The Lorentzian Anti-de Sitter Plane*

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Abstract

In this paper the two-dimensional Lorentzian problem on the anti-de Sitter plane is studied. Using methods of geometric control theory and differential geometry, we describe the reachable set, investigate the existence of Lorentzian length maximizers, compute extremal trajectories, construct an optimal synthesis, and characterize Lorentzian distance and spheres.

1 Introduction

Lorentzian geometry serves as the mathematical foundation of general relativity [1, 9, 10]. Unlike Riemannian geometry, here information can spread only along curves with velocity vectors lying within a certain pointed cone A natural problem in this context is finding curves that maximize a length-like functional along admissible curves. Thus, a key objective is to describe Lorentzian length maximizers for all pairs of points where the second point is reachable from the first one via an admissible curve. To the best of our knowledge, this problem has been fully investigated only in the simplest cases: for the left-invariant Lorentzian structure on $\mathbb{R}^n$ (Minkowski space $\mathbb{R}^n_1$) [1], for the 2-dimensional de Sitter plane [11], and for left-invariant Lorentzian metrics on the two-dimensional solvable non-Abelian Lie group [6].

This paper presents a description of Lorentzian length maximizers, distances, and spheres for the 2-dimensional anti-de Sitter plane — a Lorentzian space of constant negative curvature [1]. These results are obtained using methods of geometric control theory [2, 3]. Interestingly, in these problems, Lorentzian length maximizers do not exist for certain reachable pairs of points, and the Lorentzian distance may be infinite for some pairs of points. In these problems, all extremal trajectories (satisfying the Pontryagin maximum principle) are optimal, meaning there are no conjugate points or cut loci. The optimal trajectories, as well as the spheres and distances, are parametrized by elementary functions.

The paper is structured as follows. Section 2 provides the necessary definitions and basic results of Lorentzian geometry. Section 3 describes the construction of the 2-dimensional anti-de Sitter space. The main Section 4 formulates and investigates the problem of Lorentzian length maximizers in this space.

2 Definitions and Preliminary Results

We recall the basic concepts of Lorentzian geometry [1, 6].

Let M be a smooth manifold. A Lorentzian structure on M is a non-degenerate quadratic form g of index 1.

For $q \in M$, a vector $v \in T_qM$ is called:

- timelike if $g(v) < 0$,
- spacelike if $g(v) > 0$ or $v = 0$,
- lightlike (or null) if $g(v) = 0$ and $v \neq 0$,
- causal (or nonspacelike) if $g(v) \leq 0$.

A Lipschitz curve γ on M is called:

- timelike if its velocity vector is timelike almost everywhere,
- spacelike, lightlike, or causal if the corresponding condition holds for its velocity vector.

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Fix an arbitrary timelike vector field X_0 on M. A causal vector $v \in T_qM$ is called:

- future-directed if $g(v, X_0(q)) < 0$,
- past-directed if $g(v, X_0(q)) > 0$.

A future-directed timelike curve $\gamma(t)$, $t \in [0, t_1]$, is called arclength parametrized if $g(\dot\gamma(t), \dot\gamma(t)) \equiv -1$.
The Lorentzian length of a causal curve $\gamma \in \mathrm{Lip}([0, t_1], M)$ is defined as:

$$l(\gamma) = \int_0^{t_1} |g(\dot\gamma, \dot\gamma)|^{1/2}\, dt.$$

For two points $q_0, q_1 \in M$, denote by $\Omega_{q_0 q_1}$ the set of all future-directed causal curves connecting q_0 to q_1. If $\Omega_{q_0 q_1} \neq \emptyset$, the Lorentzian distance from q_0 to q_1 is:

$$d(q_0, q_1) = \sup_{\gamma \in \Omega_{q_0 q_1}} l(\gamma),$$

otherwise, $d(q_0, q_1) := 0$.

A future-directed causal curve γ is called a Lorentzian length maximizer if it realizes the maximal Lorentzian arc length between $\gamma(0) = q_0$ and $\gamma(t_1) = q_1$.

The causal future of a point $q_0 \in M$ is the set:

$$J^+(q_0) = \{q_1 \in M \mid \exists \text{ a future-directed causal curve } \gamma \text{ connecting } q_0 \text{ with } q_1\}.$$

For $q_0 \in M$ and $q_1 \in J^+(q_0)$, finding a Lorentzian length maximizer reduces to solving the following optimization problem:

$$l(\gamma) \to \max, \quad \gamma(0) = q_0, \quad \gamma(t_1) = q_1.$$

Vector fields $X_1, \ldots, X_n \in \mathrm{Vec}(M)$, where $n = \dim M$, form an orthonormal frame for the Lorentzian structure g if for all $q \in M$:

$$g_q(X_1, X_1) = -1, \quad g_q(X_i, X_i) = 1 \ (i = 2, \ldots, n), \quad g_q(X_i, X_j) = 0 \ (i \neq j).$$

Fixing the time orientation by X_1, the Lorentzian problem for a structure with orthonormal frame $X_1, \ldots, X_n$ can be formulated as an optimal control problem:

$$\dot q = \sum_{i=1}^n u_i X_i(q), \quad q \in M, \quad u \in U = \left\{(u_1, \ldots, u_n) \in \mathbb{R}^n \mid u_1 \geq \sqrt{u_2^2 + \cdots + u_n^2}\right\},$$

$$q(0) = q_0, \quad q(t_1) = q_1, \quad l(q(\cdot)) = \int_0^{t_1} \sqrt{u_1^2 - u_2^2 - \cdots - u_n^2}\, dt \to \max.$$

Remark 2.1. The Lorentzian length is invariant under strictly monotonic Lipschitz reparametrizations $t(s)$, $s \in [0, s_1]$. Thus, if $\gamma(t)$, $t \in [0, t_1]$, is a Lorentzian length maximizer, any reparametrization $\gamma(t(s))$, $s \in [0, s_1]$, is also a length maximizer.

In this paper, we primarily use:

- arclength parametrization for timelike trajectories,
- parametrization with $u_1(t) \equiv 1$ for future-directed lightlike trajectories. Alternatively, one may choose $u_1(t) \equiv 1$ for all future-directed causal trajectories.

3 The Two-Dimensional Anti-de Sitter Space

Consider the space $\mathbb{R}^3_2 = \{x = (x_1, x_2, x_3) \mid x_i \in \mathbb{R}\}$ endowed with the pseudo-Euclidean metric $ds^2 = -dx_1^2 - dx_2^2 + dx_3^2$. Define the one-sheeted hyperboloid

$$H^2_1 = \{x \in \mathbb{R}^3_2 \mid -x_1^2 - x_2^2 + x_3^2 = -1\},$$

and parametrize it as

$$x_1 = \cosh\theta \cos\varphi, \quad x_2 = \cosh\theta \sin\varphi, \quad x_3 = \sinh\theta, \qquad \theta \in \mathbb{R}, \ \varphi \in \mathbb{R}/(2\pi\mathbb{Z}), \tag{3.1}$$

with the induced Lorentzian metric $g = ds^2|_{H^2_1}$ on H^2_1.

Definition 3.1. The *two-dimensional anti-de Sitter space* [1] is the simply connected covering manifold of the hyperboloid H_1^2:

$$\widetilde{H_1^2} = \{(\varphi, \theta) \in \mathbb{R}^2\},$$

equipped with the Lorentzian metric $\tilde{g}$ induced by g.

Note that $\tilde{g}$ locally coincides with g.

Vector fields $X_1, X_2 \in \mathrm{Vec}(\widetilde{H_1^2})$ form an orthonormal frame for $\tilde{g}$ if

$$\tilde{g}(X_2, X_2) = -\tilde{g}(X_1, X_1) = 1, \qquad \tilde{g}(X_1, X_2) = 0.$$

Proposition 3.1. *(1) The metric g, and hence $\tilde{g}$, has the following form:*

$$g = -\cosh^2\theta\, d\varphi^2 + d\theta^2.$$

(2) An orthonormal frame for these metrics can be chosen as

$$X_1 = \frac{1}{\cosh\theta}\frac{\partial}{\partial\varphi}, \qquad X_2 = \frac{\partial}{\partial\theta}. \tag{3.2}$$

Proof. Both expressions follow from direct computation. From the parametrization (3.1), we obtain

$$dx_1 = \sinh\theta\cos\varphi\, d\theta - \cosh\theta\sin\varphi\, d\varphi, \quad dx_2 = \sinh\theta\sin\varphi\, d\theta + \cosh\theta\cos\varphi\, d\varphi, \quad dx_3 = \cosh\theta\, d\theta.$$

Substituting into the metric, we derive

$$g = -dx_1^2 - dx_2^2 + dx_3^2 = -\cosh^2\theta\, d\varphi^2 + d\theta^2.$$

Next, we determine the eigenvectors and normalize them with respect to the metric. The eigenvalues of (3) are $\lambda_1 = -\cosh^2\theta$ and $\lambda_2 = 1$, leading to the orthonormal frame (3.2). □

4 Lorentzian Problem on Anti-de Sitter Space

4.1 Problem Statement

The Lorentzian longest curves for the metric $\tilde{g}$ are solutions to the following optimal control problem:

$$\dot{q} = u_1 X_1(q) + u_2 X_2(q), \qquad q = (\varphi, \theta) \in M = \widetilde{H_1^2}, \tag{4.1}$$

$$u \in U = \{(u_1, u_2) \in \mathbb{R}^2 \mid u_1^2 - u_2^2 \geq 0,\ u_1 > 0\}, \tag{4.2}$$

$$q(0) = q_0 = (\varphi, \theta_0), \qquad q(t_1) = q_1 = (\varphi, \theta_1), \tag{4.3}$$

$$l = \int_0^{t_1} \sqrt{u_1^2 - u_2^2}\, dt \to \max. \tag{4.4}$$

4.2 Reachable Set from an Arbitrary Point q_0

Definition 4.1. The reachable sets of the system (4.1), (4.2) from a point $q_0 \in M$ are defined as follows:

$$\mathcal{A}_{q_0} = \{q(t_1) :\ q(t),\ t \in [0, t_1],\ \text{trajectory of the system (4.1), (4.2), s.t. } t_1 \geq 0,\ q(0) = q_0\}$$

is the reachable set for arbitrary non-negative time (the causal future of the point q_0);

$$\mathcal{A}^-_{q_0} = \{q(t_1) :\ q(t),\ t \in [t_1, 0],\ \text{trajectory of the system (4.1), (4.2), s.t. } t_1 \leq 0,\ q(0) = q_0\}$$

is the reachable set for arbitrary non-positive time (the causal past of the point q_0);

$$\mathcal{A}^{t_1}_{q_0} = \{q(t) :\ q(s),\ s \in [0, t_1],\ \text{trajectory of the system (4.1), (4.2), s.t. } t \in [0, t_1],\ q(0) = q_0\}$$

is the reachable set for time not exceeding $t_1 \geq 0$.

Theorem 4.1. *Let $q_0 = (\theta_0, \varphi_0) \in M$. Then the set $\mathcal{A}_{q_0}$ is equal to*

$$V_{q_0} := \{(\theta, \varphi) \in M : \varphi \geq \mathrm{sign}\,(\theta - \theta_0)\arctan(\sinh\theta) + \varphi_0 - \mathrm{sign}\,(\theta - \theta_0)\arctan(\sinh\theta_0)\}.$$

Proof. 1) We show that all points of the set $V = V_{q_0}$ are reachable from the point q_0. Consider constant controls $u_1 = \text{const}$, $u_2 = \text{const}$, such that $u_1 > 0$, $-u_1 \le u_2 \le u_1$, and find the corresponding trajectories of the system (4.1) with the initial condition $q_0 = (\theta_0, \varphi_0)$:

$$\begin{cases} \dot\theta = u_2, \\ \dot\varphi = \frac{u_1}{\cosh\theta}. \end{cases} \tag{4.5}$$

The solution has the form:

$$\begin{aligned} &\begin{cases} \theta(t) \equiv \theta_0, \\ \varphi(t) = \frac{u_1}{\cosh\theta_0} t + \varphi_0, \end{cases} \qquad \text{for } u_2 = 0, \\ &\begin{cases} \theta(t) = u_2 t + \theta_0, \\ \varphi(t) = \frac{u_1}{u_2} \arctan\left(\sinh\left(u_2 t + \theta_0\right)\right) + \varphi_0 - \frac{u_1}{u_2} \arctan\sinh\theta_0, \end{cases} \qquad \text{for } u_2 \ne 0. \end{aligned} \tag{4.6}$$

The trajectories for $u_2 = 0$, as well as $u_2 = \pm 1,\ u_1 \in \{1,\ 3,\ 5,\ 10,\ 20,\ 50\}$, are shown in Figs. 1, 2, 3.

Figure 1: Trajectories for $q_0 = (0, 0)$ with $u_2 = 0$, as well as $u_2 = \pm 1,\ u_1 \in \{1,\ 3,\ 5,\ 10,\ 20,\ 50\}$

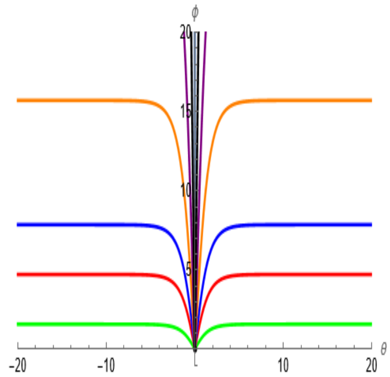


Figure 2: Trajectories for $q_0 = (0, -5)$ with $u_2 = 0$, as well as $u_2 = \pm 1,\ u_1 \in \{1,\ 3,\ 5,\ 10,\ 20,\ 50\}$

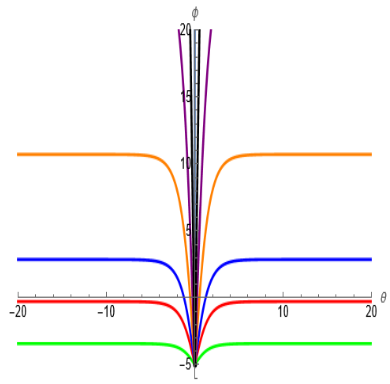
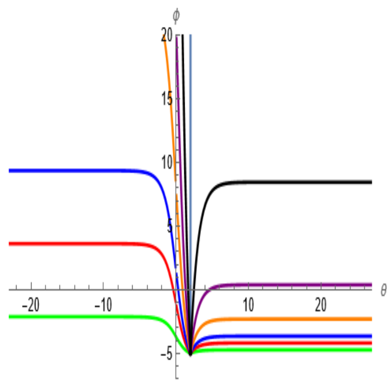


Figure 3: Trajectories for $q_0 = (2, -5)$ with $u_2 = 0$, as well as $u_2 = \pm 1,\ u_1 \in \{1,\ 3,\ 5,\ 10,\ 20,\ 50\}$



For $u_2 = 0$, we obtain the vertical ray $\theta = \theta_0$, $\varphi \geq \varphi_0$. It divides V into two disjoint sets: $V_+ = V \cap \{\theta > \theta_0\}$, $V_- = V \cap \{\theta < \theta_0\}$.

Let $u_2 = \pm 1$, then $u_1 \geq 1$ and $u_2 t = \pm t$, $\frac{u_1}{u_2} = \pm u_1 = u$, $u \geq 1$ or $u \leq -1$. Therefore,

$$\begin{cases} \theta = \pm t + \theta_0, \\ \varphi = u \arctan\left(\sinh\left(\pm t + \theta_0\right)\right) + \varphi_0 - u \arctan \sinh \theta_0, \end{cases} \Rightarrow \begin{cases} \pm t = \theta - \theta_0, \\ u = \frac{\varphi - \varphi_0}{\arctan \sinh \theta - \arctan \sinh \theta_0}. \end{cases}$$

From this, it is clear that the mapping $(t,\ u) \mapsto (t + \theta_0,\ u \arctan\left(\sinh\left(t + \theta_0\right)\right) + \varphi_0 - u \arctan \sinh \theta_0)$ establishes a bijection between the sets V_+ and $\{(t, u) : t > 0,\ u \geq 1\}$, as well as between the sets V_- and $\{(t, u) : t < 0,\ u \leq -1\}$. Hence, all points of the set V are reachable from the point q_0.

2) We show that points with $\varphi < \operatorname{sign}(\theta - \theta_0) \arctan(\sinh \theta) + \varphi_0 - \operatorname{sign}(\theta - \theta_0) \arctan(\sinh \theta_0)$ are not reachable from the point (θ_0, φ_0) in non-negative time. Direct verification shows that any vector field $u_1 X_1 + u_2 X_2$ for any admissible u at each point of the boundary V is tangent to the curve $\varphi = \operatorname{sign}(\theta - \theta_0) \arctan(\sinh \theta) + \varphi_0 - \operatorname{sign}(\theta - \theta_0) \arctan(\sinh \theta_0)$ or directed into the interior of the region $\operatorname{int}(V)$.

It follows that the reachable set from the point $\theta = \theta_0$, $\varphi = \varphi_0$ in non-negative time is V_{q_0}.

□

We immediately obtain

Corollary 4.1. *The reachable set from the point $q_0 = (\theta_0, \varphi_0) \in M$ for arbitrary non-positive time is*

$$V_{q_0}^- := \{(\theta, \varphi) \in \mathbb{R}^2 : \varphi \leq -\operatorname{sign}(\theta - \theta_0) \arctan(\sinh \theta) + \varphi_0 + \operatorname{sign}(\theta - \theta_0) \arctan(\sinh \theta_0)\}.$$

Proof. Indeed, by considering the obtained formulas (4.6) for constant controls at non-positive t and carrying out reasoning similar to that in Theorem 4.1 (constructing a bijection between the corresponding sets, as well as studying the behavior of the vector field on the boundary of the set $V_{q_0}^-$), we obtain the stated result. □

The trajectories for $u_2 = 0$, as well as $u_2 = \pm 1$, $u_1 \in \{1,\ 3,\ 5,\ 10,\ 20,\ 50\}$, are shown in Figs. 4, 5, 6.

Figure 4: Trajectories for $q_0 = (0, 0)$ with $u_2 = 0$, as well as $u_2 = \pm 1$, $u_1 \in \{1,\ 3,\ 5,\ 10,\ 20,\ 50\}$

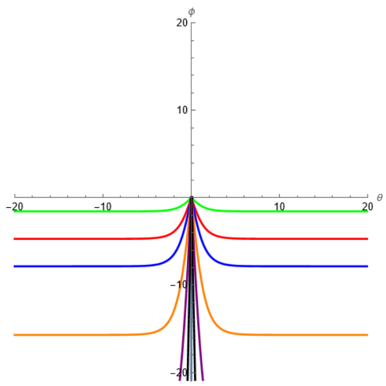


Figure 5: Trajectories for $q_0 = (0, -5)$ with $u_2 = 0$, as well as $u_2 = \pm 1$, $u_1 \in \{1,\ 3,\ 5,\ 10,\ 20,\ 50\}$

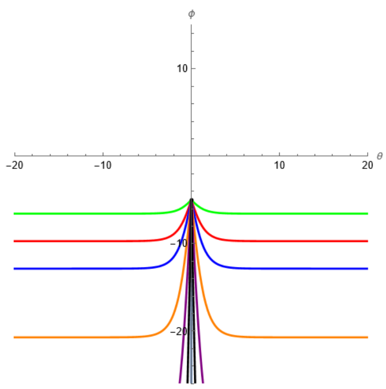
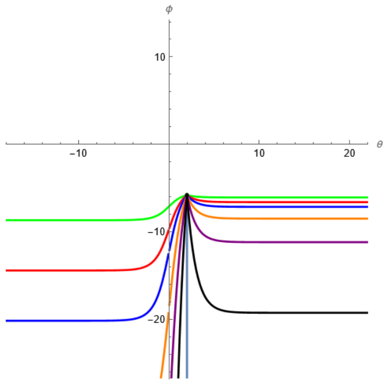


Figure 6: Trajectories for $q_0 = (2, -5)$ with $u_2 = 0$, as well as $u_2 = \pm 1,\ u_1 \in \{1,\ 3,\ 5,\ 10,\ 20,\ 50\}$



4.3 Existence of Optimal Trajectories

Consider a problem equivalent to problem (4.1)–(4.4):

$$\dot q = u_1 X_1(q) + u_2 X_2(q), \qquad q = (\varphi, \theta) \in M = \widetilde{H_1^2}, \tag{4.7}$$

$$u \in U' = \{(u_1, u_2) \in \mathbb{R}^2 \mid u_2^2 \le 1,\ u_1 = 1\}, \tag{4.8}$$

$$q(0) = q_0 = (\varphi_0, \theta_0), \qquad q(t_1) = q_1 = (\varphi_1, \theta_1), \tag{4.9}$$

$$l = \int_0^{t_1} \sqrt{1 - u_2^2}\, dt \to \max\,. \tag{4.10}$$

The solutions of problem (4.1)–(4.4) are reparametrizations of the solutions of problem (4.7)–(4.10).

Definition 4.2. The maximum motion time of trajectories of system (4.7), (4.8) from point q_0 to point q_1 is

$$T(q_0, q_1) := \sup\{t_1 > 0 :\ \exists \text{ a trajectory } q(t) \text{ of system (4.7), (4.8)},\ t \in [0, t_1],\ \text{such that } q(0) = q_0,\ q(t_1) = q_1\}.$$

Theorem 4.2. *Let $q_0 = (\theta_0, \varphi_0) \in M$ and*

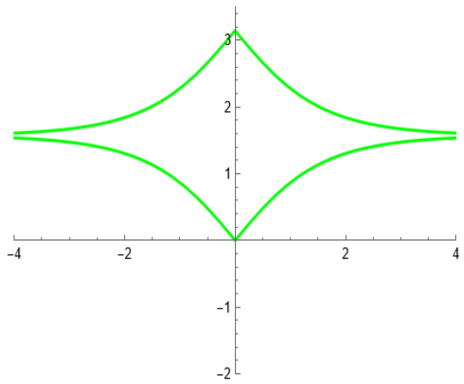
$$\mathcal{B}_{q_0} := \{(\theta, \varphi) \in M \mid \pi + \varphi_0 - \operatorname{sign}(\theta - \theta_0)(\arctan(\sinh\theta) - c_0) - 2c_0 \operatorname{sign}\theta_0 > \varphi \ge \varphi_0 + \operatorname{sign}(\theta - \theta_0)(\arctan(\sinh\theta) - c_0)\},$$

where $c_0 = \arctan(\sinh\theta_0)$.

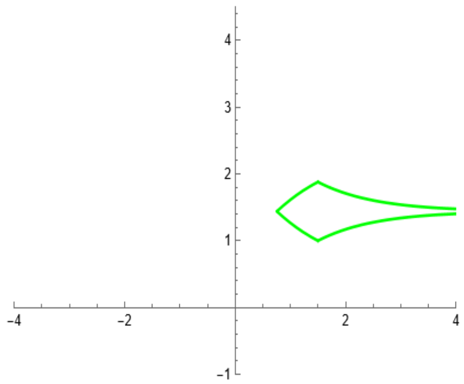
Then, for any point $q_1 \in \mathcal{B}_{q_0}$, there exists an optimal trajectory in problem (4.1)*–*(4.4)*.*

Figure 7: Boundary of the set $\mathcal{B}_{q_0}$

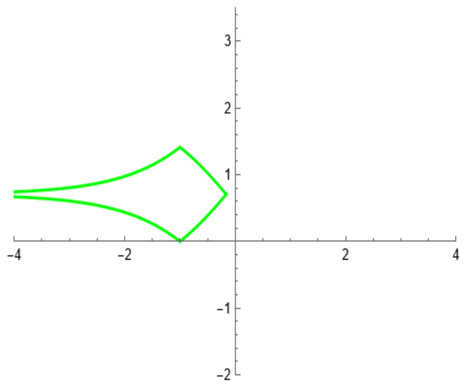
(a) For $q_0 = (0, 0)$



(b) For $q_0 = (\frac{3}{2}, 1)$



(c) For $q_0 = (-1, 0)$



Proof. We show that for any point $q_1 = (\theta_1, \varphi_1) \in \mathcal{B}_{q_0}$, the conditions of Theorem 2 from [4] are satisfied:

Theorem 4.3. *For problem* (4.7)–(4.10)*, suppose the following conditions hold:*

(1) $q_1 \in \mathcal{A}_{q_0}$*;*

(2) *The set* $\mathcal{A}_{q_0} \cap \mathcal{A}^-_{q_1}$ *is compact;*

(3) $T(q_0, q_1) < +\infty$.

Then, an optimal trajectory exists in problem (4.7)–(4.10).

(1) The first condition holds because $\mathcal{B}_{q_0} \subset V_{q_0} = \mathcal{A}_{q_0}$.

(2) We show that $V_{q_0} \cap V^-_{q_1}$ is compact. From the explicit form of these regions and the monotonicity properties of the functions defining their boundaries, it follows that the intersection $V_{q_0} \cap V^-_{q_1}$ can be enclosed in a rectangle $\{(\theta, \varphi) : \theta_- \leq \theta \leq \theta_+, \varphi_- \leq \varphi \leq \varphi_+\}$, implying boundedness.

We explicitly show that the right boundary of V_{q_0} (respectively, the left boundary) intersects with the right boundary of $V^-_{q_1}$ (respectively, the left boundary). This will imply the boundedness and closedness of the intersection.

Let $q_1 = (\theta_1, \varphi_1) \in \mathcal{B}_{q_0}$, i.e.,

$$\varphi_0 + \operatorname{sign}(\theta_1 - \theta_0)(\arctan(\sinh\theta_1) - \arctan(\sinh\theta_0)) \le \varphi_1$$
$$< \pi + \varphi_0 - \operatorname{sign}(\theta_1 - \theta_0)(\arctan(\sinh\theta_1) - \arctan(\sinh\theta_0)) - 2\arctan(\sinh|\theta_0|). \quad (4.11)$$

The equation for boundary intersections:

$$\varphi_0 + \operatorname{sign}(\theta - \theta_0)(\arctan(\sinh\theta) - \arctan(\sinh\theta_0)) = \varphi_1 - \operatorname{sign}(\theta - \theta_1)(\arctan(\sinh\theta) - \arctan(\sinh\theta_1)) \Leftrightarrow$$
$$\Leftrightarrow (\operatorname{sign}(\theta - \theta_0) + \operatorname{sign}(\theta - \theta_1)) \arctan\sinh\theta = \operatorname{sign}(\theta - \theta_0)\arctan(\sinh\theta_0) + \operatorname{sign}(\theta - \theta_1)\arctan(\sinh\theta_1) + \varphi_1 - \varphi_0.$$

The intersection of the right boundaries is determined by $\theta > \theta_1,\ \theta > \theta_0$, and the intersection of the left boundaries by $\theta < \theta_1,\ \theta < \theta_0$. The corresponding equations are:

$$\theta > \theta_1,\ \theta > \theta_0,\ 2\arctan(\sinh\theta) = \arctan(\sinh\theta_0) + \arctan(\sinh\theta_1) + \varphi_1 - \varphi_0,$$
$$\theta < \theta_1,\ \theta < \theta_0,\ 2\arctan(\sinh\theta) = \arctan(\sinh\theta_0) + \arctan(\sinh\theta_1) + \varphi_0 - \varphi_1.$$

Consider the first equation and show that its right-hand side lies in $(-\pi, \pi)$ (the second equation is analogous). Then, due to the strict monotonicity of $\arctan\sinh(\cdot)$, the solution exists and is unique.
For the proof, we use inequality (4.11). We compare:

$$\arctan(\sinh\theta_0) + \arctan(\sinh\theta_1) + \varphi_1 - \varphi_0 \bigvee \pi \quad \text{and} \quad \arctan(\sinh\theta_0) + \arctan(\sinh\theta_1) + \varphi_1 - \varphi_0 \bigvee -\pi,$$

which is equivalent to:

$$\varphi_1 \bigvee \pi + \varphi_0 - \arctan(\sinh\theta_0) - \arctan(\sinh\theta_1) \quad \text{and} \quad \varphi_1 \bigvee -\pi + \varphi_0 - \arctan(\sinh\theta_0) - \arctan(\sinh\theta_1). \quad (4.12)$$

We analyse four cases for the first inequality in (4.12):

- $\theta_1 > \theta_0$, $\theta_0 \ge 0$: The right-hand side of (4.11) is $\pi + \varphi_0 - \arctan(\sinh\theta_1) - \arctan(\sinh\theta_0)$, so
$$\varphi_1 - \varphi_0 + \arctan(\sinh\theta_1) + \arctan(\sinh\theta_0) < \pi.$$
- $\theta_1 > \theta_0$, $\theta_0 < 0$: The right-hand side of (4.11) is $\pi + \varphi_0 + \arctan(\sinh\theta_1) + 3\arctan(\sinh\theta_0)$. Comparing:
$$\pi + \varphi_0 - \arctan(\sinh\theta_1) + 3\arctan(\sinh\theta_0) \bigvee \pi + \varphi_0 - \arctan(\sinh\theta_0) - \arctan(\sinh\theta_1),$$
which reduces to $4\arctan(\sinh\theta_0) \bigvee 0$. Since $\theta_0 < 0$,
$$\varphi_1 - \varphi_0 + \arctan(\sinh\theta_1) + \arctan(\sinh\theta_0) < \pi.$$
- $\theta_1 < \theta_0$, $\theta_0 \ge 0$: The right-hand side of (4.11) is $\pi + \varphi_0 + \arctan(\sinh\theta_1) - 3\arctan(\sinh\theta_0)$. Comparing:
$$\pi + \varphi_0 + \arctan(\sinh\theta_1) - 3\arctan(\sinh\theta_0) \bigvee \pi + \varphi_0 - \arctan(\sinh\theta_0) - \arctan(\sinh\theta_1),$$
which reduces to $2\arctan(\sinh\theta_1) \bigvee 2\arctan(\sinh\theta_0)$. Since $\theta_1 < \theta_0$ and $\arctan(\cdot)$, $\sinh(\cdot)$ are strictly monotonic,
$$\varphi_1 - \varphi_0 + \arctan(\sinh\theta_1) + \arctan(\sinh\theta_0) < \pi.$$
- $\theta_1 < \theta_0$, $\theta_0 < 0$: The right-hand side of (4.11) is $\pi + \varphi_0 + \arctan(\sinh\theta_1) + \arctan(\sinh\theta_0)$. Comparing:
$$\pi + \varphi_0 + \arctan(\sinh\theta_1) + \arctan(\sinh\theta_0) \bigvee \pi + \varphi_0 - \arctan(\sinh\theta_0) - \arctan(\sinh\theta_1),$$
which reduces to $2\arctan\sinh\theta_1 + 2\arctan(\sinh\theta_0) \bigvee 0$. Since $\theta_1 < \theta_0 < 0$,
$$\varphi_1 - \varphi_0 + \arctan(\sinh\theta_1) + \arctan(\sinh\theta_0) < \pi.$$

For the right inequality in (4.12), consider two cases:

- $\theta_1 \ge \theta_0$: The left-hand side of (4.11) is $\varphi_0 + \arctan(\sinh\theta_1) - \arctan(\sinh\theta_0)$. Since $\arctan(\sinh\theta) > -\pi/2$,
$$\varphi_0 + \arctan(\sinh\theta_1) - \arctan(\sinh\theta_0) > -\pi + \varphi_0 - \arctan(\sinh\theta_0) - \arctan(\sinh\theta_1),$$
so
$$-\pi < \varphi_1 - \varphi_0 + \arctan(\sinh\theta_1) + \arctan(\sinh\theta_0) < \pi.$$

- $\theta_1 < \theta_0$: The left-hand side of (4.11) is $\varphi_0 - \arctan(\sinh\theta_1) + \arctan(\sinh\theta_0)$. Since $\arctan(\sinh\theta) > -\pi/2$,

$$\varphi_0 - \arctan(\sinh\theta_1) + \arctan(\sinh\theta_0) > -\pi + \varphi_0 - \arctan(\sinh\theta_0) - \arctan(\sinh\theta_1),$$

so

$$-\pi < \varphi_1 - \varphi_0 + \arctan(\sinh\theta_1) + \arctan(\sinh\theta_0) < \pi.$$

Thus, condition (2) of Theorem 4.3 is satisfied.

(3) We show that for any $q_1 \in \mathcal{B}_{q_0}$, the following holds:

$$\sup\{t_1 > 0 : \exists \text{ a trajectory } q(t) \text{ of system } (4.7), (4.8),\ t \in [0, t_1] : q(0) = q_0,\ q(t_1) = q_1\} < +\infty.$$

Note that $\dot{\varphi} = \frac{u_1}{\cosh\theta} = \frac{1}{\cosh\theta} > 0$, so φ increases.

By the geometric properties of the reachable set, there exists $q_2 = (\theta_2, \varphi_2) \in V_{q_0}$ with maximal $|\theta_2|$ such that $q_1 \in V_{q_2}$. The point q_2 can be explicitly computed since, by the maximality condition, it lies on the lower boundary of V_{q_0}, and q_1 lies on the lower boundary of V_{q_2}.

Since $q_1 \in \mathcal{B}_{q_0}$, inequality (4.11) holds. As q_2 lies on the lower boundary of V_{q_0}, we have:

$$\varphi_2 = \varphi_0 + \operatorname{sign}(\theta_2 - \theta_0)(\arctan(\sinh\theta_2) - \arctan(\sinh\theta_0)). \tag{4.13}$$

Since q_1 lies on the lower boundary of V_{q_2}, we have:

$$\varphi_1 = \varphi_2 + \operatorname{sign}(\theta_1 - \theta_2)(\arctan(\sinh\theta_1) - \arctan(\sinh\theta_2)). \tag{4.14}$$

Substituting (4.13) into (4.14) gives the equation for θ_2:

$$\varphi_1 = \varphi_0 + \operatorname{sign}(\theta_2 - \theta_0)(\arctan(\sinh\theta_2) - \arctan(\sinh\theta_0)) + \operatorname{sign}(\theta_1 - \theta_2)(\arctan(\sinh\theta_1) - \arctan(\sinh\theta_2)). \tag{4.15}$$

We show that a solution exists by considering the cases:

- $\theta_2 > \theta_0,\ \theta_2 > \theta_1$ (right lower boundary of V_{q_0}, left lower boundary of V_{q_2}),
- $\theta_2 < \theta_0,\ \theta_2 < \theta_1$ (left lower boundary of V_{q_0}, right lower boundary of V_{q_2}).

The right-hand sides of the resulting equations lie in $(-\pi, \pi)$, ensuring existence and uniqueness.

This allows us to bound the rate of change of φ, proving finite time to reach q_1:

$$|\theta| \le C \Rightarrow \cosh\theta \le \tilde{C} \Rightarrow \frac{1}{\cosh\theta} \ge \bar{C} > 0 \Rightarrow t_1 \le \frac{\varphi_1 - \varphi_0}{\bar{C}}.$$

All conditions of Theorem 4.3 are satisfied, so Theorem 4.2 is proven. □

4.4 Extremals of Pontryagin's Maximum Principle

We apply Pontryagin's maximum principle (PMP) [2,3,8] to the optimal control problem (4.1)–(4.4).

The Hamiltonian of the PMP, where $\nu \in \{-1, 0\}$, is of the form

$$h_u^\nu(\lambda) = h_1 u_1 + h_2 u_2 - \nu\sqrt{u_1^2 - u_2^2}, \qquad \lambda \in T^*M,$$
$$h_1(\lambda) = \langle \lambda, X_1(q) \rangle = \frac{\xi_2}{\cosh\theta}, \qquad h_2(\lambda) = \langle \lambda, X_2(q) \rangle = \xi_1,$$

here ξ_i are the canonical coordinates in the cotangent bundle T^*M. The Hamiltonian system of the PMP is

$$\begin{cases} \dot{\xi}_1 = \xi_2 u_1 \frac{\sinh\theta}{\cosh^2\theta}, \\ \dot{\xi}_2 = 0, \\ \dot{\theta} = u_2, \\ \dot{\varphi} = \frac{u_1}{\cosh\theta}. \end{cases}$$

4.4.1 Abnormal trajectories

Consider the abnormal case $\nu = 0$.

Proposition 4.1. *Abnormal trajectories are light-like trajectories, and up to reparametrization*

$$\begin{aligned}
&u_1 = \pm u_2 = 1, \\
&\varphi(t) = \pm \arctan\{\sinh(\pm t + \theta_0)\} + \varphi_0 \mp \arctan\{\sinh \theta_0\}, \qquad \theta(t) = \pm t + \theta_0.
\end{aligned}$$

Proof. Consider two cases $u_1 = \pm u_2 = 1$. Then the Hamiltonian system looks like this:

$$\begin{cases}
\dot{\xi}_1 = \xi_2 \frac{\sinh \theta}{\cosh^2 \theta}, \\
\dot{\xi}_2 = 0, \\
\dot{\theta} = \pm 1, \\
\dot{\varphi} = \frac{1}{\cosh \theta}.
\end{cases}$$

From the second equation it immediately follows that $\xi_2 \equiv \text{const} = c_2$, and from the third equation we obtain $\theta(t) = \pm t + \theta_0$. By dividing the first equation by the third and the fourth equation by the third, we can find ξ_1 and φ as functions of θ, so we get:

$$\begin{cases}
\xi_2 \equiv c_2, \\
\xi_1(t) = \mp \frac{c_2}{\cosh(\pm t + \theta_0)} \pm \frac{c_2}{\cosh \theta_0} + \xi_1(0), \\
\theta(t) = \pm t + \theta_0, \\
\varphi(t) = \pm \arctan(\pm t + \theta_0) + \varphi_0 \mp \arctan \sinh \theta_0.
\end{cases}$$

□

Abnormal trajectories form the boundary of the reachability set $\mathcal{A}_{q_0}$.

4.4.2 Normal trajectories

Now consider the normal case $\nu = -1$. From the maximality condition

$$h_1 u_1 + h_2 u_2 + \sqrt{u_1^2 - u_2^2} \to \max_{u \in U}$$

we obtain:

$$\begin{aligned}
h_1^2 - h_2^2 = 1,\ h_1 < 0, \\
u_1^2 - u_2^2 = 1,\ u_1 > 0,
\end{aligned}$$

and then, writing in hyperbolic coordinates $h_1 = -\cosh \psi,\ h_2 = \sinh \psi$ and taking into account that $\xi_2 = h_2 = \text{const}$, we obtain that the Hamiltonian system for normal extremals takes type:

$$\begin{cases}
\dot{\psi} = -\cosh \psi \frac{\sinh \theta}{\cosh \theta}, \\
\dot{\theta} = \sinh \psi, \\
\dot{\varphi} = \frac{\cosh \psi}{\cosh \theta}.
\end{cases} \tag{4.16}$$

Dividing the first equation by the second and integrating, we obtain the first integral:

$$\cosh \psi \cosh \theta = \text{const}.$$

Proposition 4.2. *Normal extremals with initial condition $\theta(0) = 0,\ \varphi(0) = 0,\ \psi(0) = \psi_0$ have the following form:*

(1) *for $\psi_0 = 0,\ t \in \mathbb{R}$*

$$\begin{cases}
\theta(t) \equiv 0, \\
\psi(t) \equiv 0, \\
\varphi(t) = t,
\end{cases} \tag{4.17}$$

(2) *for* $\psi_0 \neq 0,\ t \in (-\pi/2, \pi/2)$

$$\begin{cases} \theta(t) = \operatorname{arsinh}(\sinh\psi_0 \sin t), \\ \psi(t) = \operatorname{arsinh}\left(\frac{\sinh\psi_0 \cos t}{\sqrt{\cos^2 t + \cosh^2\psi_0 \sin^2 t}}\right), \\ \varphi(t) = \arctan(\cosh\psi_0 \tan t), \end{cases} \tag{4.18}$$

which continues for all $t \in (-\infty, +\infty)$ *by formulae*

$$\begin{cases} \theta(t) = \operatorname{arsinh}\Big(\sinh\psi_0 \sin t\Big), \\ \psi(t) = \operatorname{arsinh}\left(\frac{\sinh\psi_0 \cos t}{\sqrt{\cos^2 t + \cosh^2\psi_0 \sin^2 t}}\right), \\ \varphi(t) = n\pi + \varphi_0(t - n\pi), \qquad t \in [n\pi - \pi/2, n\pi + \pi/2), \end{cases} \tag{4.19}$$

where $n \in \mathbb{Z}$ *and*

$$\varphi_0(t) = \begin{cases} -\pi/2, & t = -\pi/2, \\ \arctan(\cosh\psi_0 \tan t), & t \in (-\pi/2, \pi/2), \\ \pi/2, & t = \pi/2. \end{cases}$$

Proof. Using the first integral $\cosh\psi\cosh\theta \equiv D$, we first consider the case $D = 1 = \cosh\psi_0$, and obtain the solution (4.17). In the case $D > 1$, at the energy level we express $\sinh\psi$ as a function of θ, and then by integrating and substituting the initial condition we obtain the solution (4.18) for $t \in (-\pi/2, \pi/2)$.

Consider the equation for the function φ, $\varphi(0) = 0$, $\theta(0) = 0$, obtained after finding $\theta(t)$:

$$\dot{\varphi} = \frac{\cosh\psi}{\cosh\theta} = \frac{D}{\cosh^2\theta} = \frac{\cosh\theta_0 \cosh\psi_0}{1 + \sinh^2\psi_0 \sin^2 t} = \frac{\cosh\psi_0}{1 + \sinh^2\psi_0 \sin^2 t}.$$

The right-hand side is a smooth bounded function for all $t \geq 0$, so the solution $\varphi(t)$ is a smooth function for all $t \geq 0$:

$$\varphi(t) = \int_0^t \frac{\cosh\psi_0}{1 + \sinh^2\psi_0 \sin^2\tau}\, d\tau.$$

We have evaluated this integral and obtained a formula that is true on the interval $t \in (-\pi/2, \pi/2)$:

$$\varphi_0(t) = \arctan\Big(\cosh\psi_0 \tan t\Big).$$

On the other hand, note that the derivative $\dot{\varphi}(t)$ is a periodic function with period π.
At points $k\pi$, $k \in \mathbb{Z}$ the function $\dot{\varphi}(t)$ reaches its maximum value, and at point $n\pi/2$, $n = 2l + 1, l \in \mathbb{Z}$ — its minimum value. Moreover, it is even on each interval $[(k-1)\pi, k\pi]$, $k \in \mathbb{Z}$, relative to the midpoint $(2k-1)\pi/2$. Due to the continuity of the solution, we glue it on each interval

$$\lim_{t \to \pm\pi/2 \mp 0} \arctan\Big(\cosh\psi_0 \tan t\Big) = \pm\pi/2.$$

By continuity, $\lim\limits_{t \to \pm\pi/2 \pm 0} \varphi(t) = \pm\pi/2$.
Since the derivative with respect to the point $\pi/2$ is even, we extend the solution to the segment $[\pi/2, \pi]$ using the formula:

$$t \in [0, \pi/2],\ \varphi(\pi/2 + t) = \pi/2 + \pi/2 - \arctan\Big(\cosh\psi_0 \tan(\pi/2 - t)\Big),$$

where the first term is from the condition that $\varphi(t) = \pi/2$, and

$$\pi/2 - \arctan\Big(\cosh\psi_0 \tan(\pi/2 - t)\Big) = \pi/2 - \varphi_0(\pi/2 - t) = \int_{\pi/2}^{\pi/2 + t} \frac{\cosh\psi_0}{1 + \sinh^2\psi_0 \sin^2\tau}\, d\tau.$$

Accordingly, $\varphi(\pi) = \pi$, and we get the general formula, where $n \in \mathbb{Z}$:

$$\varphi(t) = n\pi + \varphi_0(t - n\pi),\ t \in [n\pi - \pi/2, n\pi + \pi/2].$$

Return to the formulas (4.18) for the solution on the interval $t \in (-\pi/2, \pi/2)$ and write the new function $\varphi(t)$, $t \in [0, +\infty)$ through $\varphi_0(t) = \arctan\Big(\cosh(\psi(0)) \tan t\Big)$, and get (4.19). □

Figure 8: Extremal trajectory for $\psi(0) = 1$

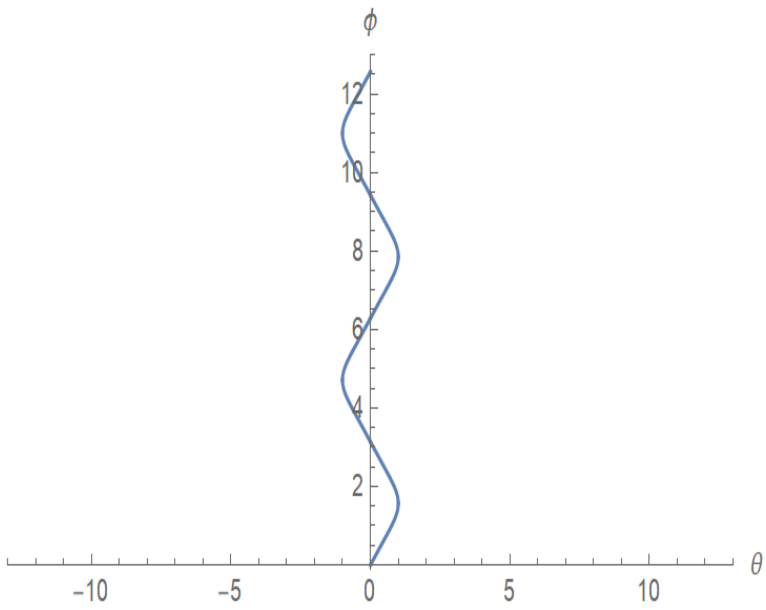
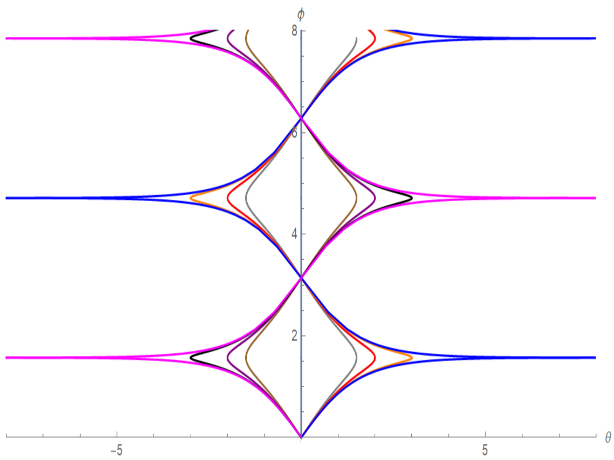


Figure 9: Extremal trajectories for $\psi(0) \in \{0,\ \pm 1.5,\ \pm 2,\ \pm 3,\ \pm 8\}$



4.5 Exponential Map and Its Properties

We define the exponential map

$$\mathrm{Exp}\ :\ N \to M, \qquad N = \left(T^*_{q_0} M \cap \left\{h_1 = -\sqrt{1+h_2^2}\right\}\right)_{\psi_0} \times (0,\pi)_t = \{(\psi_0, t) \mid \psi_0 \in \mathbb{R},\ t \in (0,\pi)\},$$

$$\mathrm{Exp}(\psi_0, t) = q(t) = (\theta(t), \varphi(t)),$$

where

$$\begin{cases} \theta(t) = \operatorname{arsinh}\Big(\sinh\psi_0 \sin t\Big), \\ \varphi(t) = \begin{cases} \arctan\Big(\cosh\psi_0 \tan t\Big), & t \in (0, \pi/2), \\ \pi/2, & t = \pi/2, \\ \pi + \arctan\left(\cosh\psi_0 (\tan(t-\pi))\right), & t \in (\pi/2, \pi). \end{cases} \end{cases} \tag{4.20}$$

Theorem 4.4. *The exponential map defines a homeomorphism of regions*

$$A' = \{(\psi_0,\ t) : \psi_0 \in \mathbb{R},\ t \in (0,\pi)\}, \qquad C' = \{(\theta,\varphi) \in \mathbb{R}^2 : \arctan\sinh|\theta| < \varphi < \pi - \arctan\sinh|\theta|\}$$

and a diffeomorphism of regions

$$A = \{(\psi_0,\ t) : \psi_0 \in \mathbb{R},\ t \in (0,\pi/2)\}, \qquad C = \{(\theta,\varphi) \in \mathbb{R}^2 : \arctan\sinh|\theta| < \varphi < \pi/2\},$$

as well as regions

$$\tilde{A} = \{(\psi_0,\ t) : \psi_0 \in \mathbb{R},\ t \in (\pi/2, \pi)\}, \qquad \tilde{C} = \{(\theta, \varphi) \in \mathbb{R}^2 : \pi/2 < \varphi < \pi - \arctan\sinh|\theta|\}.$$

Proof. We show explicitly that on the set C' there exists an inverse mapping to the mapping defined by the formulas (4.20), and we will see from the obtained formulas that it is continuous in both directions. First, we prove this for sets A and C by considering the following mapping:

$$\begin{cases} \theta(t) = \operatorname{arsinh}\Big(\sinh\psi_0 \sin t\Big), \\ \varphi(t) = \arctan\Big(\cosh\psi_0 \tan t\Big), \qquad t \in (0, \pi/2). \end{cases} \tag{4.21}$$

To do this, we take an intermediate step: let $X = \sinh\theta$, $Y = \tan\varphi$ and first prove that the domains A and $B = \{(X,Y) \in \mathbb{R}^2 | Y > |X|\}$ using the following formulas:

$$\begin{cases} X = \sinh\psi_0 \sin t, \\ Y = \cosh\psi_0 \tan t. \end{cases}$$

Expressing $\sinh\psi_0$ from the first equation, substituting into the second and using the formulas for trigonometric and hyperbolic functions, we obtain

$$t = \arcsin\sqrt{\frac{Y^2 - X^2}{1 + Y^2}},\ \psi_0 = \operatorname{arsinh}\left(X\sqrt{\frac{1+Y^2}{Y^2 - X^2}}\right) - \text{ smooth functions on the set } B.$$

Therefore, the inverse mapping to the mapping given by the formulas (4.21) is expressed on the set C by smooth functions:

$$t = \arcsin\sqrt{\frac{\tan^2\varphi - \sinh^2\theta}{1 + \tan^2\varphi}}, \qquad \psi_0 = \operatorname{arsinh}\left(\sinh\theta\sqrt{\frac{1 + \tan^2\varphi}{\tan^2\varphi - \sinh^2\theta}}\right). \tag{4.22}$$

We have thus proven that the domains A and C are diffeomorphic due to the exponential mapping. Now we will show the diffeomorphism of the domains $\tilde{A}$ and $\tilde{C}$ due to this mapping. Expressing explicitly through inverse mappings of elementary functions from the formulas (4.20), we obtain:

$$\begin{cases} \theta(t) = \operatorname{arsinh}(\sinh\psi_0 \sin t) \\ \varphi(t) = \pi + \arctan(\cosh\psi_0 \tan(\pi - t)) \end{cases} \Leftrightarrow \begin{cases} \sinh\theta = \sinh\psi_0 \sin t \\ \tan(\pi - \varphi) = \cosh\psi_0 \tan(\pi - t) \end{cases} \tag{4.23}$$

Since $t \in (\pi/2,\ \pi)$, then $(\pi - t) \in (0, \pi/2) \Rightarrow \cos(\pi - t) = -\cos t > 0,\ \sin(\pi - t) = -\sin t > 0 \Rightarrow \tan(\pi - t) = \tan t$. For similar reasons, $\tan(\pi - \varphi) = \tan\varphi$, since $\varphi \in (\pi/2, \pi)$. Therefore, the formulas (4.23) are equivalent to the following:

$$\begin{cases} \sinh\theta = \sinh\psi_0 \sin t, \\ \tan\varphi = \cosh\psi_0 \tan t. \end{cases}$$

These are exactly the formulas for the previous case $t \in (0, \pi/2)$, $\psi_0 \in \mathbb{R}$, but we must take into account that $t \in (\pi/2, \pi)$ and $\psi_0 \in \mathbb{R}$, whence we obtain:

$$t = \pi - \arcsin\left(\sqrt{\frac{\tan^2\varphi - \sinh^2\theta}{1 + \tan^2\varphi}}\right), \qquad \psi_0 = \operatorname{arsinh}\left(\sinh\theta\sqrt{\frac{1 + \tan^2\varphi}{\tan^2\varphi - \sinh^2\theta}}\right). \tag{4.24}$$

The continuity of the exponential mapping in the domain A' follows from the continuous dependence of the solution of the differential equation on the initial conditions.

It remains to show the continuity of the mapping inverse to the exponential mapping on the line $\varphi = \pi/2$, $\theta \in \mathbb{R}$ (corresponding to $t = \pi/2$, $\psi_0 \in \mathbb{R}$). To do this, we compare the left and right limits at $\varphi \to \pi/2$ for the mappings 4.22 and 4.24, respectively. For ψ_0 nothing needs to be checked, but for t it is necessary:

$$\lim_{\varphi \longrightarrow \pi/2 - 0} \arcsin\sqrt{\frac{\tan^2\varphi - \sinh^2\psi_0}{1 + \tan^2\varphi}} = \arcsin 1 = \pi/2 = \pi - \arcsin 1 = \lim_{\varphi \longrightarrow \pi/2 + 0}\left(\pi - \arcsin\sqrt{\frac{\tan^2\varphi - \sinh^2\psi_0}{1 + \tan^2\varphi}}\right).$$

□

Remark 4.1. In Theorem 4.4 we proved the homeomorphism of the domains A' and C' by virtue of the exponential mapping. In the Proposition 4.3 the analyticity of the mapping $t_0(\theta, \varphi)$ will be shown.

4.6 Optimal synthesis on the set $\mathcal{B}_{(0,0)}$

Theorem 4.5. (1) *If the point $q_1 = (\theta_1, \varphi_1)$ belongs to the lower boundary of the set $\mathcal{B}_{(0,0)} = \{(\theta,\varphi) \in M : \pi - \operatorname{arctan}\sinh|\theta| > \varphi \geq \operatorname{arctan}\sinh|\theta|\}$, then the optimal trajectory connecting the origin and q_1 exists, is unique and is an abnormal extremal, and the Lorentzian distance from $(0,0)$ to q_1 is 0.*

(2) *If a point $q_1 = (\theta_1, \varphi_1) \in \operatorname{int}\mathcal{B}_{(0,0)}$, then the optimal trajectory connecting the origin and q_1 exists, is unique, and is a normal extremal with initial condition ψ_{q_1}, and the Lorentzian distance from $(0,0)$ to q_1 is t_{q_1}, where*

$$\psi_{q_1} = \operatorname{arsinh}\left(\sinh\theta_1\sqrt{\frac{1+\tan^2\varphi_1}{\tan^2\varphi_1 - \sinh^2\theta_1}}\right), \tag{4.25}$$

$$t_{q_1} = \begin{cases} \arcsin\sqrt{\frac{\tan^2\varphi_1 - \sinh^2\theta_1}{1+\tan^2\varphi_1}}, & \varphi_1 \in (0,\pi/2), \\ \pi/2, & \varphi_1 = \pi/2, \\ \pi - \arcsin\sqrt{\frac{\tan^2\varphi_1 - \sinh^2\theta_1}{1+\tan^2\varphi_1}}, & \varphi_1 \in (\pi/2,\pi). \end{cases} \tag{4.26}$$

The uniqueness of the optimal trajectory is meant up to reparametrization.

Proof. It follows from Theorem 4.2 that for any point $q_1 \in \mathcal{B}_{(0,0)}$ there exists an optimal trajectory. The optimal trajectory satisfies the Pontryagin maximum principle.

First, we prove point (2).

(2) From Theorem 4.4 we obtain that normal trajectories connect the origin $(0,0)$ with points q_1 lying in the interior of the set $\mathcal{B}_{(0,0)}$:

$$\operatorname{Exp}(A') = C' = \operatorname{int}\mathcal{B}_{(0,0)}.$$

From the same theorem we conclude that there exists a unique extremal trajectory, determined by the initial condition ψ_{q_1}. Since for any point $q_1 \in \operatorname{int}(\mathcal{B}_{(0,0)})$ there exists an optimal trajectory, and also since the extremal trajectory passing through q_1 is unique, it follows that for each point $q_1 \in \operatorname{int}(\mathcal{B}_{(0,0)})$ the optimal trajectory connecting it with $(0,0)$ is the only normal extremal passing through q_1. Optimal synthesis in the interior of the set is formulated as follows.

We must first find ψ_{q_1}:

$$\psi_{q_1} = \operatorname{arsinh}\left(\sinh\theta_1\sqrt{\frac{1+\tan^2\varphi_1}{\tan^2\varphi_1 - \sinh^2\theta_1}}\right).$$

Next, we find the moment at which we reach point q_1:

$$t_{q_1} = \begin{cases} \arcsin\sqrt{\frac{\tan^2\varphi_1 - \sinh^2\theta_1}{1+\tan^2\varphi_1}}, & \varphi_1 \in (0,\pi/2) \\ \pi/2, & \varphi_1 = \pi/2 \\ \pi - \arcsin\sqrt{\frac{\tan^2\varphi_1 - \sinh^2\theta_1}{1+\tan^2\varphi_1}}, & \varphi_1 \in (\pi/2,\pi) \end{cases}$$

The optimal trajectory connecting points $(0,0)$ and $q_1 = (\theta_1, \varphi_1) \in C'$ has the following form:

- If $t_{q_1} < \pi/2$, then

$$\begin{cases} \theta(t) = \operatorname{arsinh}(\sinh\psi_{q_1}\sin t), \\ \varphi(t) = \arctan(\cosh\psi_{q_1}\tan t), & t \in (0, t_{q_1}). \end{cases} \tag{4.27}$$

- If $t_{q_1} = \pi/2$, then

$$\begin{cases} \theta(t) = \operatorname{arsinh}(\sinh\psi_{q_1}\sin t), \\ \varphi(t) = \begin{cases} \arctan(\cosh\psi_{q_1}\tan t), & t \in (0,\pi/2), \\ \pi/2, & t = \pi/2. \end{cases} \end{cases} \tag{4.28}$$

- If $\pi/2 < t_{q_1} < \pi$, then

$$\begin{cases} \theta(t) = \operatorname{arsinh}(\sinh\psi_{q_1}\sin t), \\ \varphi(t) = \begin{cases} \arctan(\cosh\psi_{q_1}\tan t), & t \in (0,\pi/2), \\ \pi/2, & t = \pi/2, \\ \pi - \arctan(\cosh\psi_{q_1}\tan(\pi - t)), & t \in (\pi/2, t_{q_1}). \end{cases} \end{cases} \tag{4.29}$$

Point (2) is proved.

Now we prove point (1).

Since normal trajectories connect $(0,0)$ only with points in the interior of $\mathcal{B}_{(0,0)}$, the optimal trajectories coming to the boundary of the set $\mathcal{B}_{(0,0)}$ are abnormal. Uniqueness follows from the fact that φ increases along the trajectories of the control system. Indeed, our control system is defined by differential equations (4.5) and the set of admissible controls (4.2).

That is, if we choose a point on the boundary with $\theta_1 > 0$, we can move along the lower boundary to the right and cannot return to the origin due to the monotonicity of the boundary itself along the φ coordinate.

From the Proposition 4.1 we obtain that abnormal trajectories connect the origin $(0,0)$ with each point q_1 lying on the lower boundary of the set $\mathcal{B}_{(0,0)}$. The optimal synthesis on the lower bound of the set is formulated as follows:

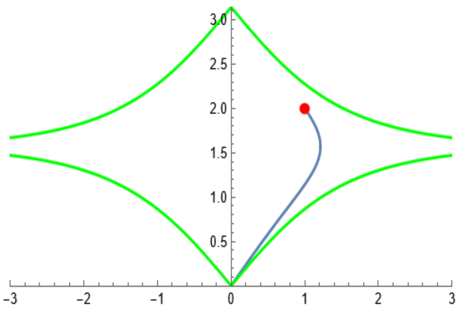
- If $\theta_1 > 0$, then
$$\theta(t) = t, \qquad \varphi(t) = \arcsin \sinh t, \qquad u_1 = u_2, \ t \in [0, \theta_1],$$
- If $\theta_1 < 0$, then
$$\theta(t) = -t, \qquad \varphi(t) = -\arcsin \sinh(-t), \qquad u_1 = -u_2, \ t \in [0, \theta_1],$$

Note that the length of abnormal trajectories is 0, since $u_1^2 - u_2^2 = 0$ along them.

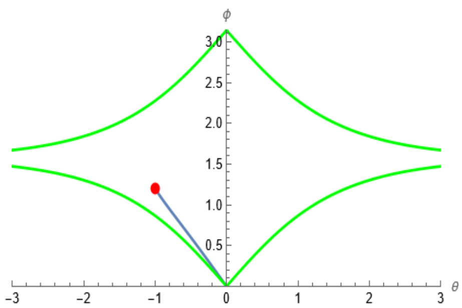
□

Figure 10: Optimal trajectories

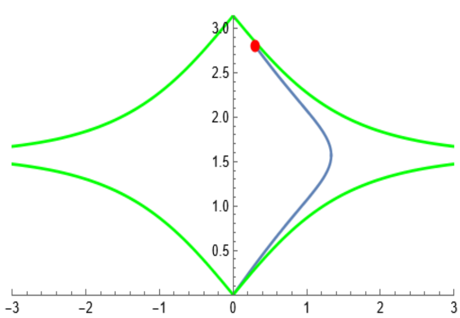
(a) For $q_1 = (1, 2)$



(b) For $q_1 = (-1, 1.2)$



(c) For $q_1 = (0.3, 2.8)$



4.7 Points above the upper boundary of $\mathcal{B}_{(0,0)}$

Theorem 4.6. *For points $(\theta_1, \varphi_1) \in M$ such that $\varphi_1 > \pi - \arctan(\sinh|\theta_1|)$, there is no optimal trajectory starting at $(\theta_0, \varphi_0) = (0,0)$. The Lorentzian distance from this point to the point (θ_1, φ_1) is $+\infty$.*

Proof. As shown in the Theorems 4.2 and 4.5, the optimal trajectories are contained in the set $\mathcal{B}_{(0,0)}$.

We will show that the Lorentzian distance for points above the upper boundary $\mathcal{B}_{(0,0)}$ is $+\infty$. We construct a family of admissible (piecewise smooth with $(u_1, u_2) \in U$) curves depending on the parameter $\alpha > 0$, connecting the origin with the point $q_1 = (\theta_1, \varphi_1)$, such that $\varphi_1 > \pi - \arctan(\sinh|\theta|)$, and calculate the limit of the lengths of these curves as $\alpha \to +\infty$. Each curve of the family consists of 3 parts:

1) We move along the curve $\varphi = \arctan\sinh\theta,\ \theta > 0$ to the point $(\theta, \varphi) = (\alpha, \arctan(\sinh\alpha)),\ \alpha > \theta_1$, if $\theta_1 > 0$, along the curve $\varphi = -\arctan\sinh\theta,\ \theta < 0$ to the point $(-\alpha, -\arctan(\sinh(-\alpha))),\ -\alpha < \theta_1$, if $\theta_1 < 0$;

2) Move vertically upward until we intersect the curve $\varphi = \varphi_1 + \pi - \arctan\sinh(\theta - \theta_1)$ when $\theta_1 > 0$, until we intersect the curve $\varphi = \varphi_1 + \pi + \arctan\sinh(\theta - \theta_1)$ when $\theta_1 < 0$;

3) Move along the curve $\varphi = \varphi_1 + \pi - \arctan\sinh|\theta - \theta_1|$ until we reach the point (θ_1, φ_1).

Figure 11: Trajectory for $q_1 = (1, 4)$, $\alpha = 2$

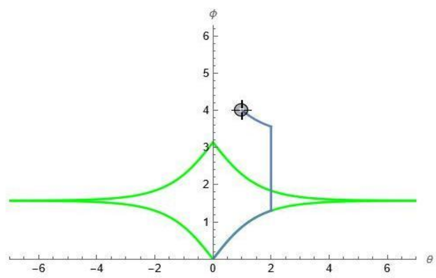


Figure 12: Trajectory for $q_1 = (1, 4)$, $\alpha = 4$

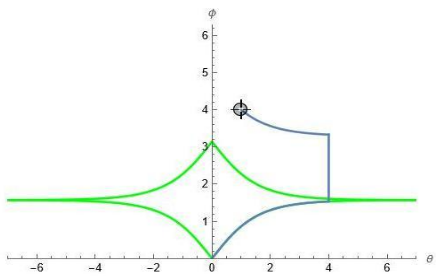


Figure 13: Trajectory for $q_1 = (1, 4)$, $\alpha = 6$

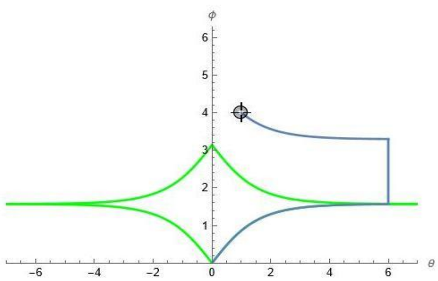


Figure 14: Trajectory for $q_1 = (-1, 4)$, $\alpha = 2$

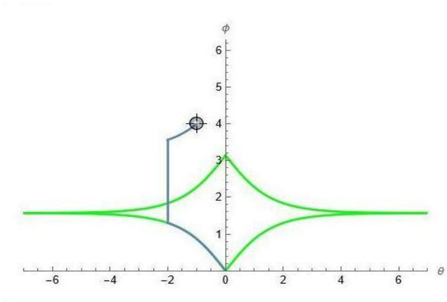


Figure 15: Trajectory for $q_1 = (-1, 4)$, $\alpha = 4$

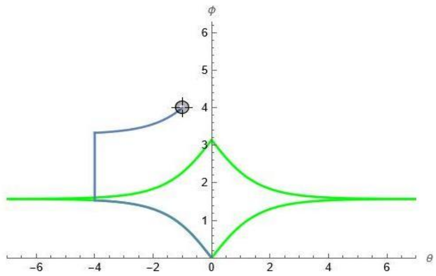
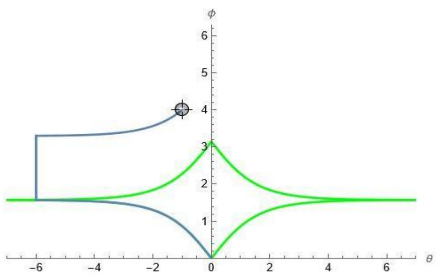


Figure 16: Trajectory for $q_1 = (-1, 4)$, $\alpha = 6$



These parts are defined by the following controls:

1) $u_1 = 1,\ u_2 = 1$ for $\theta_1 > 0$, $u_1 = 1,\ u_2 = -1$ for $\theta_1 < 0$, $t \in [0, t_1]$;

2) $u_1 = 1,\ u_2 = 0$, $t \in [t_1, t_2]$;

3) $u_1 = 1,\ u_2 = -1$ for $\theta_1 > 0$, $u_1 = 1,\ u_2 = 1$ for $\theta_1 < 0$, $t \in [t_2, t_3]$.

We calculate the value of the length functional on such a curve:

$$\int_0^{t_3} \sqrt{u_1^2 - u_2^2}\, dt = \left(\int_0^{t_1} + \int_{t_1}^{t_2} + \int_{t_2}^{t_3} \right) \sqrt{u_1^2 - u_2^2}\, dt = \int_{t_1}^{t_2} dt = t_2 - t_1.$$

It remains to find t_2 and t_1.

- First, for $\theta_1 > 0$:

  1) With the initial condition $\theta(0) = 0,\ \varphi(0) = 0$ we obtain the solution

  $$(\theta(t), \varphi(t)) = (t, \arctan \sinh t)$$

  on the segment $t \in [0, \alpha]$;

2) With the initial condition $\theta(\alpha) = \alpha,\ \varphi(\alpha) = \arctan(\sinh\alpha)$ we obtain the solution

$$(\theta(t), \varphi(t)) = \left(\alpha, \frac{t}{\cosh\alpha} - \frac{\alpha}{\cosh\alpha} + \arctan(\sinh\alpha)\right)$$

on the segment $t \in [\alpha, t_2]$;

3) With the initial condition $\theta(t_3) = \theta_1,\ \varphi(t_3) = \varphi_1$ we obtain the solution

$$(\theta(t), \varphi(t)) = (-t + t_3 + \theta_1, -\arctan(\sinh(-t + t_3 + \theta_1)) + \varphi_1 + \arctan(\sinh(\theta_1)))$$

on the segment $t \in [t_2, t_3]$.

So, $t_1 = \alpha$ we have explicitly found.

It remains to find t_2 from the intersection of the vertical line 2) with the curve 3). At point t_2 vertical line 2) reaches the point with ordinate $-\arctan(\sinh\alpha) + \varphi_1 + \arctan(\sinh(\theta_1))$. We compose the corresponding equation and solve it:

$$\frac{t_2}{\cosh\alpha} - \frac{\alpha}{\cosh\alpha} + \arctan(\sinh\alpha) = -\arctan(\sinh\alpha) + \varphi_1 + \arctan(\sinh(\theta_1)) \Leftrightarrow$$

$$\Leftrightarrow t_2 = \left[\frac{\alpha}{\cosh\alpha} - 2\arctan(\sinh\alpha) + \varphi_1 + \arctan(\sinh(\theta_1))\right]\cosh\alpha.$$

And now we obtain the length of the curve:

$$t_2 - t_1 = \left[\frac{\alpha}{\cosh\alpha} - 2\arctan(\sinh\alpha) + \varphi_1 + \arctan(\sinh(\theta_1))\right]\cosh\alpha - \alpha =$$
$$= [-2\arctan(\sinh\alpha) + \varphi_1 + \arctan(\sinh(\theta_1))]\cosh\alpha =: L(\alpha).$$

We calculate the limit

$$\lim_{\alpha\to+\infty} L(\alpha) = \lim_{\alpha\to+\infty} ([-2\arctan(\sinh\alpha) + \varphi_1 + \arctan(\sinh(\theta_1))]\cosh\alpha) = +\infty,$$

since $\varphi_1 + \arctan(\sinh\theta_1) - \pi > 0$.

- Now let $\theta_1 < 0$:

1) With the initial condition $\theta(0) = 0,\ \varphi(0) = 0$ we obtain the solution

$$(\theta(t), \varphi(t)) = (-t, -\arctan\sinh(-t))$$

on the segment $t \in [0, \alpha]$;

2) With the initial condition $\theta(\alpha) = -\alpha,\ \varphi(\alpha) = -\arctan(\sinh(-\alpha))$ we obtain the solution

$$(\theta(t), \varphi(t)) = \left(-\alpha, \frac{t}{\cosh(-\alpha)} + \frac{\alpha}{\cosh(-\alpha)} - \arctan(\sinh(-\alpha))\right)$$

on the segment $t \in [\alpha, t_2]$;

3) With the initial condition $\theta(t_3) = \theta_1,\ \varphi(t_3) = \varphi_1$ we obtain the solution

$$(\theta(t), \varphi(t)) = (t - t_3 + \theta_1, \arctan(\sinh(t - t_3 + \theta_1)) + \varphi_1 - \arctan(\sinh(\theta_1)))$$

on the segment $t \in [t_2, t_3]$.

So, $t_1 = \alpha$ we have explicitly found.

It remains to find t_2 from the intersection of the vertical line 2) with the curve 3). At point t_2 vertical line 2) reaches the point with ordinate

$$t_2 = \left[-\frac{\alpha}{\cosh(-\alpha)} + 2\arctan(\sinh(-\alpha)) + \varphi_1 - \arctan(\sinh(\theta_1))\right]\cosh(-\alpha).$$

And now we obtain the length of the curve:

$$t_2 - t_1 = \left[-2\frac{\alpha}{\cosh\alpha} - 2\arctan(\sinh\alpha) + \varphi_1 - \arctan(\sinh(\theta_1))\right]\cosh\alpha =: L(\alpha).$$

We calculate the limit:

$$\lim_{\alpha\to+\infty} L(\alpha) = \lim_{\alpha\to+\infty} \left(\left[-2\frac{\alpha}{\cosh\alpha} - 2\arctan(\sinh\alpha) + \varphi_1 - \arctan(\sinh(\theta_1))\right]\cosh(\alpha)\right) = +\infty,$$

since $\theta_1 < 0$, $\varphi_1 - \arctan(\sinh\theta_1) - \pi > 0$.

- Now we look at the case when $\theta_1 = 0,\ \varphi_1 > \pi$. Consider the curves that go to the right of the origin (as in the case $\theta_1 > 0$).

  1) With the initial condition $\theta(0) = 0,\ \varphi(0) = 0$ we obtain the solution

  $$(\theta(t), \varphi(t)) = (t, \arctan \sinh t)$$

  on the segment $t \in [0, \alpha]$;
  2) With the initial condition $\theta(\alpha) = \alpha,\ \varphi(\alpha) = \arctan(\sinh \alpha)$ we obtain the solution

  $$(\theta(t), \varphi(t)) = (\alpha, \frac{t}{\cosh \alpha} - \frac{\alpha}{\cosh \alpha} + \arctan(\sinh \alpha))$$

  on the segment $t \in [\alpha, t_2]$;
  3) With the initial condition $\theta(t_3) = 0,\ \varphi(t_3) = \varphi_1$ we obtain the solution

  $$(\theta(t), \varphi(t)) = (-t + t_3, -\arctan(\sinh(-t + t_3)) + \varphi_1)$$

  on the segment $t \in [t_2, t_3]$.

  Thus, we have explicitly found $t_1 = \alpha$.

  It remains to find t_2 from the intersection of the vertical line 2) with the curve 3). At point t_2 vertical line 2) reaches the point with ordinate $-\arctan(\sinh \alpha) + \varphi_1 + \arctan(\sinh(\theta_1))$.

  $$t_2 = \Big[\frac{\alpha}{\cosh \alpha} - 2 \arctan(\sinh \alpha) + \varphi_1\Big] \cosh \alpha.$$

  And now we obtain the length of the curve:

  $$t_2 - t_1 = [-2 \arctan(\sinh \alpha) + \varphi_1 + \arctan(\sinh(\theta_1))] \cosh(\alpha) =: L(\alpha).$$

  We calculate the limit:

  $$\lim_{\alpha \to +\infty} L(\alpha) = \lim_{\alpha \to +\infty} ([-2 \arctan(\sinh \alpha) + \varphi_1] \cosh(\alpha)) = +\infty,$$

  since $\varphi_1 > \pi$.

The theorem is proved. □

4.8 Upper Boundary Points of $\mathcal{B}_{(0,0)}$

Theorem 4.7. *For points $\tilde{q} = (\tilde{\theta}, \tilde{\varphi}) \in M$, i.e., $\tilde{\varphi} = \pi - \arctan(\sinh|\tilde{\theta}|)$, the Lorentzian distance from $(0, 0)$ is π. For $(0, \pi)$, there is a continuum of optimal trajectories from $(0, 0)$. For other points on this curve, there are no optimal trajectories.*

Proof. For $(\tilde{\theta}, \tilde{\varphi}) \in M$, i.e., $\tilde{\varphi} = \pi - \arctan(\sinh|\tilde{\theta}|)$, $(\tilde{\theta}, \tilde{\varphi}) \neq (0, \pi)$, there are no extremal trajectories connecting $(0, 0)$, so there are no optimal ones.

Now we show that the Lorentz distance from the point $(0, 0)$ is π. To do this, we use the following Lemma 4.4 from [1].

Lemma 4.1. *Let M be a Lorentzian manifold with distance d.*
If $d(p, q) < \infty$, $p_n \to p$, and $q_n \to q$, then $d(p, q) \leq \liminf d(p_n, q_n)$.
If $d(p, q) = \infty$, $p_n \to p$, and $q_n \to q$, then $\lim\limits_{n\to\infty} d(p_n, q_n) = \infty$.

Denote $q_0 := (0, 0)$. Consider a sequence of points $q_n \to \tilde{q}$ in M. Assume that $d(q_0, \tilde{q}) = +\infty$. Then by Lemma 4.1 $\lim\limits_{n\to\infty} d(q_0, q_n) = +\infty$. But the length of the optimal curve is expressed by the time t of motion along it, and according to our calculations, for all $q_n \in \mathcal{B}_{(0,0)}$ there is an optimal trajectory, and we showed in Theorem 4.5 that $0 < t < \pi$. Hence, $\lim\limits_{n\to\infty} d(q_0, q_n) \leq \pi$. Consequently, $d(q_0, \tilde{q}) \leq \pi$.

Now we take the sequence $q_n \in \mathcal{B}_{(0,0)}$, $q_n \to \tilde{q} = (\tilde{\theta}, \pi - \arctan(\sinh|\tilde{\theta}|))$: $q_n = (\theta_n, \varphi_n) = (\tilde{\theta}, \pi - \arctan(\sinh|\tilde{\theta}|) - \frac{1}{n})$, $n \in \mathbb{N}$, $n \geq n_0$, choosing the initial n_0 such that $q_n \in \mathcal{B}_{(0,0)}$, for example, from the condition that $\pi - \arctan(\sinh|\tilde{\theta}|) - \frac{1}{n_0} \geq \pi/2 \Leftrightarrow n_0(\pi/2 - \arctan(\sinh|\tilde{\theta}|)) \geq 1 \Leftrightarrow n_0 \geq \frac{1}{\pi/2 - \arctan(\sinh|\tilde{\theta}|)}$. Now, according to the formulas of the Theorem 4.5,

$$d(q_0, q_n) = \pi - \arcsin \sqrt{\frac{\tan^2 \varphi_n - \sinh^2 \theta_n}{1 + \tan^2 \varphi_n}} = \pi - \arcsin \sqrt{\frac{\tan^2 [\arctan(\sinh|\tilde{\theta}|) + \frac{1}{n}] - \sinh^2 \tilde{\theta}}{1 + \tan^2 [\arctan(\sinh|\tilde{\theta}|) + \frac{1}{n}]}} \to \pi, \qquad n \to \infty.$$

Therefore, for any point $\tilde{q} \in \{(\theta, \varphi) : \theta \in \mathbb{R},\ \varphi = \pi - \arctan(\sinh|\theta|)\}$ we have $d(q_0, \tilde{q}) = \pi$.

As for the optimal trajectories, all the extremal trajectories filling the interior of the set $\mathcal{B}_{(0,0)}$ continue to $t = \pi$, ending up at the point $(0, \pi)$, as can be seen from the explicit formulas (4.29), and the issue of continuation was discussed in the Proposition 4.2. Their lengths are exactly equal to π, as a result of which they are all optimal. □

4.9 Properties of the distance function and Lorentzian spheres

4.9.1 Analyticity of the distance inside the set $\mathcal{B}_{(0,0)}$ and its asymptotics near the boundary of this set

Proposition 4.3. (1) *The distance from a point $q_0 = (0, 0)$ to any point $q_1 \in \operatorname{int} \mathcal{B}_{(0,0)}$ is given by the real-analytic function*

$$d(q_0, q_1) = t_{q_1} = \arccos(\cos\varphi_1 \cosh\theta_1). \tag{4.30}$$

(2) *Let the point $q_1 = (\theta_1, \varphi_1)$ satisfy the condition $\varphi_1 = \arctan\sinh|\theta_1|$, i.e., it belongs to the lower part of the boundary $\partial\mathcal{B}_{(0,0)}$. We define the 1-form $l = -\sinh 2\theta_1 \cos^2\varphi_1 d\theta + \sin 2\varphi_1 \cosh^2\theta_1 d\varphi$. If $q = (\theta, \varphi) \in \operatorname{int} \mathcal{B}_{(0,0)}$ and $q \to q_1$ so that*

$$\frac{(\Delta\theta, \Delta\varphi)}{\sqrt{(\Delta\theta)^2 + (\Delta\varphi)^2}} \to v, \qquad \Delta\theta = \theta - \theta_1, \quad \Delta\varphi = \varphi - \varphi_1,$$
$$l(v) \neq 0,$$

then

$$d(q_0, q) = \sqrt{l(\Delta\theta, \Delta\varphi)}(1 + o(1)).$$

Proof. (1) The distance $d(q_0, q)$ in our parametrization is the time of movement along the optimal trajectory from q_0 to q. We have obtained the formula (4.26) for it. We prove the formula (4.30) on this basis. For this, we consider the composition $\sin(\pi/2 - t_{q_1})$.

On the interval $\varphi \in (0, \pi/2)$:

$$\sin\left(\pi/2 - \arcsin\sqrt{\frac{\tan^2\varphi - \sinh^2\theta}{1 + \tan^2\varphi}}\right) = \cos\left(\arcsin\sqrt{\frac{\tan^2\varphi - \sinh^2\theta}{1 + \tan^2\varphi}}\right) = \sqrt{1 - \frac{\tan^2\varphi - \sinh^2\theta}{1 + \tan^2\varphi}} = \sqrt{\frac{1 + \sinh^2\theta}{1 + \tan^2\varphi}} =$$
$$= \cos\varphi \cosh\theta.$$

On the interval $\varphi \in (\pi/2, \pi)$:

$$\sin\left(\pi/2 - \left(\pi - \arcsin\sqrt{\frac{\tan^2\varphi - \sinh^2\theta}{1 + \tan^2\varphi}}\right)\right) = -\sin\left(\pi/2 - \arcsin\sqrt{\frac{\tan^2\varphi - \sinh^2\theta}{1 + \tan^2\varphi}}\right) = -\cosh\theta|\cos\varphi| = \cos\varphi\cosh\theta.$$

Since $\pi/2 - t_{q_1} \in (-\pi/2, \pi/2)$, we get:

$$\sin(\pi/2 - t_{q_1}) = \cos\varphi\cosh\theta \Leftrightarrow \pi/2 - t_{q_1} = \arcsin(\cos\varphi\cosh\theta) \Leftrightarrow t_{q_1} = \pi/2 - \arcsin(\cos\varphi\cosh\theta) = \arccos(\cos\varphi\cosh\theta).$$

The formula (4.30) is proven. In particular, the distance is a real-analytic function in the domain $\operatorname{int}(\mathcal{B}_{(0,0)})$.

(2) We calculate the asymptotics of the Lorentzian distance near the lower boundary of the reachable set. Let $\rho = ((\Delta\theta)^2 + (\Delta\varphi)^2)^{1/2} \to 0$. Then

$$d(q_0, q) = \arccos(\cos\varphi\cosh\theta) = \arcsin\sqrt{1 - \cos^2\varphi\cosh^2\theta} = \sqrt{1 - \cos^2\varphi\cosh^2\theta}(1 + o(1)).$$

Next,

$$\begin{aligned} 1 - \cos^2\varphi\cosh^2\theta &= 1 - (1 - 2\cos\varphi_1\sin\varphi_1\cosh^2\theta_1\Delta\varphi + 2\cos^2\varphi_1\cosh\theta_1\sinh\theta_1\Delta\theta) = l(\Delta\theta, \Delta\varphi) + o(\rho) = \\ &= \rho\left(l\left(\Delta\theta/\rho, \Delta\varphi/\rho\right) + o(1)\right) = \rho\left(l\left(v\right) + o(1)\right) = \rho l\left(v\right)\left(1 + o(1)\right) = l\left(\Delta\theta, \Delta\varphi\right)\left(1 + o(1)\right), \end{aligned}$$

and point (2) is proven. □

Figure 17: Graph of distance function

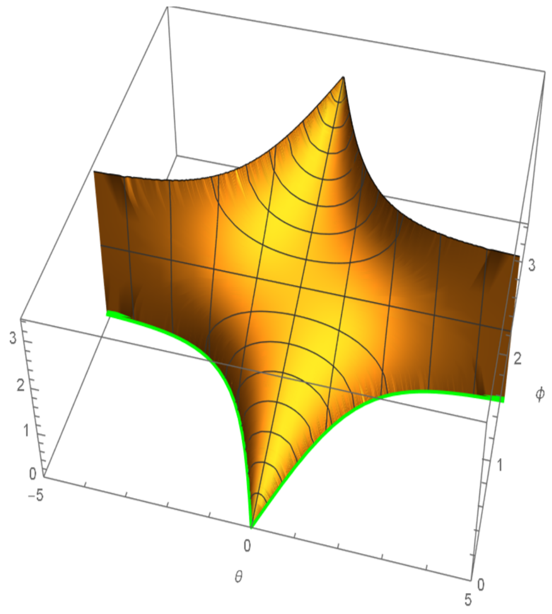


Figure 18: Graph of distance function

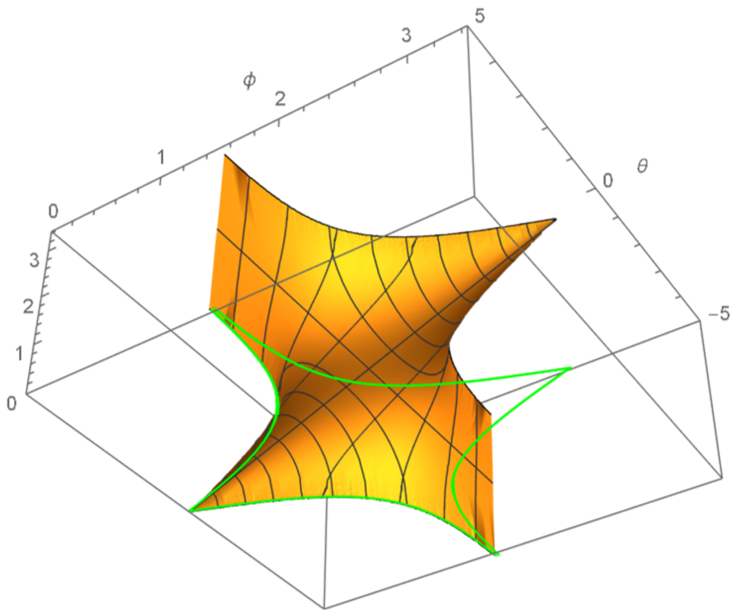
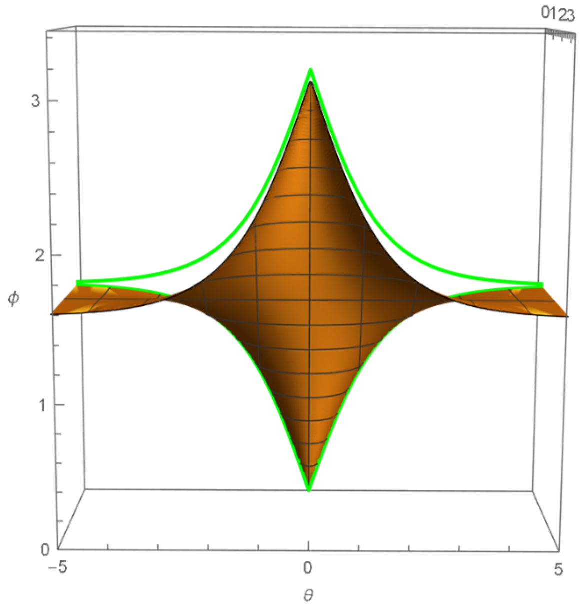


Figure 19: Graph of distance function, top view



4.9.2 Lorentzian spheres

From point (1) of the Proposition 4.3 we obtain

Corollary 4.2. *Lorentzian sphere of radius $r \in (0, \pi)$ is given by the equation*

$$\varphi = \arccos\left(\frac{\cos r}{\cosh\theta}\right).$$

Figure 20: Sphere of radius 1

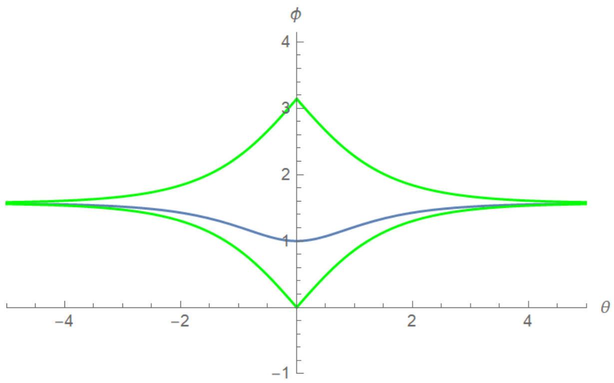


Figure 21: Sphere of radius 2

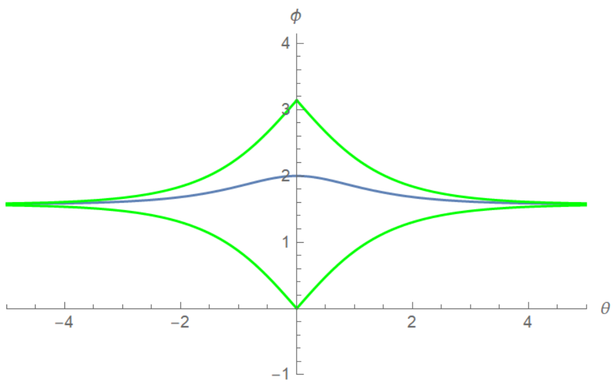
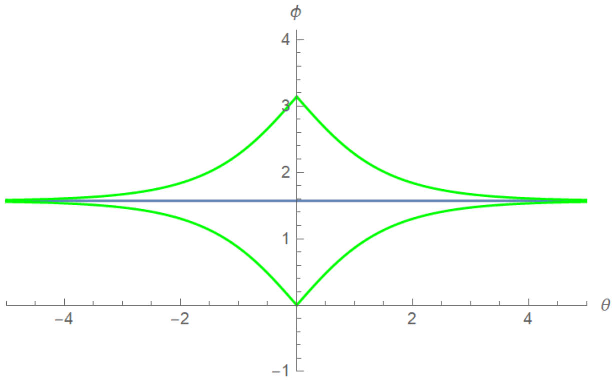


Figure 22: Sphere of radius $\pi/2$



4.10 Killing Fields

In this section, we calculate the infinitesimal symmetries of the Lorentzian structure and their flows, and on this basis we obtain an expression for the Lorentzian distance $d(q_0, q_1)$ for arbitrary points $q_0, q_1 \in M$.

4.11 Basis of the Lie algebra of Killing fields

Definition 4.3. [5, Chapter 9, Definition 22]

A vector field X is called *a Killing field of a metric* g if the Lie derivative of the metric along it is zero, i.e., $L_X g = 0$.

Proposition 4.4. *[1] A vector field $X \in \mathrm{Vec}(M)$ is a Killing field of metric g if and only if it satisfies the equality*

$$X(g(V, W)) = g([X, V], W) + g(V, [X, W]), \tag{4.31}$$

where V, W are arbitrary vector fields on M.

Proposition 4.5. *[1] The Killing vector fields form a Lie subalgebra in the Lie algebra of all vector fields on a manifold, and for a connected Lorentzian manifold of constant curvature the dimension of this Lie algebra is $n(n+1)/2$, where n is the dimension of the manifold.*

Theorem 4.8. (1) *The Lie algebra of Killing fields of the anti-de Sitter space $\widetilde{H_1^2}$ with metric $\tilde{g}$ is three-dimensional and its basis vector fields can be chosen as*

$$\hat{X}_1 = \cosh\theta X_1 = \partial_\varphi, \qquad \hat{X}_2 = \sinh\theta\cos\varphi X_1 + \sin\varphi X_2, \qquad \hat{X}_3 = -\sinh\theta\sin\varphi X_1 + \cos\varphi X_2.$$

(2) *The following relations hold for their commutators:*

$$[\hat{X}_1, \hat{X}_2] = \hat{X}_3, \qquad [\hat{X}_2, \hat{X}_3] = -\hat{X}_1, \qquad [\hat{X}_3, \hat{X}_1] = \hat{X}_2.$$

(3) *The Lie algebra of Killing fields is isomorphic to $\mathfrak{sl}(2)$.*

Proof. (1) First, note that our anti-de Sitter manifold $\widetilde{H_1^2}$ satisfies the conditions of Proposition 4.5 and has dimension 2, so its Lie algebra of Killing fields has dimension 3.

To find Killing fields of the metric $\tilde{g}$, we use equation (4.31).

Using the statement 3.1, we write out the required fields in terms of the basis of eigenvectors of the metric $g = d\theta^2 - \cosh^2\theta d\varphi^2$ (locally coinciding with $\tilde{g}$), $X = c_1 X_1 + c_2 X_2 = c_1 \frac{1}{\cosh\theta}\frac{\partial}{\partial\varphi} + c_2\frac{\partial}{\partial\theta}$.

We compose 3 equations for the unknown functional coefficients. As the fields V and W we take X_1 and X_2 in various combinations. But first we need to calculate the corresponding commutators $[X, X_1]$ and $[X, X_2]$:

$$[X, X_1] = [c_1 X_1 + c_2 X_2, X_1] = \left[\frac{c_1}{\cosh\theta}\frac{\partial}{\partial\varphi} + c_2\frac{\partial}{\partial\theta}, \frac{1}{\cosh\theta}\frac{\partial}{\partial\varphi}\right] =$$
$$= \left(c_2\partial_\theta\left(\frac{1}{\cosh\theta}\right) - \frac{1}{\cosh\theta}\partial_\varphi\left(\frac{c_1}{\cosh\theta}\right)\right)\frac{\partial}{\partial\varphi} + \left(-\frac{1}{\cosh\theta}\partial_\varphi(c_2)\right)\frac{\partial}{\partial\theta} =$$
$$= -\frac{c_2\sinh\theta}{\cosh^2\theta}\frac{\partial}{\partial\varphi} - \frac{\partial_\varphi c_1}{\cosh^2\theta}\frac{\partial}{\partial\varphi} - \frac{\partial_\varphi c_2}{\cosh\theta}\frac{\partial}{\partial\theta} = -\frac{1}{\cosh\theta}(c_2\sinh\theta + \partial_\varphi c_1)X_1 - \frac{\partial_\varphi c_2}{\cosh\theta}X_2.$$

$$[X, X_2] = \left[\frac{c_1}{\cosh\theta}\frac{\partial}{\partial\varphi} + c_2\frac{\partial}{\partial\theta}, \frac{\partial}{\partial\theta}\right] =$$
$$= \left(-\frac{\partial_\theta c_1}{\cosh\theta} + c_1\frac{\tanh\theta}{\cosh\theta}\right)\frac{\partial}{\partial\varphi} - \partial_\theta c_2\frac{\partial}{\partial\theta} = (-\partial_\theta c_1 + c_1\tanh\theta)X_1 - \partial_\theta c_2 X_2.$$

We form the first equation, taking X_1 as V and W:

$$X\left(g(X_1, X_1)\right) = g\left([X, X_1], X_1\right) + g(X_1, [X, X_1]) \Leftrightarrow 0 = 2g\left([X, X_1], X_1\right) \Leftrightarrow$$
$$\Leftrightarrow 0 = 2g\left(-\frac{1}{\cosh\theta}(c_2\sinh\theta + \partial_\varphi c_1)X_1 - \frac{\partial_\varphi c_2}{\cosh\theta}X_2, X_1\right) \Leftrightarrow$$
$$\Leftrightarrow 0 = \frac{2}{\cosh\theta}\left(c_2\sinh\theta + \partial_\varphi c_1\right).$$

For the second equation, we take the field X_2 as V and W:

$$X\left(g(X_2, X_2)\right) = g\left([X, X_2], X_2\right) + g(X_2, [X, X_2]) \Leftrightarrow 0 = 2g\left([X, X_2], X_2\right) \Leftrightarrow$$
$$\Leftrightarrow 0 = 2\left((-\partial_\theta c_1 + c_1\tanh\theta)X_1 - \partial_\theta c_2 X_2, X_2\right) \Leftrightarrow$$
$$\Leftrightarrow 0 = -2\partial_\theta c_2.$$

Finally, for the third equation we take $V = X_1$, $W = X_2$:

$$X\left(g(X_1, X_2)\right) = g\left([X, X_1], X_2\right) + g\left(X_1, [X, X_2]\right) \Leftrightarrow$$
$$\Leftrightarrow 0 = g\left(-\frac{1}{\cosh\theta}(c_2\sinh\theta + \partial_\varphi c_1)X_1 - \frac{\partial_\varphi c_2}{\cosh\theta}X_2, X_2\right) + g\left(X_1, (-\partial_\theta c_1 + c_1\tanh\theta)X_1 - \partial_\theta c_2 X_2\right) \Leftrightarrow$$
$$\Leftrightarrow 0 = -\frac{\partial_\varphi c_2}{\cosh\theta} - \left(-\partial_\theta c_1 + c_1\tanh\theta\right).$$

We obtain a system on functional coefficients c_1, c_2:

$$\begin{cases} c_2\sinh\theta + \partial_\varphi c_1 = 0, \\ \partial_\theta c_2 = 0, \\ \partial_\varphi c_2 = \cosh\theta\partial_\theta c_1 - c_1\sinh\theta. \end{cases} \tag{4.32}$$

From the second equation (4.32) it follows that $c_2(\varphi, \theta) = c_2(\varphi)$. Next, we can integrate the first equation (4.32):

$$\partial_\varphi c_1 = -c_2(\varphi)\sinh\theta \Leftrightarrow c_1 = -\sinh\theta\int_0^\varphi c_2(s)\,ds + f(\theta) = -\sinh\theta u(\varphi) + f(\theta).$$

We substitute into the third equation (4.32):

$$u''(\varphi) = \cosh\theta[-\cosh\theta u(\varphi) + f'(\theta)] - \sinh\theta[-\sinh\theta u(\varphi) + f(\theta)] \Leftrightarrow$$
$$\Leftrightarrow u''(\varphi) = [-\cosh^2\theta + \sinh^2\theta]u(\varphi) + \cosh\theta f'(\theta) - \sinh\theta f(\theta) \Leftrightarrow$$
$$\Leftrightarrow u''(\varphi) + u(\varphi) = \cosh\theta f'(\theta) - \sinh\theta f(\theta).$$

We see that the left side of the resulting equation depends only on φ, and the right side depends only on θ. This means that the left and right sides are equal to a constant. We get two equations: for $u(\varphi)$ and for $f(\theta)$:

$$u''(\varphi) + u(\varphi) = A = \cosh\theta f'(\theta) - \sinh\theta f(\theta).$$

Both equations are linear ODEs. The solution to the first one is found almost instantly:

$$u(\varphi) = B_1\cos\varphi + B_2\sin\varphi + A.$$

The second is solved by the method of variation of the constant:

$$f(\theta) = A\sinh\theta + B\cosh\theta.$$

So the coefficients look like this:

$$c_1(\varphi, \theta) = -\sinh\theta(B_1\cos\varphi + B_2\sin\varphi + A) + A\sinh\theta + B\cosh\theta = -\sinh\theta(B_1\cos\varphi + B_2\sin\varphi) + B\cosh\theta$$
$$c_2 = u'(\varphi) = -B_1\sin\varphi + B_2\cos\varphi$$

And we get Killing fields:

$$c_1X_1 + c_2X_2 = (-\sinh\theta(B_1\cos\varphi + B_2\sin\varphi) + B\cosh\theta)X_1 + (-B_1\sin\varphi + B_2\cos\varphi)X_2.$$

The basis can be chosen from 3 vector fields:

$$\hat{X}_1 = \cosh\theta X_1 = \partial_\varphi, \qquad \hat{X}_2 = \sinh\theta\cos\varphi X_1 + \sin\varphi X_2, \qquad \hat{X}_3 = -\sinh\theta\sin\varphi X_1 + \cos\varphi X_2.$$

(2) We calculate the commutators of the basis vectors of the Killing field algebra $\hat{X}_1$, $\hat{X}_2$, $\hat{X}_3$ obtained in the previous section:

$$[\hat{X}_1, \hat{X}_2] = \left[\partial_\varphi, \frac{\sinh\theta\cos\varphi}{\cosh\theta}\partial_\varphi + \sin\varphi\partial_\theta\right] = -\frac{\sinh\theta\sin\varphi}{\cosh\theta}\partial_\varphi + \cos\varphi\partial_\theta = \hat{X}_3,$$
$$[\hat{X}_1, \hat{X}_3] = \left[\partial_\varphi, -\frac{\sinh\theta\sin\varphi}{\cosh\theta}\partial_\varphi + \cos\varphi\partial_\theta\right] = \frac{\sinh\theta\cos\varphi}{\cosh\theta}\partial_\varphi - \sin\varphi\partial_\theta = -\hat{X}_2,$$
$$[\hat{X}_2, \hat{X}_3] = \left[\frac{\sinh\theta\cos\varphi}{\cosh\theta}\partial_\varphi + \sin\varphi\partial_\theta, -\frac{\sinh\theta\sin\varphi}{\cosh\theta}\partial_\varphi + \cos\varphi\partial_\theta\right] =$$
$$= \left(\tanh\theta\cos\varphi\tanh\theta(-\cos\varphi) + \sin\varphi(-\sin\varphi)\frac{1}{ch^2\theta} + \tanh\theta\sin\varphi\tanh\theta(-\sin\varphi) - \cos\varphi\cos\varphi\frac{1}{\cosh^2\theta}\right)\partial_\varphi +$$
$$+ \left(\tanh\theta\cos\varphi(-\sin\varphi) - \tanh\theta(-\sin\varphi)\cos\varphi\right)\partial_\theta =$$
$$= \left(-\tanh^2\theta(\cos^2\varphi + \sin^2\varphi) - \frac{1}{\cosh^2\theta}(\sin^2\varphi + \cos^2\varphi)\right)\partial_\varphi = -\frac{\sinh^2\theta + 1}{\cosh^2\theta}\partial_\varphi = -\partial_\varphi = -\hat{X}_1.$$

We obtained the following relations:

$$[\hat{X}_1, \hat{X}_2] = \hat{X}_3, \qquad [\hat{X}_2, \hat{X}_3] = -\hat{X}_1, \qquad [\hat{X}_3, \hat{X}_1] = \hat{X}_2.$$

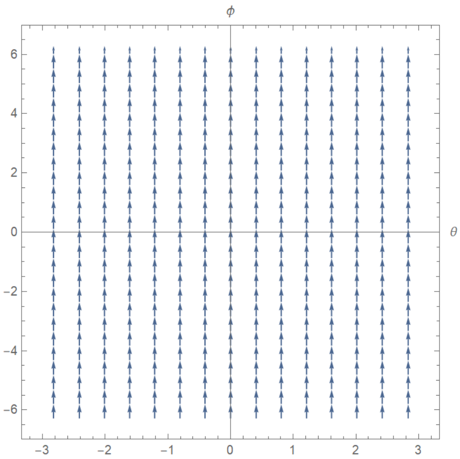
(3) The Lie algebra of Killing fields is isomorphic to $\mathfrak{sl}(2)$, which follows from the expressions for commutators obtained in the second section and the theorem on the classification of three-dimensional Lie algebras (see [7]). □

4.11.1 Phase portraits of the Killing fields

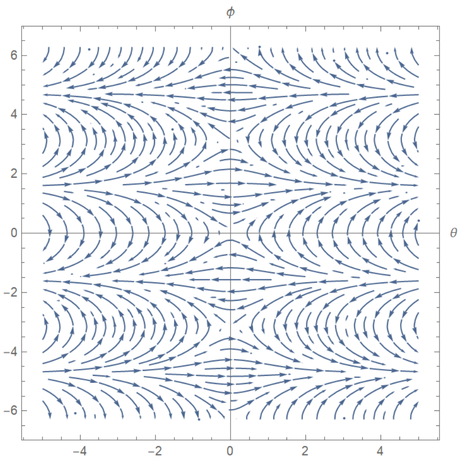
Phase portraits of the fields were obtained using the StreamPlot function in Wolfram Mathematica.

Figure 23: Killing field trajectories

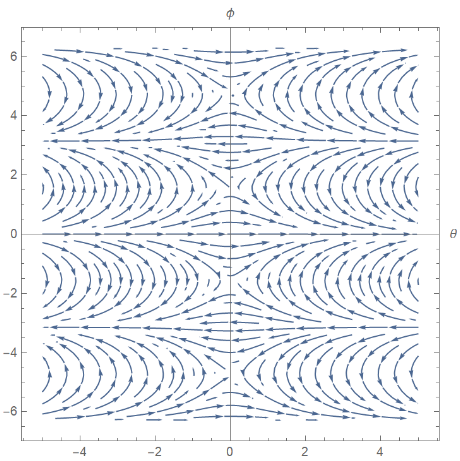
(a) Trajectories of $\hat{X}_1$



(b) Trajectories of $\hat{X}_2$



(c) Trajectories of $\hat{X}_3$



4.11.2 Lorentz distance between two arbitrary points of $\widetilde{H_1^2}$

Suppose we want to calculate the distance between points q_0, $q_1 \in \widetilde{H_1^2}$. Then we transfer point q_0 to $(0,0) = q_0'$ along the trajectories of the Killing fields along a certain route, and then — point q_1 to point q_1' parallel to this route. Since we know the Lorentz distance from the point $(0,0)$ to any point of $\widetilde{H_1^2}$ (theorems 4.5, 4.6 and 4.7 of this paper), and $d(q_0, q_1) = d(q_0', q_1')$, since the value of the metric does not change when transferred along the Killing fields, we can calculate the Lorentz distance between an arbitrary pair of points of the manifold.

Note that the field $\hat{X}_1 = \partial_\varphi$ allows moving up and down. Also, for $\varphi = 0$, the field $\hat{X}_3$ has the first coordinate equal to 0 and the second equal to 1, which allows moving left and right along this line. Therefore, our route will look like this:

1. $\theta_0 = 0$.
   We move along the field $\hat{X}_1$ (if $\varphi_0 < 0$) or $-\hat{X}_1$ (if $\varphi_0 > 0$) until the point $(0, 0)$.
2. $\varphi_0 = 0$.
   We move along the field $\hat{X}_3$ (if $\theta_0 < 0$) or $-\hat{X}_3$ (if $\theta_0 > 0$) until the point $(0, 0)$.
3. $\theta_0 < 0, \ \varphi_0 < 0$.
   We move along the field $\hat{X}_1$ until the intersection with the line $\varphi = 0$. We move along the field $\hat{X}_3$ until the point $(0, 0)$.
4. $\theta_0 < 0, \ \varphi_0 \geq 0$.
   We move along the field $-\hat{X}_1$ until the intersection with the line $\varphi = 0$. We move along the field $\hat{X}_3$ until the point $(0, 0)$.
5. $\theta_0 \geq 0, \ \varphi_0 < 0$.
   We move along the field $\hat{X}_1$ until the intersection with the line $\varphi = 0$. We move along the field $-\hat{X}_3$ until the point $(0, 0)$.
6. $\theta_0 \geq 0, \ \varphi_0 \geq 0$.
   We move along the field $-\hat{X}_1$ until the intersection with the line $\varphi = 0$. We move along the field $-\hat{X}_3$ until the point $(0, 0)$.

4.11.3 Killing Field Trajectories

Proposition 4.6. *The system of ODEs defined by the vector field $\hat{X}_1$:*

$$\dot{\theta} = 0, \qquad \dot{\varphi} = 1 \tag{4.33}$$

has the following solutions with initial conditions $\theta(0) = \theta_0$, $\varphi(0) = \varphi_0$:

$$\theta(t) \equiv \theta_0, \qquad \varphi(t) = t + \varphi_0.$$

Theorem 4.9. *The system of ODEs defined by the vector field $\hat{X}_2$:*

$$\dot{\theta} = \sin\varphi, \qquad \dot{\varphi} = \cos\varphi \tanh\theta \tag{4.34}$$

has a first integral $C = \cos\varphi \cosh\theta \in \mathbb{R}$. Let $s_1 = \operatorname{sign}\cos\varphi_0$, $s_2 = \operatorname{sign}\sin\varphi_0$, $s_3 = \operatorname{sign}\theta_0$, $n = [(\varphi_0 + \pi/2)/(2\pi)]$. Then the system (4.34) has the following solutions with initial conditions $\theta(0) = \theta_0$, $\varphi(0) = \varphi_0$ depending on the value of C.

1) If $C = 0$, then $\varphi(t) \equiv \varphi_0$, $\theta(t) = \theta_0 + s_2 t$.

2) Let $C^2 = 1$.

2.1) If $\sin\varphi_0 = 0$, then $\varphi(t) \equiv \varphi_0$, $\theta(t) \equiv \theta_0$.

2.2) If $\sin\varphi_0 \neq 0$, then

$$\begin{aligned} &\theta(t) = \operatorname{arsinh}(\sinh\theta_0 \exp(s_2 s_3 t)), \\ &\varphi(t) = \begin{cases} s_2 s_3 \arcsin(\tanh\theta(t)) + 2\pi n & \text{with } s_1 = 1, \\ \pi - s_2 s_3 \arcsin(\tanh\theta(t)) + 2\pi n & \text{with } s_1 = -1. \end{cases} \end{aligned} \tag{4.35}$$

3) If $C^2 \in (0, 1)$, then

$$\begin{aligned} &\theta(t) = \operatorname{arsinh}(\sqrt{1 - C^2}\sinh\tau), \qquad \tau = s_2 t + \operatorname{arsinh}(\sinh\theta_0/\sqrt{1 - C^2}), \\ &\varphi(t) = \begin{cases} s_2 \arcsin\sqrt{1 - C^2/\cosh^2\theta(t)} + 2\pi n & \text{for } s_1 = 1, \\ \pi - s_2 \arcsin\sqrt{1 - C^2/\cosh^2\theta(t)} + 2\pi n & \text{when } s_1 = -1. \end{cases} \end{aligned} \tag{4.36}$$

4) If $C^2 > 1$, then

$$\varphi(t) = \begin{cases} \arcsin x + 2\pi n & \textit{for } s_1 = 1, \\ \pi - \arcsin x + 2\pi n & \textit{with } s_1 = -1, \end{cases} \tag{4.37}$$
$$\theta(t) = s_3 \operatorname{arcosh}(C/\cos\varphi(t)),$$

where $x = (|s| - C^2 + 1)/(2\sqrt{s})$, $s = (e^\tau s_+ + s_-)/(1 + e^\tau)$, $s_\pm = C^2 + 1 \pm 2|C|$, $\tau = s_1 s_3 (s_+ - s_-)t/(2C) + \ln((s_0 - s_-)/(s_+ - s_0))$, $s_0 = r^2$, $r = \sqrt{x_0^2 + C^2 - 1} + x_0$, $x_0 = \sin\varphi_0$.

Remark 4.2. Since the field $\hat{X}_3$ is obtained from the field $\hat{X}_2$ by shifting along φ by $\pi/2$, it suffices to integrate only $\hat{X}_2$.

Denote by $e^{tX} : M \to M$ the flow of the vector field X on the manifold M.

4.11.4 Distance in terms of Killing fields flows

Theorem 4.10. *The Lorentzian distance between two arbitrary points $q_0, q_1 \in \widetilde{H_1^2}$ is $d(\tilde{q}_0, \tilde{q}_1)$, where $\tilde{q}_0 = (0,0) = e^{(-\theta_0 \hat{X}_3)} \circ e^{(-\varphi_0 \hat{X}_1)}(q_0)$ and, correspondingly, $\tilde{q}_1 = e^{(-\theta_0 \hat{X}_3)} \circ e^{(-\varphi_0 \hat{X}_1)}(q_1)$.*

Proof. Since $\hat{X}_1$ and $\hat{X}_3$ are Killing fields, shifting along the flows of these fields preserves the distance between points in our Lorentzian metric d.

The field $\hat{X}_1 = \partial_\varphi$ has the trajectories $\varphi(t) = t + \varphi_0$, $\theta(t) \equiv \theta_0$ as solutions, and the field $\hat{X}_3 = -\sinh\theta \sin\varphi X_1 + \cos\varphi X_2$ has the trajectory $\varphi(t) \equiv 0$, $\theta(t) = t + C$ as one of its solutions. This allows us to construct a trajectory consisting of the composition $e^{t_2 \hat{X}_3} \circ e^{t_1 \hat{X}_1}$, taking any point q_0 to the origin $\tilde{q}_0 = (0,0)$.

It remains to show that t_1 and t_2 have the stated form. Consider the following cases.

1. $\theta_0 = 0$.

   If $\varphi_0 < 0$, then we move along the field $\hat{X}_1$:
   $$\dot{\theta} = 0,\ \dot{\varphi} = 1 \Leftrightarrow \theta(t) \equiv C_\theta,\ \varphi(t) = t + C_\varphi.$$
   Initial condition $\theta(0) = \theta_0 = 0 = C_\theta$, $\varphi(0) = \varphi_0 = C_\varphi$. Therefore, t_1 is found from the condition $\varphi(t_1) = 0$, so we obtain the equation: $0 = t_1 + \varphi_0$, which means $t_1 = -\varphi_0$. And $t_2 = 0 = \theta_0$. We get: $e^{(-\varphi_0)\hat{X}_1}(q_0) = (0,0)$.

   If $\varphi_0 > 0$, then we move along the field $-\hat{X}_1$:
   $$\dot{\theta} = 0,\ \dot{\varphi} = -1 \Leftrightarrow \theta(t) \equiv C_\theta,\ \varphi(t) = -t + C_\varphi.$$
   Initial condition $\theta(0) = \theta_0 = 0 = C_\theta$, $\varphi(0) = \varphi_0 = C_\varphi$. Therefore, t_1 is found from the condition $\varphi(t_1) = 0$, so we obtain the equation: $0 = -t_1 + \varphi_0$, which means $t_1 = \varphi_0$. And $t_2 = 0 = \theta_0$. We get: $e^{(-\varphi_0 \hat{X}_1)}(q_0) = (0,0)$.

2. $\varphi_0 = 0$.

   If $\theta_0 < 0$, then $t_1 = 0 = \varphi_0$, and we move only along the field $\hat{X}_3$, namely along the solution $\theta(t) = t + C_\theta$, $\varphi(t) \equiv 0$. Initial condition $\theta(0) = \theta_0 = C_\theta$. Therefore, we find t_2 from the condition $\theta(t_2) = 0$, from which we obtain the equation: $0 = t_2 + \theta_0$, so $t_2 = -\theta_0$. We get: $e^{(-t_2)\hat{X}_3}(q_0) = (0,0)$.

   If $\theta_0 > 0$, then $t_1 = 0 = \varphi_0$, and we move only along the field $-\hat{X}_3$, namely along the solution $\theta(t) = -t + C_\theta$, $\varphi(t) \equiv 0$. The initial condition $\theta(0) = \theta_0 = C_\theta$. Therefore, we find t_2 from the condition $\theta(t_2) = 0$, from which we get the equation: $0 = -t_2 + \theta_0$, so $t_2 = \theta_0$. We get: $e^{(-t_2 \hat{X}_3)}(q_0) = (0,0)$.

3. $\theta_0 < 0,\ \varphi_0 < 0$.

   Combining the calculations of the first and second points, we obtain the time of movement along the field $\hat{X}_1$: $t_1 = -\varphi_0$, and then the time of movement along the field $\hat{X}_3$: $t_2 = -\theta_0$. We obtain: $e^{(-\theta_0)\hat{X}_3} \circ e^{(-\varphi_0)\hat{X}_1}(q_0) = (0,0)$.

4. $\theta_0 < 0,\ \varphi_0 > 0$.

   Combining the calculations of the first and second points, we obtain the time of motion along the field $-\hat{X}_1$: $t_1 = \varphi_0$, and then the time of motion along the field $\hat{X}_3$: $t_2 = -\theta_0$. We obtain: $e^{(-\theta_0 \hat{X}_3)} \circ e^{(-\varphi_0)\hat{X}_1}(q_0) = (0,0)$.

5. $\theta_0 > 0,\ \varphi_0 < 0$.

   Combining the calculations of the first and second points, we obtain the time of motion along the field $\hat{X}_1$: $t_1 = -\varphi_0$, and then the time of motion along the field $-\hat{X}_3$: $t_2 = \theta_0$. We get: $e^{(-\theta_0)\hat{X}_3} \circ e^{(-\varphi_0 \hat{X}_1)}(q_0) = (0,0)$.

6. $\theta_0 > 0,\ \varphi_0 > 0$.

   Combining the calculations of the first and second points, we get the time of movement along the field $-\hat{X}_1$: $t_1 = \varphi_0$, and then the time of movement along the field $-\hat{X}_3$: $t_2 = \theta_0$. We get: $e^{(-\theta_0 \hat{X}_3)} \circ e^{(-\varphi_0 \hat{X}_1)}(q_0) = (0, 0)$.

□

5 Conclusion

The methods of geometric control theory have proven to be very fruitful for the study of the Lorentzian anti-de Sitter plane. It would be interesting to apply them to more complex Lorentzian structures of variable curvature, for example, to the Schwarzschild and Kerr spaces [1].

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