frac{\partial f_1}{\partial x}, \ldots, \frac{\partial f_{n-k}}{\partial x}\right) = n - k,$$

(b) or by a regular parametrization:

$$x = \Phi(y), \qquad y \in \mathbb{R}^k, \quad x \in \mathbb{R}^n,$$
$$\operatorname{rank} \frac{\partial \Phi}{\partial y} = k.$$

An abstract smooth *manifold* M (not embedded into $\mathbb{R}^n$) is defined via a system of charts (local coordinates) that mutually agree.

Tangent vectors and spaces

A mapping between smooth manifolds is called *smooth* if it is smooth (of class C^∞) in local coordinates.

The *tangent space* to a smooth submanifold $M \subset \mathbb{R}^n$ at a point $x \in M$ is defined as follows for the two definitions above of a submanifold:

(a) $T_x M = \operatorname{Ker} \frac{\partial f}{\partial x}(x)$,

(b) $T_x M = \operatorname{Im} \frac{\partial \Phi}{\partial y}(y)$, $x = \Phi(y)$.

Given a smooth mapping $F : M \to N$ between smooth manifolds, for any $q \in M$ the *differential* $F_{*q} : T_q M \to T_{F(q)} N$ is defined as follows:

$$F_{*q} v = \left. \frac{d}{dt} \right|_{t=0} F(\gamma(t)),$$

where $\gamma : (-\varepsilon, \varepsilon) \to M$ is a smooth curve such that $\gamma(0) = q$, $\dot{\gamma}(0) = v$.

Smooth vector fields and their commutativity

- A smooth *vector field* on a manifold M is a smooth mapping $M \ni q \mapsto V(q) \in T_qM$. Notation: $V \in \mathrm{Vec}(M)$.
- A *trajectory of a vector field* V through a point $q_0 \in M$ is a solution to the Cauchy problem $\dot{q}(t) = V(q(t)), \quad q(0) = q_0$.
- Suppose that a trajectory $q(t)$ exists for all times $t \in \mathbb{R}$, then we denote $e^{tV}(q_0) := q(t)$. The one-parameter group of diffeomorphisms $e^{tV} : M \to M$ is the *flow* of the vector field V.
- We say that vector fields V and W *commute* if their flows commute:

$$e^{tV} \circ e^{sW} = e^{sW} \circ e^{tV}, \qquad t, s \in \mathbb{R}.$$

- In the general case vector fields V and W do not commute and the curve

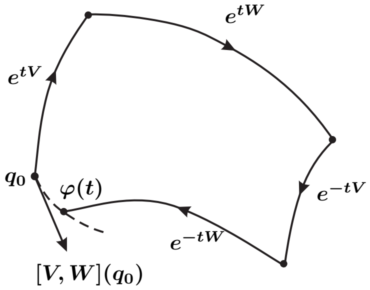
$$\varphi(t) = e^{-tW} \circ e^{-tV} \circ e^{tW} \circ e^{tV}(q_0)$$

  satisfies the inequality $\varphi(t) \neq q_0$, $t \in \mathbb{R}$.
- The leading nontrivial term of the Taylor expansion of $\varphi(t)$, $t \to 0$, is taken as the measure of noncommutativity of vector fields V and W.

Lie brackets of vector fields

- The *commutator* (*Lie bracket*) of the vector fields V, W at the point q_0 is defined as $[V, W](q_0) := \frac{1}{2}\ddot{\varphi}(0)$, so that

$$\varphi(t) = q_0 + t^2[V, W](q_0) + o(t^2), \qquad t \to 0.$$



- In local coordinates $[V, W] = \frac{\partial W}{\partial x} V - \frac{\partial V}{\partial x} W$.

Example: Car in the plane

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{pmatrix} = u \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix} + v \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \qquad V = \cos\theta \frac{\partial}{\partial x} + \sin\theta \frac{\partial}{\partial y}, \quad W = \frac{\partial}{\partial \theta}.$$

$$[V, W] = \frac{\partial W}{\partial q} V - \frac{\partial V}{\partial q} W = 0 \cdot V - \begin{pmatrix} 0 & 0 & -\sin\theta \\ 0 & 0 & \cos\theta \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \sin\theta \\ -\cos\theta \\ 0 \end{pmatrix}.$$

Another way of computing Lie brackets, via commutator of differential operators:

$$[V, W] = V \circ W - W \circ V = \left(\cos\theta \frac{\partial}{\partial x} + \sin\theta \frac{\partial}{\partial y} \right) \frac{\partial}{\partial \theta} - \frac{\partial}{\partial \theta} \left(\cos\theta \frac{\partial}{\partial x} + \sin\theta \frac{\partial}{\partial y} \right)$$

$$= \sin\theta \frac{\partial}{\partial x} - \cos\theta \frac{\partial}{\partial y}.$$

Example: Car in the plane

- Notice the visual meaning of the vector fields $V, W, [V, W]$ for the car in the plane:
  - V generates the motion forward
  - W generates rotations of the car
  - $[V, W]$ generates motion of the car in the direction perpendicular to its orientation, thus physically forbidden.
- Choosing alternating motions of the car:

$$\text{forward} \rightarrow \text{rotation counter-clockwise} \rightarrow \text{backward} \rightarrow \text{rotation clockwise},$$

we can move the car infinitesimally in the forbidden direction. So the Lie bracket $[V, W]$ is generated by a car during parking manoeuvres in a limited space.