Symmetries and Maxwell points in the plate-ball problem

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Nonholonomic days in Pereslavl

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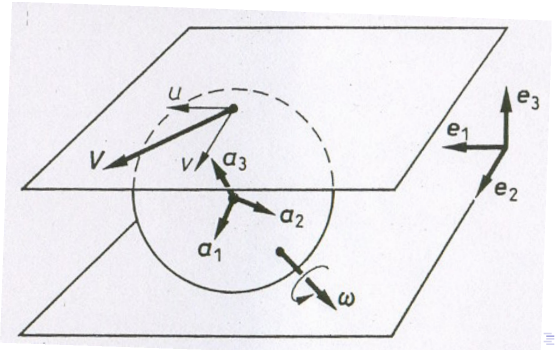
The plate-ball problem

Rolling of sphere on plane without slipping or twisting

Given: $A, B \in \mathbb{R}^2$, frames (a_1, a_2, a_3), (b_1, b_2, b_3) in $\mathbb{R}^3$.

Find: $\gamma(t) \in \mathbb{R}^2$, $t \in [0, t_1]$, — the shortest curve s.t.:
$\gamma(0) = A$, $\gamma(t_1) = B$,
by rolling along $\gamma(t)$, orientation of the sphere transfers from (a_1, a_2, a_3) to (b_1, b_2, b_3).



State and control variables

- Contact point $(x, y) \in \mathbb{R}^2$
- Orientation of sphere $R : a_i \mapsto e_i, \ i = 1, 2, 3, \quad R \in \mathrm{SO}(3)$
- State of the system $Q = (x, y, R) \in \mathbb{R}^2 \times \mathrm{SO}(3) = M$
- Boundary conditions $Q(0) = Q_0, \ Q(t_1) = Q_1$
- Controls $u_1 = u/2, \ u_2 = v/2$
- Cost functional
$$I(\gamma) = \int_0^{t_1} \sqrt{\dot{x}^2 + \dot{y}^2}\, dt = \int_0^{t_1} \sqrt{u_1^2 + u_2^2}\, dt \to \min$$

Control system

$$\dot{x} = u_1, \qquad \dot{y} = u_2, \qquad (x,y) \in \mathbb{R}^2, \quad (u_1,u_2) \in \mathbb{R}^2,$$

$$\dot{R} = R\Omega, \qquad R \in \mathrm{SO}(3), \qquad \Omega = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix},$$

$$\omega = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} \text{ angular velocity vector.}$$

No twisting $\quad \Rightarrow \quad \omega_3 = 0.$

No slipping $\quad \Rightarrow \quad \omega_1 = u_2,\ \omega_2 = -u_1.$

$$\Omega = \begin{pmatrix} 0 & 0 & -u_1 \\ 0 & 0 & -u_2 \\ u_1 & u_2 & 0 \end{pmatrix}$$

History of the problem

1894 H. Hertz: rolling sphere as a nonholonomic mechanical system.

1983 J.M. Hammersley: statement of the plate-ball problem.

1986 A.M. Arthur, G.R.Walsh: integrability of Hamiltonian system of PMP in quadratures.

1990 Z. Li, E. Canny: complete controllability of the control system.

1993 V. Jurdjevic:

- projections of extremal curves $(x(t), y(t))$ — Euler elasticae,
- description of qualitative types of extremal trajectories,
- quadratures for evolution of Euler angles along extremal trajectories.

New results

- Continuous and discrete symmetries
- Fixed points of symmetries (Maxwell points)
- Necessary optimality conditions
- Asymptotics of extremal trajectories and limit behavior of Maxwell points for sphere rolling on sinusoids of small amplitude
- Global structure of the exponential mapping

Existence of solutions

- Left-invariant sub-Riemannian problem:

$$\dot{Q} = u_1 X_1(Q) + u_2 X_2(Q), \qquad (u_1, u_2) \in \mathbb{R}^2,$$
$$Q(0) = Q_0, \qquad Q(t_1) = Q_1, \qquad Q \in M = \mathbb{R}^2 \times \mathrm{SO}(3),$$
$$l = \int_0^{t_1} \sqrt{u_1^2 + u_2^2}\, dt \to \min .$$

- Complete controllability by Rashevskii-Chow theorem:

$$\operatorname{span}_Q(X_1, X_2, X_3, X_4, X_5) = T_Q M \quad \forall\ Q \in M,$$
$$X_3 = [X_1, X_2], \qquad X_4 = [X_1, X_3], \qquad X_5 = [X_2, X_3].$$

- Filippov's theorem: $\forall Q_0, Q_1 \in M$ optimal trajectory exists.
- $Q_0 = (0, 0, \mathrm{Id}) \in \mathbb{R}^2 \times \mathrm{SO}(3)$.

Pontryagin maximum principle

- Abnormal extremal trajectories: rolling of sphere along straight lines.
- Normal extremals:

$$\dot\theta = c, \quad \dot c = -r\sin\theta, \quad \dot\alpha = \dot r = 0, \tag{1}$$

$$\dot x = \cos(\theta+\alpha), \quad \dot y = \sin(\theta+\alpha), \tag{2}$$

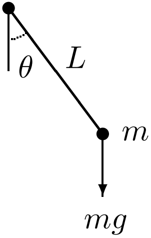
$$\dot R = R(\sin(\theta+\alpha)A_1 - \cos(\theta+\alpha)A_2), \tag{3}$$

$$A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix},$$

$$A_3 = [A_1, A_2] = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

(1) mathematical pendulum,
(2) Euler elasticae.

Mathematical pendulum $\dot{\theta} = c$, $\dot{c} = -r \sin\theta$

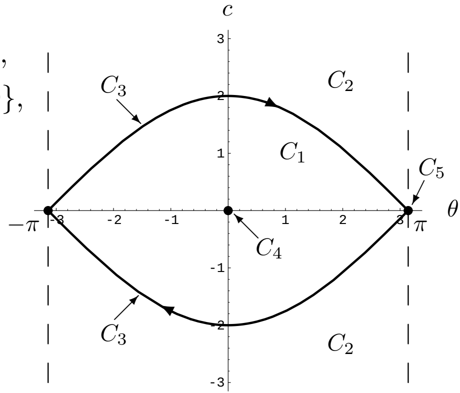


- $\lambda = (\theta, c, r) \in C = \{\theta \in S^1,\ c \in \mathbb{R},\ r \geq 0\}$,
- Energy $E = c^2/2 - r\cos\theta = \text{const} \in [-r, +\infty)$,
- $r = g/L \geq 0$.

Decomposition of the phase cylinder C of pendulum

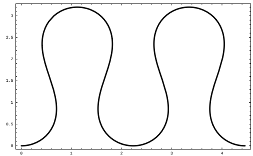
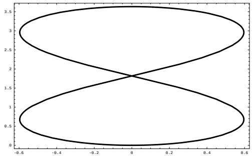
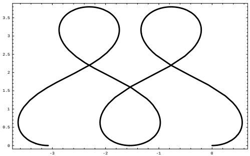
$$C = \cup_{i=1}^{7} C_i, \qquad C_i \cap C_j = \emptyset, \quad i \neq j$$

$$C_1 = \{\lambda \in C \mid E \in (-r, r),\ r > 0\},$$
$$C_2 = \{\lambda \in C \mid E \in (r, +\infty),\ r > 0\},$$
$$C_3 = \{\lambda \in C \mid E = r > 0,\ c \neq 0\},$$
$$C_4 = \{\lambda \in C \mid E = -r,\ r > 0\},$$
$$C_5 = \{\lambda \in C \mid E = r > 0,\ c = 0\},$$
$$C_6 = \{\lambda \in C \mid r = 0,\ c \neq 0\},$$
$$C_7 = \{\lambda \in C \mid r = 0,\ c = 0\}.$$

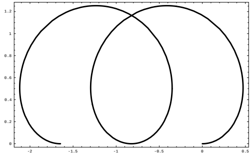


Euler elasticae $\dot{x} = \cos(\theta + \alpha)$, $\dot{y} = \sin(\theta + \alpha)$

C_1: inflectional elasticae

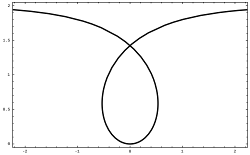




C_2: non-inflectional elasticae



Euler elasticae $\dot{x} = \cos(\theta + \alpha)$, $\dot{y} = \sin(\theta + \alpha)$

C_3: critical elasticae



C_4, C_5, C_7: straight lines

C_6: circles

Integration of normal Hamiltonian system of PMP

$$\dot\theta = c, \qquad \dot c = -r\sin\theta, \qquad \dot x = \cos(\theta+\alpha), \qquad \dot y = \sin(\theta+\alpha),$$
$$\dot R = R(\sin(\theta+\alpha)A_1 - \cos(\theta+\alpha)A_2).$$

- $\theta(t)$, $c(t)$, $x(t)$, $y(t)$: Jacobi's functions cn, sn, dn, E,

$$\operatorname{cn}(u,k) = \cos(\operatorname{am}(u,k)),$$
$$\varphi = \operatorname{am}(u,k) \iff u = \int_0^\varphi \frac{dt}{\sqrt{1-k^2\sin^2 t}} = F(\varphi,k).$$

- $R(t) = e^{(\alpha-\varphi_3^0)A_3} e^{-\varphi_2^0 A_2} e^{\varphi_1(t)A_3} e^{\varphi_2(t)A_2} e^{(\varphi_3(t)-\alpha)A_3}$,
  $\varphi_i(t)$: Jacobi's functions + elliptic integral of the 3-rd kind

$$\Pi(n,u,k) = \int_0^u \frac{dt}{(1-n\sin^2 t)\sqrt{1-k^2\sin^2 t}}.$$

Optimality of extremal trajectories

- Short arcs of extremal trajectories $Q(s)$ are optimal
- Cut time along $Q(s)$:

$$t_{\text{cut}} = \sup\{t > 0 \mid Q(s),\ s \in [0, t], \ \text{ is optimal } \}.$$

- Maxwell time:

$$\exists\ \tilde{Q}(s) \not\equiv Q(s), \quad Q(0) = \tilde{Q}(0) = Q_0,$$
$$Q(t) = \tilde{Q}(t) \text{ Maxwell point,}$$
$$t = t_{\text{Max}} \text{ Maxwell time.}$$

- Upper bound on cut time:

$$t_{\text{cut}} \leq t_{\text{Max}}.$$

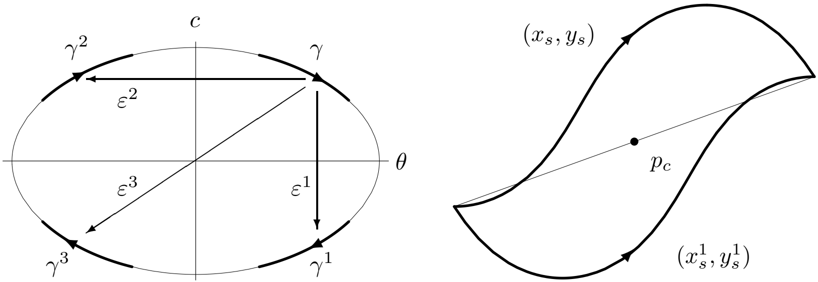
Rotations Φ^β, $\beta \in S^1$

$$(\theta, c, r, \alpha) \mapsto (\theta, c, r, \alpha + \beta),$$

$$\begin{pmatrix} x_s \\ y_s \end{pmatrix} \mapsto \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix} \begin{pmatrix} x_s \\ y_s \end{pmatrix},$$

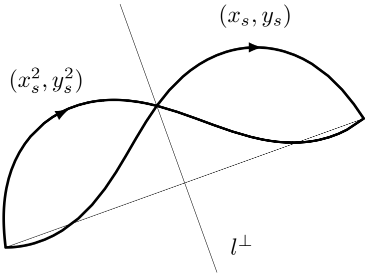
$$R_s \mapsto e^{\beta A_3} R_s e^{-\beta A_3}.$$

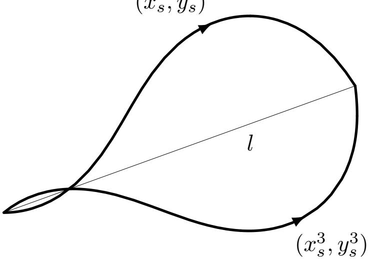
Reflections ε^i



ε^1: $(\theta_s, c_s) \mapsto (\theta_{t-s}, -c_{t-s}),\ s \in [0, t]$
$(x_s, y_s) \mapsto (x_s^1, y_s^1) = (x_{t-s} - x_t, y_{t-s} - y_t)$
$R_s \mapsto (R_t)^{-1} R_{t-s}$

Reflections ε^i





ε^2: $(\theta_s, c_s) \mapsto (-\theta_{t-s}, c_{t-s})$, $s \in [0, t]$
$(x_s, y_s) \mapsto (x_s^2, y_s^2) = (x_{t-s} - x_t, y_t - y_{t-s})$
$R_s \mapsto I_2(R_t)^{-1} R_{t-s} I_2$, $I_2 = e^{\pi A_2}$.

ε^3: $(\theta_s, c_s) \mapsto (-\theta_s, -c_s)$, $s \in [0, t]$
$(x_s, y_s) \mapsto (x_s^3, y_s^3) = (x_s, -y_s)$
$R_s \mapsto I_2 R_s I_2$.

Exponential mapping and its symmetries

- Group of symmetries
  $G = \langle \Phi^\beta, \varepsilon^1, \varepsilon^2, \varepsilon^3 \rangle = \{\Phi^\beta,\ \Phi^\beta \circ \varepsilon^i \mid \beta \in S^1,\ i = 1,2,3\}$
- Exponential mapping

$$\mathrm{Exp}(\lambda, s) = Q_s = (x_s, y_s, R_s) \in M = \mathbb{R}^2 \times \mathrm{SO}(3),$$
$$\lambda = (\theta, c, \alpha, r) \in C, \qquad s > 0.$$

- Symmetries of exponential mapping

$$\begin{tikzcd} C \times \mathbb{R}_+ \arrow[r, "\mathrm{Exp}"] \arrow[d, "\varepsilon^i \circ \Phi^\beta"] & M \arrow[d, "\varepsilon^i \circ \Phi^\beta"] \\ C \times \mathbb{R}_+ \arrow[r, "\mathrm{Exp}"] & M \end{tikzcd} \qquad \begin{tikzcd} (\lambda, t) \arrow[r, mapsto, "\mathrm{Exp}"] \arrow[d, mapsto, "\varepsilon^i \circ \Phi^\beta"] & Q_t \arrow[d, mapsto, "\varepsilon^i \circ \Phi^\beta"] \\ (\lambda^{i,\beta}, t) \arrow[r, mapsto, "\mathrm{Exp}"] & Q_t^{i,\beta} \end{tikzcd}$$

Maxwell sets corresponding to reflections

- $\mathrm{MAX}^i = \{(\lambda, t) \mid \exists \beta \in S^1 \ : \ \lambda^{i,\beta} \neq \lambda, \ Q_t = Q_t^{i,\beta}\}$, $i = 1, 2, 3$.

- Necessary optimality conditions:

$$(\lambda, t) \in \mathrm{MAX}^i \quad \Rightarrow \quad \begin{array}{l} Q_s = \mathrm{Exp}(\lambda, s) \text{ not optimal for } s > t, \\ t_{\mathrm{cut}}(\lambda) \leq t. \end{array}$$

Representation of rotations in $\mathbb{R}^3$ by quaternions

- $\mathbb{H} = \{q = q_0 + iq_1 + jq_2 + kq_3 | q_0, \dots, q_3 \in \mathbb{R}\}$
- $S^3 = \{q \in \mathbb{H} | |q|^2 = q_0^2 + q_1^2 + q_2^2 + q_3^2 = 1\}$
- $I = \{q \in \mathbb{H} | \mathrm{Re}\, q = q_0 = 0\}$
- $q \in S^3 \Rightarrow R_q(a) = qaq^{-1}, \quad a \in I, \quad R_q \in \mathrm{SO}(3) \cong \mathrm{SO}(I)$
- lift of the system $\dot{R} = R\Omega$ from SO(3) to S^3:

$$\begin{cases} \dot{q}_0 = \frac{1}{2}(q_2 u_1 - q_1 u_2), \\ \dot{q}_1 = \frac{1}{2}(q_3 u_1 + q_0 u_2), \\ \dot{q}_2 = \frac{1}{2}(-q_0 u_1 + q_3 u_2), \\ \dot{q}_3 = \frac{1}{2}(-q_1 u_1 - q_2 u_2), \end{cases} \qquad q \in S^3, \quad (u_1, u_2) \in \mathbb{R}^2,$$

$$q(0) = 1.$$

Necessary optimality conditions in terms of MAX^1

Theorem

Let $Q_s = (x_s, y_s, R_s) = \mathrm{Exp}(\lambda, s)$, $t > 0$ *satisfy the conditions:*

(1) $q_3(t) = 0$,

(2) *elastica* $\{(x_s, y_s) \mid s \in [0, t]\}$ *is nondegenerate and not centered at inflection point.*

Then $(\lambda, t) \in \mathrm{MAX}^1$, *thus for any* $t_1 > t$ *the trajectory* Q_s, $s \in [0, t_1]$, *is not optimal.*

$$q_3(t) = 0 \iff \text{axis of rotation } (q_1(t), q_2(t), q_3(t)) \parallel \mathbb{R}^2$$

Necessary optimality conditions in terms of MAX^2

Theorem

Let $Q_s = (x_s, y_s, R_s) = \mathrm{Exp}(\lambda, s)$, $t > 0$ satisfy the conditions:

(1) $(xq_1 + yq_2)(t) = 0$,

(2) *elastica $\{(x_s, y_s) \mid s \in [0, t]\}$ is nondegenerate and not centered at vertex.*

Then $(\lambda, t) \in \mathrm{MAX}^2$, thus for any $t_1 > t$ the trajectory Q_s, $s \in [0, t_1]$, is not optimal.

$$(xq_1 + yq_2)(t) = 0 \iff (q_1(t), q_2(t), q_3(t)) \perp (x(t), y(t), 0)$$

Asymptotics of extremal trajectories near $(\theta, c) = (0, 0)$

- $m = \sqrt{r}, \quad s = mt, \quad d = c/m,$
  $\begin{pmatrix} x \\ y \end{pmatrix} = A(\alpha) \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix}, \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = A(\alpha) \begin{pmatrix} \bar{q}_1 \\ \bar{q}_2 \end{pmatrix}$
  $A(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}, \ q_0 = \bar{q}_0, \ q_3 = \bar{q}_3.$
- $\rho_0 = \sqrt{\theta_0^2 + d_0^2} \to 0.$
- oscillations of pendulum $\theta' = d, \ d' = -\sin\theta$:

$$\theta(s) = \theta_0 \cos s + d_0 \sin s + O(\rho_0^2).$$

- elasticae $\approx$ sinusoids of small amplitude:

$$\bar{x}(s) = \frac{s}{m} + O(\rho_0^2),$$
$$\bar{y}(s) = \frac{1}{m}(\theta_0 \sin s + d_0(1 - \cos s)) + O(\rho_0^2).$$

Asymptotics of extremal trajectories near $(\theta, c) = (0, 0)$

$$\bar{q}_0(s) = \cos\frac{s}{2m} + O(\rho_0^2),$$

$$\bar{q}_1(s) = \frac{1}{2(m^2-1)}\left(m\cos\frac{s}{2m}\sin s - (1+\cos s)\sin\frac{s}{2m}\right)\theta_0 + \\ + \frac{1}{2(m^2-1)}\left(m(1-\cos s)\cos\frac{s}{2m} - \sin s\sin\frac{s}{2m}\right)d_0 + O(\rho_0^2),$$

$$\bar{q}_2(s) = -\sin\frac{s}{2m} + O(\rho_0^2),$$

$$\bar{q}_3(s) = \frac{1}{2(m^2-1)}\left((-1+\cos s)\cos\frac{s}{2m} + m\sin s\sin\frac{s}{2m}\right)\theta_0 + \\ + \frac{1}{2(m^2-1)}\left(\sin s\cos\frac{s}{2m} - m(1+\cos s)\sin\frac{s}{2m}\right)d_0 + O(\rho_0^2).$$

Limit behavior of Maxwell set MAX^1 as $\rho_0 \to 0$

- Th.: $q_3(t) = 0$, elastica not centered at inflection point $\Rightarrow \quad (\lambda, t) \in \mathrm{MAX}^1$.
- $q_3 = \frac{d_0 \cos p - \theta_0 \sin p}{m^2 - 1} \cdot g_1(p, m) + O(\rho_0^2), \ p = mt/2.$
- $g_1(p, m) = \cos \frac{p}{m} \sin p - m \cos p \sin \frac{p}{m}$
- $p_1(m) = \min\{p > 0 | g_1(p, m) = 0\}$
- $Q_t = \mathrm{Exp}(\lambda, t), \ \lambda = (\theta_0, d_0, m, \alpha) \in C_1,$
  $\rho_0 \to 0, \ \ m \to \bar{m} \quad \Rightarrow$

  $\forall \varepsilon > 0 \ \exists t \in (t_1 + \varepsilon, t_1 + \varepsilon)$ such that $(\lambda, t) \in \mathrm{MAX}^1$,
  trajectory Q_t not optimal at $t \in [0, t_1 + \varepsilon], \ t_1 = 2p_1(\bar{m})/\bar{m}$.

Bounds of the first root of equation $g_1(p, m) = 0$

Theorem

For any $m \in (0,1) \cup (1, +\infty)$,

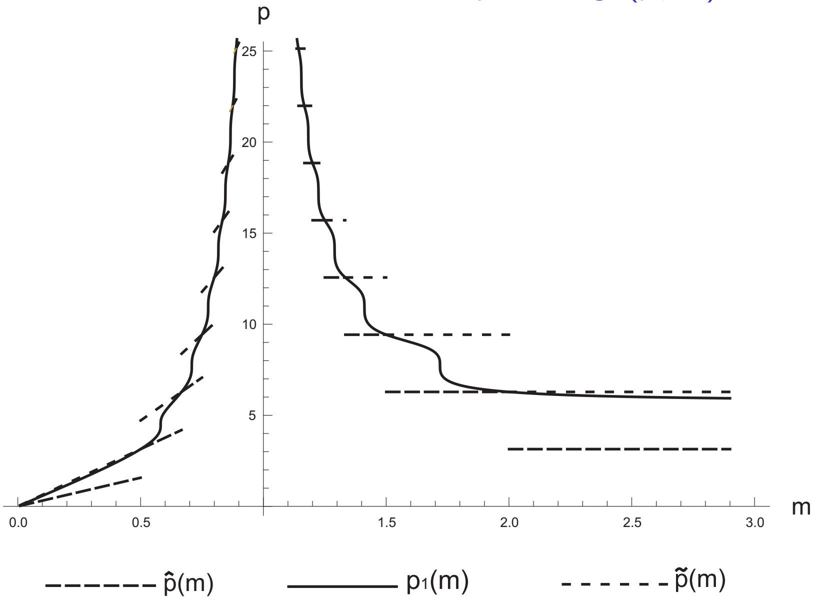
$$\hat{p}(m) \leq p_1(m) < \tilde{p}(m),$$

where

$$\hat{p}(m) = \begin{cases} m\rho, \ \ \textit{for } m \in (0, \frac{1}{2}], \\ m\pi([\frac{m}{1-m}] + 1), \ \ \textit{for } m \in (\frac{1}{2}, 1), \\ \pi([\frac{1}{m-1}] + 1), \ \ \textit{for } m \in (1, 2], \\ \rho, \ \ \textit{for } m > 2, \quad \rho = \tan\rho, \ \ \rho \in \left(\pi, \frac{3\pi}{2}\right) \end{cases}$$

$$\tilde{p}(m) = \begin{cases} m\pi([\frac{m}{1-m}] + 2), \ \ \textit{for } m \in (0, 1), \\ \pi([\frac{1}{m-1}] + 2), \ \ \textit{for } m \in (1, +\infty). \end{cases}$$

Bounds of the first root of equation $g_1(p, m) = 0$



Properties of the first root of equation $g_1(p, m) = 0$

Theorem

a) $\lim_{m \to 1} p_1(m) = +\infty$;

b) $p_1(m)$ *grows for* $m \in (0, 1)$ *and decreases for* $m \in (1, +\infty)$;

c) $p_1(1 + \frac{1}{n}) = \pi(n + 1)$, $p_1(\frac{n}{n+1}) = \pi n$,
$p_1(1 + \frac{2}{2n+1}) = \pi(n + \frac{3}{2})$, $p_1(\frac{2n+1}{2n+3}) = \pi(n + \frac{1}{2})$;

d) $p_1(m)$ *is continuous for* $m \in (0, 1) \cup (1, +\infty)$;

e) $p_1(m)$ *is smooth for*

$$m \in (0, 1) \cup (1, +\infty) \backslash (\{\frac{n}{n+1} | n \in \mathbb{N}\} \cup \{\frac{n+1}{n} | n \in \mathbb{N}\}).$$

$$\forall\ n \in \mathbb{N} :\ \ p_1'(\tfrac{n}{n+1}) = +\infty,\ p_1'(\tfrac{n+1}{n}) = -\infty.$$

Limit behavior of Maxwell set MAX^1

- $d = \rho\cos\chi$, $\theta = \rho\sin\chi$
- $\bar\chi \in S^1$, $\bar m > 0$, $\bar m \neq 1$
- $D_\delta = \{\lambda = (\rho, \chi, m, \alpha) \in C_1 | 0 < \rho < \delta,\ |\chi - \bar\chi| < \delta,\ |m - \bar m| < \delta\}$.
- $t_1 = t_1(\bar m) = 2p_1(\bar m)/\bar m$
- $I_\varepsilon = \{t > 0 | t_1 - \varepsilon < t < t_1 + \varepsilon\}, \quad \varepsilon > 0.$

Theorem

Let $\bar m \neq 1$, $\cos(\bar\chi + t_1\bar m/2) \neq 0$, *then*

$$\forall \varepsilon > 0\ \exists \delta = \delta(\epsilon) > 0\ \forall \lambda \in D_\delta\ \exists t \in I_\varepsilon \text{ such that } (\lambda, t) \in \mathrm{MAX}^1 .$$

Bounds on cut time near $(\theta, c) = (0, 0)$

Theorem

Let $\lambda = (\rho, \chi, m, \alpha) \in C_1$ *and* $\bar{m} > 0$, $\bar{m} \neq 1$. *Then*

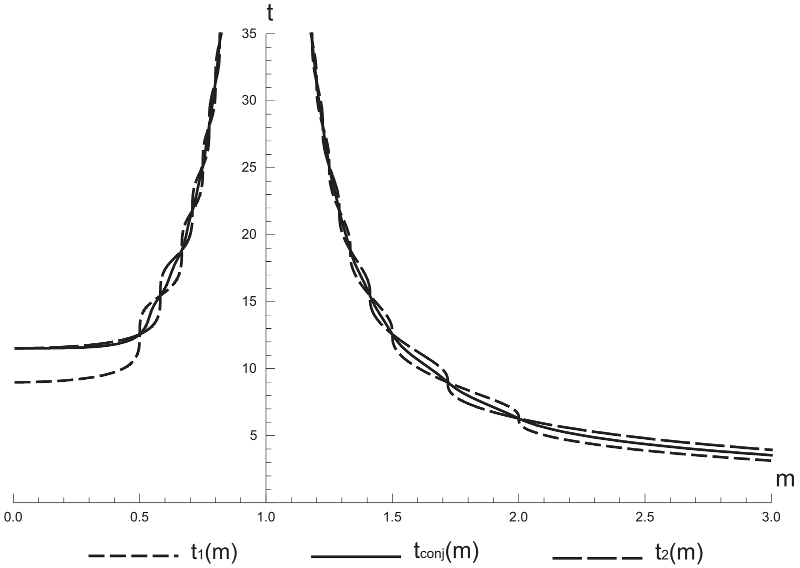
$$\overline{\lim_{\rho \to 0,\, m \to \bar{m}}} t_{\mathrm{cut}}(\lambda) \leq t_1(\bar{m}).$$

Theorem

Let $\lambda = (\rho, \chi, m, \alpha) \in C_1$. *Let* $K \subset \{m \in \mathbb{R} \mid m > 0,\ m \neq 1\}$ *be a compact. Then*

$$\overline{\lim_{\rho \to 0,\, m \in K}} t_{\mathrm{cut}}(\lambda) \leq \max_{m \in K} t_1(m).$$

Maxwell times and conjugate time as $(\theta, c) \to 0$



Global structure of exponential mapping

- $\mathrm{Exp} : N \to M$,
  $N = C \times \mathbb{R}_+ = \{(\theta, c, \alpha, r, t) \mid \theta \in S^1,\ c \in \mathbb{R},\ \alpha \in S^1,\ r \geq 0,\ t > 0\}$,
  $M = \mathbb{R}^2 \times \mathrm{SO}(3)$

- $\forall Q_1 \in M \setminus Q_0 \ \ \exists (\lambda, t) \in N$ such that $Q_s = \mathrm{Exp}(\lambda, s)$ optimal,
  $Q_1 = \mathrm{Exp}(\lambda, t)$
  $t \leq t_{\mathrm{cut}}(\lambda) \leq t^1_{\mathrm{Max}} = \inf\{s \mid (\lambda, s) \in \mathrm{MAX}^1 \cup \mathrm{MAX}^2\}$
  $(\lambda, t) \in \widehat{N} = \{(\lambda, s) \in N \mid \lambda \in C,\ 0 < s \leq t^1_{\mathrm{Max}}\}$
  $\mathrm{Exp} : \widehat{N} \to M \setminus Q_0$ surjective

Global structure of exponential mapping

- Decomposition in the preimage of Exp:
  $\widehat{N} \supset \cup_{i=1}^4 N_i$, $\mathrm{cl}(\cup_{i=1}^4 N_i) \supset \widehat{N}$,
  $N_i = \{(\lambda, t) \in D_i \mid 0 < t < t^1_{\mathrm{Max}}(\lambda)\}$,
  $D_i = \{(\lambda, t) \in N \mid \operatorname{sgn} c_{t/2} = \pm 1, \ \operatorname{sgn} \sin \theta_{t/2} = \pm 1\}$.

- Decomposition in the image of Exp:
  $M \supset M_1 \cup M_2$, $\mathrm{cl}(M_1 \cup M_2) = M$,
  $M_i = \{(x, y, Q) \in M \mid q_3 > 0, \ \operatorname{sgn}(xq_1 + yq_2) = \pm 1\}$.

- Conjecture:
  $\mathrm{Exp} : N_1, \ N_3 \to M_1$ are diffeomorphisms,
  $\mathrm{Exp} : N_2, \ N_4 \to M_2$ are diffeomorphisms.

Steps required to prove the conjecture

- N_i, M_i connected, open (proved: diffeomorphic to $\mathbb{R}^4 \times S^1$),
- $N_i/\{\Phi^\beta \mid \beta \in S^1\}$, $M_i/\{\Phi^\beta \mid \beta \in S^1\}$ simply connected (proved : diffeomorphic to $\mathbb{R}^4$),
- $\mathrm{Exp}(N_1), \mathrm{Exp}(N_3) \subset M_1$, $\mathrm{Exp}(N_2), \mathrm{Exp}(N_4) \subset M_2$ (proved),
- $\mathrm{Exp}(\partial N_i) \subset \partial M_1 \cup \partial M_2$ (proved),
- $\mathrm{Exp} : N_1,\ N_3 \to M_1$, $\mathrm{Exp} : N_2,\ N_4 \to M_2$ proper (partially proved),
- $\mathrm{Exp}|_{N_i}$ nondegenerate (numerical evidence).

Algorithm for solution to the problem (modulo the conjecture)

- $Q_1 \in M_1 \cup M_2 \quad \Rightarrow \quad$ optimal trajectory $Q_s = ?$
- $Q_1 \in M_1 \quad \Rightarrow$

  $\exists!(\lambda_1, t_1) \in N_1$ such that $\text{Exp}(\lambda_1, t_1) = Q_1$,
  $\exists!(\lambda_3, t_3) \in N_3$ such that $\text{Exp}(\lambda_3, t_3) = Q_1$,

  $$\begin{aligned} t_1 < t_3 \quad &\Rightarrow \quad Q_s^1 = \text{Exp}(\lambda_1, s) \text{ optimal},\\ t_1 > t_3 \quad &\Rightarrow \quad Q_s^3 = \text{Exp}(\lambda_3, s) \text{ optimal},\\ t_1 = t_3 \quad &\Rightarrow \quad Q_s^1,\ Q_s^3 \text{ optimal}. \end{aligned}$$
- $Q_1 \in M_2 \quad \Rightarrow \quad$ similarly for $(\lambda_2, t_2) \in N_2$, $(\lambda_4, t_4) \in N_4$.

Results and plans

- parameterization of extremal trajectories,
- symmetries and Maxwell points,
- upper bound on cut time,
- asymptotic case $(\theta, c) \to (0, 0)$,
- global structure of the exponential mapping,
- software for numerical solution to the plate-ball problem (movies for MAX^i).