Flat sub-Lorentzian structures on Martinet distribution *

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Abstract

Two flat sub-Lorentzian problems on the Martinet distribution are studied. For the first one, the attainable set has a nontrivial intersection with the Martinet plane, but for the second one it does not. Attainable sets, optimal trajectories, sub-Lorentzian distances and spheres are described.

Keywords: Sub-Lorentzian geometry, geometric control theory, Martinet distribution, sub-Lorentzian length maximizers, sub-Lorentzian distance, sub-Lorentzian sphere

Contents

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1 Introduction

Sub-Riemannian geometry studies manifolds M in which the distance between points $q_0, q_1 \in M$ is the infimum of the lengths of all curves tangent to a given distribution $\Delta \subset TM$ and connecting q_0 to q_1 [2, 18]. In particular, each subspace $\Delta_q \subset T_qM$ of the distribution is equipped with a scalar product g_q and the length of a curve $q(t)$, $t \in [0, t_1]$, with tangents in Δ, is defined as in Riemannian geometry by the familiar integral expression $l(q) = \int_0^{t_1} \sqrt{g(\dot{q}(t), \dot{q}(t))}dt$. If in each space $\Delta_q \subset T_qM$ we define a non-degenerate quadratic form g_q of index 1, then a sub-Lorentzian structure (Δ, g) will be defined on the manifold M. Here the natural problem is to find the longest relative to g curve connecting given points. Sub-Lorentzian geometry strives to build a theory similar to the rich theory of sub-Riemannian geometry, and is at the beginning of its development. The foundations of sub-Lorentzian geometry were laid in the works of M. Grochowski [7–12], see also [6, 13–15].

Just as in sub-Riemannian geometry, the simplest sub-Lorentzian problem arises on the Heisenberg group; it has been fully studied [9, 22]. The next most important model of sub-Riemannian geometry after the Heisenberg group arises on the Martinet distribution [2, 3, 18, 23].

The purpose of this work is to consider two flat sub-Lorentzian problems on the Martinet distribution: to describe the optimal synthesis, distance and spheres. In the first problem, the future cone has a non-trivial intersection with the tangent space to the Martinet surface; in the second case this intersection is trivial. Accordingly, in the first case the sub-Lorentzian geometry is much more complicated, see Conclusion.

The structure of this work is as follows. In Section 2 we recall the basic facts of sub-Lorentzian geometry required in the sequel.

The main Sections 3 and 4 are devoted respectively to the first and the second flat sub-Lorentzian problems on the Martinet distribution; they have identical structure as follows. First we find an invariant set (a candidate attainable set) via the geometric statement of Pontryagin maximum principle. Then we describe explicitly abnormal and normal extremal trajectories; normal trajectories are parametrized by the sub-Lorentzian exponential mapping. We prove diffeomorphic properties of the exponential mapping via Hadamard's global diffeomorphism theorem. On this basis we show that the above-mentioned invariant set is indeed the attainable set, and prove a theorem on existence of optimal trajectories. After that we study optimality of extremal trajectories and construct an optimal synthesis, i.e., for any point q_1 reachable from a fixed initial point q_0 we find an optimal trajectory connecting q_0 to q_1. We complete our study by describing the main properties of sub-Lorentzian distances and spheres. It turns out that for the first problem (where the future cone intersects nontrivially the tangent space of the Martinet surface Π) the optimal synthesis has cut points and is two-valued on Π.

Some additional features of the two problems are presented in the concluding Sec. 5.

Symbolic computations and generation of pictures were performed in Wolfram Mathematica.

2 Sub-Lorentzian geometry

A sub-Lorentzian structure on a smooth manifold M is a pair (Δ, g) consisting of a vector distribution $\Delta \subset TM$ and a Lorentzian metric g on Δ, i.e., a nondegenerate quadratic form g of negative inertia index 1. Let us recall some basic definitions of sub-Lorentzian geometry. A vector $v \in T_qM$, $q \in M$, is called horizontal if $v \in \Delta_q$. A horizontal vector v is called:

- timelike if $g(v) < 0$,
- spacelike if $g(v) > 0$ or $v = 0$,
- lightlike if $g(v) = 0$ and $v \neq 0$,
- nonspacelike if $g(v) \leq 0$.

A Lipschitz curve in M is called timelike if it has timelike velocity vector a.e.; spacelike, lightlike and nonspacelike curves are defined similarly.

A time orientation X is an arbitrary timelike vector field in M. A nonspacelike vector $v \in \Delta_q$ is future directed if $g(v, X(q)) < 0$, and past directed if $g(v, X(q)) > 0$.

A future directed timelike curve $q(t)$, $t \in [0, t_1]$, is called arclength parametrized if $g(\dot{q}(t), \dot{q}(t)) \equiv -1$. By a simple change of variables, any future directed timelike curve can be parametrized by arclength.

The length of a nonspacelike curve $\gamma \in \mathrm{Lip}([0, t_1], M)$ is

$$l(\gamma) = \int_0^{t_1} |g(\dot{\gamma}, \dot{\gamma})|^{1/2} dt.$$

For points $q_1, q_2 \in M$ denote by $\Omega_{q_1 q_2}$ the set of all future directed nonspacelike curves in M that connect q_1 to q_2. In the case $\Omega_{q_1 q_2} \neq \emptyset$ denote the sub-Lorentzian distance from the point q_1 to the point q_2 as

$$d(q_1, q_2) = \sup\{l(\gamma) \mid \gamma \in \Omega_{q_1 q_2}\}. \tag{2.1}$$

And if $\Omega_{q_1 q_2} = \emptyset$ then $d(q_1, q_2) := 0$. A future directed nonspacelike curve γ is called a sub-Lorentzian length maximizer if it realizes the supremum in (2.1) between its endpoints $\gamma(0) = q_1$, $\gamma(t_1) = q_2$.

The causal future of a point $q_0 \in M$ is the set $J^+(q_0)$ of points $q_1 \in M$ for which there exists a future directed nonspacelike curve γ that connects q_0 and q_1.

Let $q_0 \in M$, $q_1 \in J^+(q_0)$. The search for sub-Lorentzian length maximizers that connect q_0 with q_1 reduces to the search for future directed nonspacelike curves γ that solve the problem

$$l(\gamma) \to \max, \qquad \gamma(0) = q_0, \quad \gamma(t_1) = q_1. \tag{2.2}$$

A set of vector fields $X_1, \dots, X_k \in \mathrm{Vec}(M)$ is an orthonormal frame for a sub-Lorentzian structure (Δ, g) if for all $q \in M$

$$\begin{aligned}
&\Delta_q = \mathrm{span}(X_1(q), \dots, X_k(q)),\\
&g_q(X_1, X_1) = -1, \qquad g_q(X_i, X_i) = 1, \quad i = 2, \dots, k,\\
&g_q(X_i, X_j) = 0, \quad i \neq j.
\end{aligned}$$

Assume that a time orientation is defined by a timelike vector field $X \in \mathrm{Vec}(M)$ for which $g(X, X_1) < 0$ (e.g., $X = X_1$). Then the sub-Lorentzian causal future problem for the sub-Lorentzian structure with the orthonormal frame $X_1, \dots, X_k$ is stated as the following optimal control problem:

$$\begin{aligned}
&\dot{q} = \sum_{i=1}^{k} u_i X_i(q), \qquad q \in M,\\
&u \in U = \left\{ (u_1, \dots, u_k) \in \mathbb{R}^k \mid u_1 \geq \sqrt{u_2^2 + \cdots + u_k^2} \right\},\\
&q(0) = q_0, \qquad q(t_1) = q_1,\\
&l(q(\cdot)) = \int_0^{t_1} \sqrt{u_1^2 - u_2^2 - \cdots - u_k^2}\, dt \to \max .
\end{aligned}$$

Remark 1. *The sub-Lorentzian length is preserved under monotone Lipschitz time reparametrizations $t(s)$, $s \in [0, s_1]$. Thus if $q(t)$, $t \in [0, t_1]$, is a sub-Lorentzian length maximizer, then so are any of its reparametrizations $q(t(s))$, $s \in [0, s_1]$.*

In this paper we choose primarily the following parametrization of trajectories: the arclength parametrization ($u_1^2 - u_2^2 - \cdots - u_k^2 \equiv 1$) for timelike trajectories, and the parametrization with $u_1(t) \equiv 1$ for future directed lightlike trajectories.

3 The first problem

Let $M = \mathbb{R}^3_{x,y,z}$, $X_1 = \frac{\partial}{\partial x}$, $X_2 = \frac{\partial}{\partial y} + \frac{x^2}{2} \frac{\partial}{\partial z}$. The distribution $\Delta = \mathrm{span}(X_1, X_2)$ is called the Martinet distribution [2, 3, 18, 23]. The plane $\Pi = \{x = 0\}$ is called the Martinet surface. The distribution Δ has growth vector $(2, 3)$ outside of Π, and growth vector $(2, 2, 3)$ on Π. The Lie algebra generated by the vector fields X_1, X_2 is a Carnot algebra (Engel algebra), the nonzero Lie brackets of these vector fields are: $[X_1, X_2] = x X_3$, $[X_1, x X_3] = X_3 := \frac{\partial}{\partial z}$.

In this section we study a sub-Lorentzian problem in which the interior of the future cone intersects nontrivially with the tangent space to the Martinet plane Π.

3.1 Problem statement

The first flat sub-Lorentzian problem on the Martinet distribution is stated as the following optimal control problem [4, 21]:

$$\dot{q} = u_1 X_1 + u_2 X_2, \qquad q \in M, \tag{3.1}$$

$$u = (u_1, u_2) \in U_1 = \{u_2 \geq |u_1|\}, \tag{3.2}$$

$$q(0) = q_0 = (0,0,0), \qquad q(t_1) = q_1, \tag{3.3}$$

$$l = \int_0^{t_1} \sqrt{u_2^2 - u_1^2 dt} \to \max, \tag{3.4}$$

see Fig. 1.

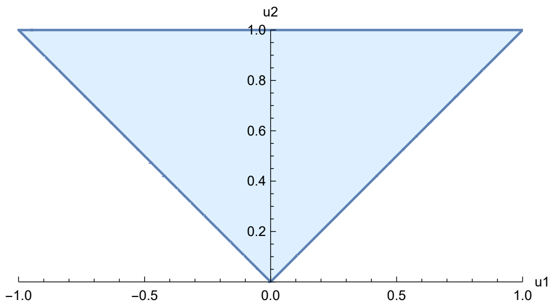

Figure 1: The set U_1

3.2 Invariant set

In this subsection we compute an invariant set $\mathcal{B}_1$ of system (3.1), (3.2). Later, in Th. 3, we prove that $\mathcal{B}_1$ is the attainable set $\mathcal{A}_1$ of system (3.1), (3.2) from the point q_0 for arbitrary nonnegative time (the causal future of the point q_0).

By the geometric statement of the Pontryagin maximum principle (PMP) for free time ([4], Th. 12.8), if a trajectory $q(t)$ corresponding to a control $u(t)$, $t \in [0, t_1]$, satisfies the inclusion $q(t_1) \in \partial \mathcal{A}_1$, then there exists a Lipschitz curve $\lambda_t \in T^*_{q(t)} M$, $\lambda_t \neq 0$, $t \in [0, t_1]$, such that

$$\dot{\lambda}_t = \vec{h}_{u(t)}(\lambda_t), \tag{3.5}$$

$$h_{u(t)}(\lambda_t) = \max_{u \in U_1} h_u(\lambda_t), \tag{3.6}$$

$$h_{u(t)}(\lambda_t) = 0$$

for almost all $t \in [0, t_1]$. Here $h_u(\lambda) = u_1 h_1(\lambda) + u_2 h_2(\lambda)$, $h_i(\lambda) = \langle \lambda, X_i(\pi(\lambda)) \rangle$, $i = 1, 2$, and $\pi : T^*M \to M$ is the canonical projection of the cotangent bundle, $\pi(\lambda) = q$, $\lambda \in T^*_q M$. Moreover, $\vec{h}_u(\lambda)$ is the Hamiltonian vector field on the cotangent bundle T^*M with the Hamiltonian $h_u(\lambda)$.

We have $[X_1, X_1] = x X_3$, $X_3 = \frac{\partial}{\partial z}$, and if we denote $h_3(\lambda) = \langle \lambda, X_3(\pi(\lambda)) \rangle$, then the Hamiltonian system (3.5) reads as

$$\dot{h}_1 = -u_2 x h_3, \quad \dot{h}_2 = u_1 x h_3, \quad \dot{h}_3 = 0, \quad \dot{q} = u_1 X_1 + u_2 X_2.$$

The maximality condition (3.6) implies that up to reparametrization there can be two cases:

a) $u(t) = (\pm 1, 1)$,

b) $u(t) = (0, 1)$, $x(t) = 0$.

Take any $0 \leq t_1 \leq t_2$ and compute trajectories with one switching corresponding to the following controls:

1) Let

$$u(t) = \begin{cases} (1,1), & t \in [0, t_1], \\ (-1,1), & t \in [t_1, t_2]. \end{cases}$$

Then $x(t) = t$, $y(t) = t$, $z(t) = t^3/6$ for $t \in [0, t_1]$, $x(t) = 2t_1 - t$, $y(t) = t$, $z(t) = t_1^3/6 + (4t_1^2(t - t_1) - 2t_1(t^2 - t_1^2) + (t^3 - t_1^3)/3)$ for $t \in [t_1, t_2]$, thus $x(t_2) = 2t_1 - t_2$, $y(t_2) = t_2$, $z(t_2) = -t_1^3 + 2t_1^2 t_2 - t_1 t_2^2 + t_2^3/6$. Thus the endpoint $q(t_2)$ satisfies the equality

$$z = (-3x^3 + 3x^2 y + 3xy^2 + y^3)/24. \tag{3.7}$$

2) Let

$$u(t) = \begin{cases} (-1,1), & t \in [0, t_1], \\ (1,1), & t \in [t_1, t_2]. \end{cases}$$

Then $x(t) = -t$, $y(t) = t$, $z(t) = t^3/6$ for $t \in [0, t_1]$, $x(t) = t - 2t_1$, $y(t) = t$, $z(t) = ((t^3 - t_1^3)/3 - 2t_1(t^2 - t_1^2) + 4t_1^2(t - t_1))/2$ for $t \in [t_1, t_2]$, thus $x(t_2) = t_2 - 2t_1$, $y(t_2) = t_2$, $z(t_2) = -t_1^3 + 2t_1^2 t_2 - t_1 t_2^2 + t_2^3/6$. Thus the endpoint $q(t_2)$ satisfies the equality

$$z = (3x^3 + 3x^2 y - 3xy^2 + y^3)/24. \tag{3.8}$$

3) Let

$$u(t) = \begin{cases} (0,1), & t \in [0, t_1], \\ (1,1), & t \in [t_1, t_2]. \end{cases}$$

Then $x(t) = 0$, $y(t) = t$, $z(t) = 0$ for $t \in [0, t_1]$, $x(t) = t - t_1$, $y(t) = t$, $z(t) = (t - t_1)^3/6$ for $t \in [t_1, t_2]$, thus $x(t_2) = t_2 - t_1$, $y(t_2) = t_2$, $z(t_2) = (t_2 - t_1)^3/6$. Thus the endpoint $q(t_2)$ satisfies the equality

$$z = x^3/6. \tag{3.9}$$

4) Finally, let

$$u(t) = \begin{cases} (0,1), & t \in [0, t_1], \\ (-1,1), & t \in [t_1, t_2]. \end{cases}$$

Then $x(t) = 0$, $y(t) = t$, $z(t) = 0$ for $t \in [0, t_1]$, $x(t) = t_1 - t$, $y(t) = t$, $z(t) = (t - t_1)^3/6$ for $t \in [t_1, t_2]$, thus $x(t_2) = t_1 - t_2$, $y(t_2) = t_2$, $z(t_2) = (t_2 - t_1)^3/6$. Thus the endpoint $q(t_2)$ satisfies the equality

$$z = -x^3/6. \tag{3.10}$$

Consider the surfaces S_1–S_4 given by Eqs. (3.7)–(3.10) respectively,

$$
\begin{aligned}
&S_1 \ : \ z = (-3x^3 + 3x^2y + 3xy^2 + y^3)/24, && x \geq 0,\\
&S_2 \ : \ z = (3x^3 + 3x^2y - 3xy^2 + y^3)/24, && x \leq 0,\\
&S_3 \ : \ z = x^3/6, && x \geq 0,\\
&S_4 \ : \ z = -x^3/6, && x \leq 0.
\end{aligned}
$$

Introduce the homogeneous coordinates on the set $\{y \neq 0\}$ induced by the one-parameter group of dilations $\delta_\alpha \ : \ (x, y, z) \mapsto (\alpha x, \alpha y, \alpha^3 z)$, $\alpha > 0$:

$$\xi = \frac{x}{y}, \qquad \eta = \frac{24z - 3x^2y - y^3}{24y^3}. \tag{3.11}$$

Then the surfaces S_1–S_4 are given as follows:

$$
\begin{aligned}
&S_1 \ : \ \eta = \frac{\xi(1-\xi^2)}{8} =: \varphi_1(\xi), && \xi \in [0, 1],\\
&S_2 \ : \ \eta = \frac{\xi(\xi^2-1)}{8} =: \varphi_2(\xi) = \varphi_1(-\xi), && \xi \in [-1, 0],\\
&S_3 \ : \ \eta = \frac{\xi^3}{6} - \frac{3\xi^2+1}{24} =: \varphi_3(\xi), && \xi \in [0, 1],\\
&S_4 \ : \ \eta = -\frac{\xi^3}{6} - \frac{3\xi^2+1}{24} =: \varphi_4(\xi) = \varphi_3(-\xi), && \xi \in [-1, 0].
\end{aligned}
$$

The surface $\cup_{i=1}^4 S_i$ bounds a domain

$$\mathcal{B}_1 = \begin{cases} \varphi_3(\xi) \leq \eta \leq \varphi_1(\xi), & 0 \leq \xi \leq 1,\\ \varphi_4(\xi) \leq \eta \leq \varphi_2(\xi), & -1 \leq \xi \leq 0, \end{cases}$$

see Figs. 2–5.

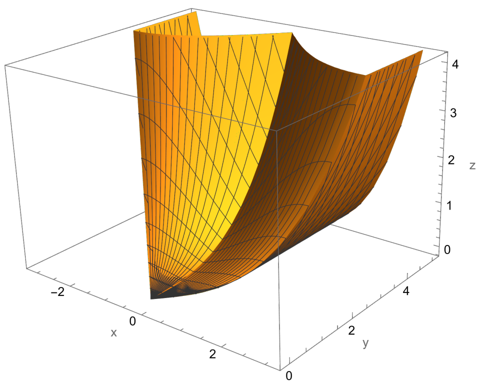

Figure 2: The boundary of $\mathcal{B}_1$

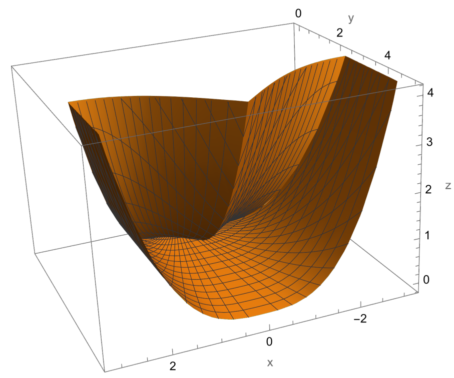

Figure 3: The boundary of $\mathcal{B}_1$

Recall that $\mathcal{A}_1$ is the attainable set of system (3.1), (3.2) from the point q_0 for arbitrary non-negative time (the causal future of the point q_0).

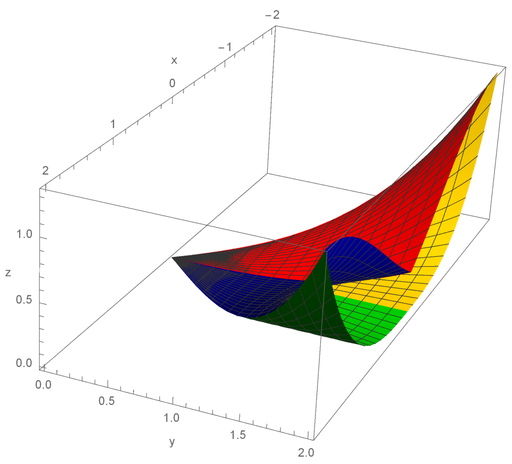

Figure 4: The boundary of $\mathcal{B}_1$

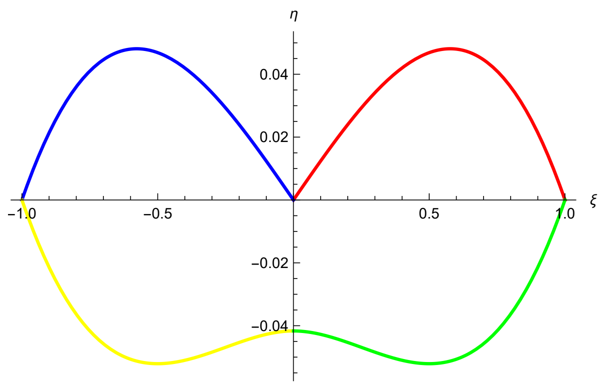

Figure 5: The boundary of $\mathcal{B}_1$ projected to (ξ, η)

Proposition 1. *The set* $\mathcal{B}_1$ *is an invariant domain of system* (3.1), (3.2). *Moreover,* $\mathcal{A}_1 \subset B_1$.

Proof. Direct computation shows that on each of the surfaces S_1–S_4 the vector field $u_1 X_1 + u_2 X_2$, $(u_1, u_2) \in U_1$, is directed inside the domain $\mathcal{B}_1$. Since $q_0 \in \mathcal{B}_1$, then $\mathcal{A}_1 \subset \mathcal{B}_1$. □

We show in Th. 3 that $\mathcal{A}_1 = \mathcal{B}_1$.

3.3 Extremal trajectories

The family of Hamiltonians determined by the Pontryagin maximum principle (PMP) [4, 19] are given by the $h_u^\nu(\lambda) = u_1 h_1(\lambda) + u_2 h_2(\lambda) - \nu\sqrt{u_2^2 - u_1^2}$, where $\lambda \in T^*M$, $(u_1, u_2) \in U_1$ and $\nu \in \{-1, 0\}$. By PMP (Th. 12.10 [4]), if $q(t)$, $t \in [0, t_1]$, is an optimal trajectory in problem (3.1)–(3.4), then there exist a Lipschitz curve $\lambda_t \in T^*_{q(t)}M$, $t \in [0, t_1]$, and a number $\nu \in \{-1, 0\}$ such that

$$\dot{\lambda}_t = \vec{h}^\nu_{u(t)}(\lambda_t),$$
$$h^\nu_{u(t)}(\lambda_t) = \max_{v \in U_1} h_v(\lambda_t),$$
$$(\lambda_t, \nu) \neq (0, 0)$$

for almost all $t \in [0, t_1]$.

3.3.1 Abnormal extremal trajectories

If $\nu = 0$, then the control satisfies, up to reparametrization, the conditions

a) $u(t) = (\pm 1, 1)$,

b) $u(t) = (0, 1)$, $x(t) = 0$,

and has up to one switching. These trajectories were computed in Subsec. 3.2, they form the boundary of the candidate attainable set $\mathcal{B}_1$.

Remark 2. *Abnormal trajectories starting from an arc on the plane* Π *change their causal type: first they are timelike (when they belong to* Π*), then lightlike. The remaining extremal trajectories preserve the causal type.*

3.3.2 Normal extremals

If $\nu = -1$, then extremals satisfy the Hamiltonian system with the Hamiltonian $H = (h_1^2 - h_2^2)/2$, $h_2 < -|h_1|$:

$$\dot{h}_1 = h_2 h_3 x, \quad \dot{h}_2 = h_1 h_3 x, \quad \dot{h}_3 = 0, \quad \dot{x} = h_1, \quad \dot{y} = -h_2, \quad \dot{z} = -h_2 x^2/2. \tag{3.12}$$

We can choose the arclength parametrization on normal extremal trajectories and thus assume that $H \equiv -1/2$. In the coordinates $h_1 = \sinh\psi$, $h_2 = -\cosh\psi$, $h_3 = c$; $\psi, c \in \mathbb{R}$, the Hamiltonian system (3.12) reads as

$$\dot{\psi} = -cx, \qquad \dot{c} = 0, \tag{3.13}$$

$$\dot{x} = \sinh\psi, \qquad \dot{y} = \cosh\psi, \qquad \dot{z} = \frac{x^2}{2}\cosh\psi. \tag{3.14}$$

This system has an energy integral $E = cx^2/2 + \cosh\psi \in [1, +\infty)$.

Remark 3. *The normal Hamiltonian system* (3.13), (3.14) *has a discrete symmetry — reflection*

$$(\psi, c, x, y, z) \mapsto (-\psi, c, -x, y, z), \tag{3.15}$$

and a one-parameter family of symmetries — dilations

$$(t, \psi, c, x, y, z) \mapsto (\alpha t, \psi, c/\alpha^2, \alpha x, \alpha y, \alpha^3 z), \qquad \alpha > 0. \tag{3.16}$$

Moreover, the parallel translations

$$(x, y, z) \mapsto (x, y + a, z + b), \qquad a, b \in \mathbb{R}, \tag{3.17}$$

are symmetries of the problem since their generating vector fields $\frac{\partial}{\partial y}$, $\frac{\partial}{\partial z}$ *commute with the vector fields* X_1, X_2 *of the orthonormal frame.*

1) If $c = 0$, then

$$\psi \equiv \psi_0, \qquad x = t\sinh\psi_0, \qquad y = t\cosh\psi_0, \qquad z = t^3/6\cosh\psi_0\sinh^2\psi_0. \tag{3.18}$$

If $c \neq 0$, then extremal trajectories in the Martinet flat case are obtained by a linear change of variables from extremal trajectories of a left-invariant sub-Lorentzian problem on the Engel group [5].

2) Let $c = l^2 > 0$.

2.1) If $\psi_0 = 0$, then $x = z \equiv 0$, $y = t$.

2.2) Let $\operatorname{sgn}\psi_0 = \pm 1$, $E = \cosh\psi_0 > 1$, $k = \sqrt{\frac{E-1}{E+1}} \in (0,1)$, $k' = \sqrt{1-k^2}$, $m = lk'$, $æ = 1/k'$, $\tau = ælt$. Then

$$x = \pm\frac{2k}{m}\operatorname{sn}\tau,$$
$$y = \frac{1}{m}(2\operatorname{E}(\tau) - k'^2\tau),$$
$$z = -\frac{2}{3m^3}\left(k'^2\tau + 2k^2\operatorname{sn}\tau\operatorname{cn}\tau\operatorname{dn}\tau - (1+k^2)\operatorname{E}(\tau)\right),$$

where $\operatorname{sn}\tau$, $\operatorname{cn}\tau$, $\operatorname{dn}\tau$ are Jacobi's elliptic functions with modulus k, and $\operatorname{E}(\tau) = \int_0^\tau \operatorname{dn}^2 t dt$ is Jacobi's epsilon function [17,25]. See Figs. 6–9.

3) Let $c = -l^2 < 0$.

3.1) If $\psi_0 = 0$, then $x = z \equiv 0$, $y = t$.

3.2) Let $\operatorname{sgn}\psi_0 = \pm 1$, $E = \cosh\psi_0 > 1$, $k = \sqrt{\frac{2}{1+E}} \in (0,1)$, $k' = \sqrt{1-k^2}$, $m = kl$, $\tau = lt/k$. Then

$$x = \pm\frac{2k'}{m}\frac{\operatorname{sn}\tau}{\operatorname{cn}\tau},$$
$$y = \frac{1}{m}\left((2-k^2)\tau + 2\frac{\operatorname{dn}\tau\operatorname{sn}\tau}{\operatorname{cn}\tau} - 2\operatorname{E}(\tau)\right),$$
$$z = -\frac{2}{3m^3}\left(2k'^2\tau + (k^2-2)\operatorname{E}(\tau) + (k^2 + (k^2-2)\operatorname{sn}^2\tau)\frac{\operatorname{dn}\tau\operatorname{sn}\tau}{\operatorname{cn}^3\tau}\right),$$
$$\tau \in [0, K(k)),$$

where $K(k)$ is the complete elliptic integral of the second kind [17,25].

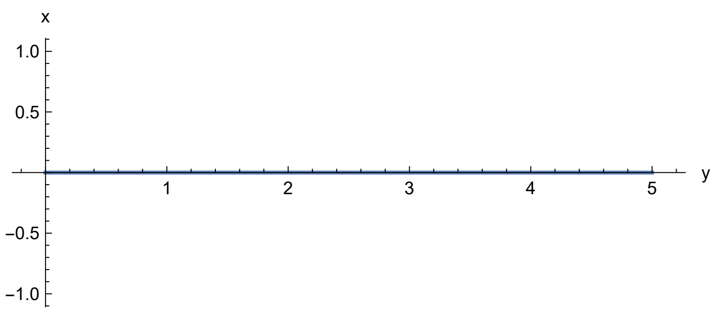

Figure 6: The curve $(x(t), y(t))$ for $c = 1$, $\psi_0 = 0$

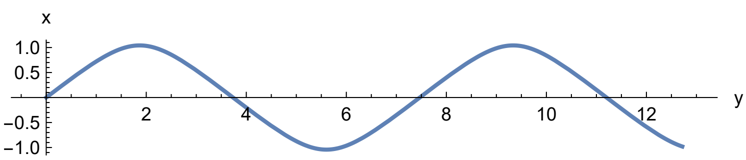

Figure 7: The curve $(x(t), y(t))$ for $c = 1$, $\psi_0 = 1$

3.4 Exponential mapping

Introduce the exponential mapping

$$\operatorname{Exp} : N \to M, \qquad \operatorname{Exp}(\lambda, t) = \pi \circ e^{t\vec{H}}(\lambda),$$
$$N = \{(\lambda, t) \in C \times \mathbb{R}_+ \mid t \in (0, +\infty) \text{ for } c \geq 0; \quad t \in (0, +kK/l) \text{ for } c < 0\},$$
$$C = T^*_{q_0}M \cap \{H = -1/2, \ h_2 < 0\}.$$

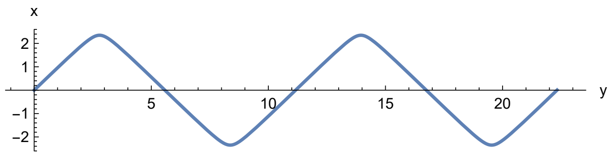

Figure 8: The curve $(x(t), y(t))$ for $c = 1$, $\psi_0 = 2$

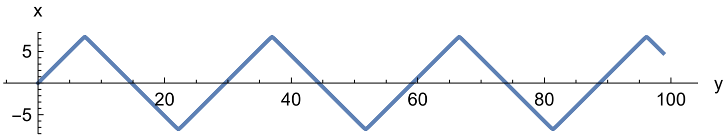

Figure 9: The curve $(x(t), y(t))$ for $c = 1$, $\psi_0 = 4$

Formulas of Subsec. 3.3.2 give an explicit parametrization of the exponential mapping.

In this subsection we describe diffeomorphic properties of the exponential mapping via the classical Hadamard's theorem on global diffeomorphism:

Theorem 1 ([16]). *Let $F : X \to Y$ be a smooth mapping of smooth manifolds,* $\dim X = \dim Y$. *Suppose that the following conditions hold:*

(1) *X and Y are connected,*

(2) *Y is simply connected,*

(3) *F is nondegenerate,*

(4) *F is proper (i.e., for any compact set $K \subset Y$, the preimage $F^{-1}(K) \subset X$ is compact).*

Then F is a diffeomorphism.

Consider the following stratification in the image of the exponential mapping:

$$\begin{aligned}
&\operatorname{int} \mathcal{B}_1 = \cup_{i=0}^{6} M_i, \\
&M_0 \ : \ x = 0, \qquad -1/24 < \eta < 0, \\
&M_1 \ : \ x > 0, \qquad \varphi_5(\xi) < \eta < \varphi_1(\xi), \\
&M_2 \ : \ x < 0, \qquad \varphi_5(\xi) < \eta < \varphi_2(\xi), \\
&M_3 \ : \ x > 0, \qquad \varphi_3(\xi) < \eta < \varphi_5(\xi), \\
&M_4 \ : \ x < 0, \qquad \varphi_2(\xi) < \eta < \varphi_4(\xi), \\
&M_5 \ : \ x > 0, \qquad \eta = \varphi_5(\xi), \\
&M_6 \ : \ x < 0, \qquad \eta = \varphi_5(\xi), \\
&\varphi_5(\xi) = (\xi^2 - 1)/24,
\end{aligned} \tag{3.19}$$

see Fig. 10.

Now define the following stratification of the subset

$$\widetilde{N} = \{(\lambda, t) \in N \mid t \in (0, 2K/(æl)) \text{ for } c > 0; \ t \in (0, +\infty) \text{ for } c = 0; \ t \in (0, kK/l) \text{ for } c < 0\}$$

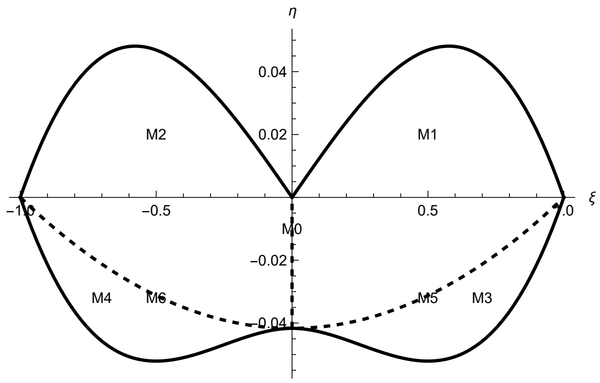

Figure 10: Stratification in the image of Exp

in the preimage of the exponential mapping:

$$
\begin{aligned}
&\widetilde{N} = \cup_{i=1}^{6} \widetilde{N}_i, \\
&\widetilde{N}_1 \ : \ c > 0, \qquad \psi_0 > 0, \qquad \tau \in (0, 2K), \\
&\widetilde{N}_2 \ : \ c > 0, \qquad \psi_0 < 0, \qquad \tau \in (0, 2K), \\
&\widetilde{N}_3 \ : \ c < 0, \qquad \psi_0 > 0, \qquad \tau \in (0, K), \\
&\widetilde{N}_4 \ : \ c < 0, \qquad \psi_0 < 0, \qquad \tau \in (0, K), \\
&\widetilde{N}_5 \ : \ c = 0, \qquad \psi_0 > 0, \\
&\widetilde{N}_6 \ : \ c = 0, \qquad \psi_0 < 0.
\end{aligned}
$$

Proposition 2. *The inclusion* $\operatorname{Exp}(\widetilde{N}_1) \subset M_1$ *holds. Moreover, the mapping* $\operatorname{Exp} : \widetilde{N}_1 \to M_1$ *is a real-analytic diffeomorphism.*

Proof. a) Let us show that $\operatorname{Exp}(\widetilde{N}_1) \subset M_1$.

Let $(\lambda, t) \in \widetilde{N}_1$. We have to prove that $(\xi^2 - 1)/24 < \eta < \xi(1 - \xi^2)/8$, i.e.,

$$
(x^2 - y^2)y \underset{1)}{<} 24z - 3x^2y - y^3 \underset{2)}{<} 3x(y^2 - x^2).
$$

Inequality 1) can be rewritten as

$$
6z > x^2 y, \tag{3.20}
$$

let us prove this inequality. Differentiating the both sides of (3.20) by virtue of ODEs (3.14), we get

$$
x \cosh \psi > y \sinh \psi. \tag{3.21}
$$

Further, differentiating the both sides of (3.21) by virtue of ODEs (3.14), we get

$$
x \sinh \psi > y \cosh \psi. \tag{3.22}
$$

Since $\cosh \psi > \sinh \psi > 0$, we have $y > x$ by the first two equalities in (3.14), and inequality (3.22) is proved. Since the both sides of this inequality vanish at $t = 0$, inequality (3.21) is proved as well by integration. Similarly, inequality (3.20) is proved, thus inequality 1) follows.

Now we prove similarly inequality 2): it is equivalent to

$$24z < y^3 + 3x^2y + 3xy^2 - 3x^3. \tag{3.23}$$

Differentiating the both sides of (3.23) by virtue of ODEs (3.14), we get

$$3x^2 < y^2 + 2xy, \tag{3.24}$$

which holds since $y > x$. Returning back from (3.24) to (3.23) by integration, we prove inequality 2).

So the inclusion $\operatorname{Exp}(\widetilde{N}_1) \subset M_1$ is proved.

b) We show that the mapping $\operatorname{Exp} : \widetilde{N}_1 \to M_1$ is nondegenerate, i.e., the Jacobian $J = \frac{\partial(x,y,z)}{\partial(c,\psi,t)}$ does not vanish on $\widetilde{N}_1$. Direct computation of this Jacobian gives

$$\begin{aligned} &J = f \cdot \frac{\partial(x,y,z)}{\partial(m,\tau,k)} = f \cdot \frac{9(1-k^2)k^2m^6}{32} J_1, \qquad f \neq 0, \\ &J_1 = J_0 + o(1), \qquad k \to 0, \\ &J_0 = \cos(3\tau) + (8\tau^2 - 1)\cos\tau - 4\tau\sin\tau. \end{aligned}$$

Since

$$\left(\frac{J_0}{\sin\tau}\right)' = -\frac{2(2\tau - \sin 2\tau)^2}{\sin^2\tau} < 0, \qquad \tau \in (0, \pi),$$

thus $J_0(\tau) < 0$, $\tau \in (0, \pi]$.

Moreover,

$$\begin{aligned} &J_1|_{\tau=2K} = 12(1-k^2)g(k), \\ &g(k) = E^2(k) - 2E(k)K(k) + (1-k^2)K^2(k). \end{aligned} \tag{3.25}$$

We have $g(k) = g_1(k)g_2(k)$, $g_1(k) = E(k) - (1+k)K(k)$, $g_2(k) = E(k) - (1-k)K(k)$. Since $g_1'(k) = \frac{E(k)}{k-1} < 0$, then $g_1(k) < 0$ for $k \in (0,1)$. Similarly, since $g_2'(k) = \frac{E(k)}{k+1} > 0$, then $g_2(k) > 0$ for $k \in (0,1)$. Thus $g(k) < 0$ and $J_2|_{\tau=2K} < 0$ for $k \in (0,1)$.

By homotopy invariance of the Maslov index (the number of conjugate points on an extremal) [1], we have $J(\tau) < 0$ for $\tau \in (0, 2K]$. Thus the mapping $\operatorname{Exp} : \widetilde{N}_1 \to M_1$ is nondegenerate.

c) We show that this mapping is proper. We have to prove that if $\widetilde{N}_1 \ni (m,k,\tau) \to \partial\widetilde{N}_1$, then $q = \operatorname{Exp}(m,k,\tau) \to \partial M_1$. This implication follows from the representation $\widetilde{N}_1 = \{m > 0,\ k \in (0,1),\ \operatorname{am}\tau \in (0,\pi)\}$, the definition (3.19) of the domain M_1, and the parametrization of the exponential mapping given in Subsec. 3.3.2.

Since $\widetilde{N}_1$ and M_1 are connected and simply connected, this mapping is a diffeomorphism by Th. 1. □

Proposition 3. *The inclusion* $\operatorname{Exp}(\widetilde{N}_3) \subset M_3$ *holds. Moreover, the mapping* $\operatorname{Exp} : \widetilde{N}_3 \to M_3$ *is a real-analytic diffeomorphism.*

Proof. A similar argument to that used in the proof of Propos. 2 proves the claim. □

Proposition 4. *The inclusions* $\operatorname{Exp}(\widetilde{N}_2) \subset M_2$ *and* $\operatorname{Exp}(\widetilde{N}_4) \subset M_4$ *hold. Moreover, the mappings* $\operatorname{Exp} : \widetilde{N}_2 \to M_2$ *and* $\operatorname{Exp} : \widetilde{N}_4 \to M_4$ *are real-analytic diffeomorphisms.*

Proof. Follows from Propositions 2 and 3 via reflection (3.15). □

Proposition 5. *The inclusions* $\mathrm{Exp}(\widetilde{N}_5) \subset M_5$ *and* $\mathrm{Exp}(\widetilde{N}_6) \subset M_6$ *hold. Moreover, the mappings* $\mathrm{Exp} : \widetilde{N}_5 \to M_5$, $\mathrm{Exp} : \widetilde{N}_6 \to M_6$ *are real-analytic diffeomorphisms.*

Proof. Follows from the parametrization (3.18) of the exponential mapping $\mathrm{Exp}|_{\widetilde{N}5}$. □

Introduce the following subsets in the preimage of the exponential mapping:

$$N_0^{\pm} = \{(\lambda, t) \in N \mid c > 0,\ \operatorname{sgn} \psi_0 = \pm 1,\ t = 2K/(\text{æ}l)\}.$$

Proposition 6. *The inclusions* $\mathrm{Exp}(N_0^{\pm}) \subset M_0$ *hold. Moreover, the mappings* $\mathrm{Exp} : N_0^{\pm} \to M_0$ *are real-analytic diffeomorphisms.*

Proof. In view of the reflection (3.15), it suffices to consider only the set N_0^+. And in view of the dilations (3.16), it is enough to consider only the case $c = 1$.

If $c = 1$, $\psi_0 > 0$, $t = 2K/(\text{æ}l) = 2K/k'$, then

$$\begin{aligned}
&x = 0,\\
&y = k'(4E(k) - 2k'^2 K(k)),\\
&z = \frac{4}{3}k'^3((1 + k^2)E(k) - k'^2 K(k)),\\
&\xi = 0,\\
&\eta = -\frac{1}{24} + \frac{(1 + k^2)E(k) - (1 - k^2)K(k)}{6(2E(k) - (1 - k^2)K(k))^3} =: \eta_1(k).
\end{aligned}$$

We have $\lim_{k\to 0} \eta_1(k) = -1/24$, $\lim_{k\to 1} \eta_1(k) = 0$ and $\eta_1'(k) = -\frac{(1-k^2)g(k)}{2k(2E(k)-(1-k^2)K(k))^4}$, where the function $g(k)$ is given by (3.25). We showed in the proof of Propos. 2 that $g(k) < 0$, $k \in (0, 1)$, thus $\eta_1 : (0, 1) \to (-1/24, 0)$ is a strictly increasing diffeomorphism.

Thus $\mathrm{Exp}(N_0^+) \subset M_0$ and $\mathrm{Exp} : N_0^+ \to M_0$ is a diffeomorphism. □

Denote

$$\begin{aligned}
N_+ &= \widetilde{N}_1 \cup \widetilde{N}_3 \cup N_5,\\
N_- &= \widetilde{N}_2 \cup \widetilde{N}_4 \cup N_6,\\
M_+ &= M_1 \cup M_3 \cup M_5 = (\operatorname{int} \mathcal{B}_1) \cap \{x > 0\},\\
M_- &= M_2 \cup M_4 \cup M_6 = (\operatorname{int} \mathcal{B}_1) \cap \{x < 0\}.
\end{aligned}$$

Theorem 2. *The inclusions* $\mathrm{Exp}(N_\pm) \subset M_\pm$ *hold. Moreover, the mappings* $\mathrm{Exp} : N_\pm \to M_\pm$ *are real-analytic diffeomorphisms.*

Proof. By virtue of the reflection (3.15), it suffices to consider only the domain N_+.

The inclusion $\mathrm{Exp}(N_+) \subset M_+$ follows from Propositions 2, 3, 5.

The mappings $\mathrm{Exp}|_{\widetilde{N}_1}$ and $\mathrm{Exp}|_{\widetilde{N}_3}$ are nondegenerate by Propositions 2 and 3 respectively. Since $\widetilde{N}_5 \subset \operatorname{cl}(\widetilde{N}_1)$ and $\mathrm{Exp}|_{\widetilde{N}_1}$ is nondegenerate, then it follows by homotopy invariance of the Maslov index that the mapping $\mathrm{Exp} : N_+ \to M_+$ is also nondegenerate at the set N_5. Summing up, $\mathrm{Exp}|_{N_+}$ is nondegenerate.

Reasoning in a way similar to argument used in Proposition 2, it follows that the mapping $\mathrm{Exp} : N_+ \to M_+$ is proper.

Then Theorem 1 implies that this mapping is a diffeomorphism. □

Proposition 7. *The mapping* $\mathrm{Exp} : N \to M$ *is a local diffeomorphism at points of* $N_0^{\pm}$*.*

Proof. By virtue of the reflection (3.15), we can consider only the case N_0^+.

Observing that $N_0^+ \subset \mathrm{cl}(\widetilde{N}_1)$ and $\mathrm{Exp} : \widetilde{N}_1 \to M$ is nondegenerate, the homotopy invariance of the Maslov index then implies that the mapping $\mathrm{Exp} : N \to M$ is a local diffeomorphism at points of N_0^+. □

3.5 Attainable sets and existence of optimal trajectories

Theorem 3. *We have* $\mathcal{A}_1 = \mathcal{B}_1$*.*

Proof. By virtue of Propos. 1, it remains to prove the inclusion $\mathcal{A}_1 \supset \mathcal{B}_1$. But Theorem 2 and Propos. 6 imply that $\mathrm{Exp}(N_+ \cup N_- \cup N_0^+) \supset M_+ \cup M_- \cup M_0 = \mathrm{int}\,\mathcal{B}_1$. Thus $\mathcal{A}_1 \supset \mathrm{int}\,\mathcal{B}_1$.

Further, each point of $\partial\mathcal{B}_1$ is reachable from q_0 by an abnormal trajectory, thus $\mathcal{A}_1 \supset \partial\mathcal{B}_1$.

Summing up, $\mathcal{A}_1 \supset \mathcal{B}_1$, and the equality $\mathcal{A}_1 = \mathcal{B}_1$ follows. □

Recall that $J^+(q)$ is the causal future of a point $q \in M$, i.e., the attainable set from q for arbitrary nonnegative time. Similarly, $J^-(q)$ is the causal past of q, i.e., the set of points attainable from q for arbitrary nonpositive time. Notice that $\mathcal{A}_1 = J^+(q_0)$.

Corollary 1. *Let* $q = (0, y_0, z_0) \in M$*. Then*

$$J^+(q) = \begin{cases} \varphi_3(\xi_0) \leq \eta_0 \leq \varphi_1(\xi_0), & 0 \leq \xi_0 \leq 1, \qquad y \geq y_0, \\ \varphi_4(\xi_0) \leq \eta_0 \leq \varphi_2(\xi_0), & -1 \leq \xi_0 \leq 0, \qquad y \geq y_0, \end{cases}$$

$$\xi_0 = \frac{x}{y - y_0}, \qquad \eta_0 = \frac{24(z - z_0) - 3x^2(y - y_0) - (y - y_0)^3}{24(y - y_0)^3},$$

$$J^-(q) = \begin{cases} \varphi_3(\xi_1) \leq \eta_1 \leq \varphi_1(\xi_1), & 0 \leq \xi_1 \leq 1, \qquad y \leq y_0, \\ \varphi_4(\xi_1) \leq \eta_1 \leq \varphi_2(\xi_1), & -1 \leq \xi_1 \leq 0, \qquad y \leq y_0, \end{cases}$$

$$\xi_1 = -\xi_0, \qquad \eta_1 = \eta_0.$$

Proof. The expression for $J^+(q)$ follows from the expression for $J^+(q_0) = \mathcal{A}_1$ via the translation (3.17). And the expression for $J^-(q)$ follows by time inversion. □

Theorem 4. *Let points* $q_0, q_1 \in M$ *satisfy the inclusion* $q_1 \in J^+(q_0)$*. Then there exists an optimal trajectory in problem* (3.1)–(3.4)*.*

Proof. By Theorem 2 and Remark 2 [24], the following conditions are sufficient for existence of an optimal trajectory connecting q_0 and q_1:

(1) $q_1 \in J^+(q_0)$,

(2) There exists a compact $K \subset M$ such that $J^+(q_0) \cap J^-_{q_1} \subset K$,

(3) $T(q_0, q_1) < +\infty$,

where $T(q_0, q_1)$ is the supremum of time required to reach q_1 from q_0 for a trajectory of system (3.1) reparametrized so that $u_1 \equiv 1$.

Condition (1) holds by assumption of this theorem.

Condition (3) holds since if $u \equiv 1$ then $\dot{y} \equiv 1$, thus $T(q_0, q_1) = y_1 - y_0 < +\infty$.

Now we prove condition (2). It is easy to see from Cor. 1 that

$$\cup_{q \in \Pi} J^+(q) = \cup_{q \in \Pi} J^-(q) = M.$$

Thus there exist $p_0, p_1 \in \Pi$ such that $p_0 \in J^+(q_0)$, $q_1 \in J^-(p_1)$. Thus $J^+(q_0) \subset J^+(p_0)$, $J^-(q_1) \subset J^-(p_1)$, so $J^+(q_0) \cap J^-(q_1) \subset J^+(p_0) \cap J^-(p_1)$. It is easy to see from Cor. 1 that the set $K := J^+(p_0) \cap J^-(p_1)$ is compact. So condition (2) above holds, and the theorem is proved. □

3.6 Optimality of extremal trajectories

Proposition 8. *Let $\lambda = (\psi_0, c) \in C$, $c > 0$, $\psi_0 \neq 0$, and let $t_1 > 2K/(æl)$. Then the extremal trajectory $q(t) = \operatorname{Exp}(\lambda, t)$, $t \in [0, t_1]$, is not optimal.*

Proof. By virtue of the reflection (3.15), we can assume that $\psi_0 > 0$. By contradiction, suppose that $q(t)$, $t \in [0, t_1]$, is optimal.

Let $\widetilde{\lambda} = (-\psi_0, c) \in C$, $\widetilde{q}(t) = \operatorname{Exp}(\widetilde{\lambda}, t)$, $t \in [0, t_1]$. Denote $\bar{t} = 2K/(æl)$, then $q(\bar{t}) = \widetilde{q}(\bar{t})$ and $l(q(\cdot)|_{[0,\bar{t}]}) = l(\widetilde{q}(\cdot)|_{[0,\bar{t}]})$, i.e., the trajectories $q(\cdot)$ and $\widetilde{q}(\cdot)$ have a Maxwell point at $t = \bar{t}$ [20]. Now consider the trajectory

$$\widehat{q}(t) = \begin{cases} \widetilde{q}(t), & t \in [0, \bar{t}], \\ q(t), & t \in [\bar{t}, t_1]. \end{cases}$$

Since $l(\widehat{q}(\cdot)|_{[0,t_1]}) = l(q(\cdot)|_{[0,t_1]})$, then the trajectory $\widehat{q}(t)$, $t \in [0, t_1]$, is optimal. But $\widehat{q}(t)$ has a corner point at $t = \bar{t}$, which is not possible for normal trajectories. Thus $\widehat{q}(t)$ is abnormal, so its support belongs to $\partial \mathcal{A}_1$. But this is impossible since the support of the normal trajectory $q(t)$ belongs to $\operatorname{int} \mathcal{A}_1$. A contradiction obtained completes the proof. □

Define the following function:

$$\begin{aligned} &\mathbf{t} : C \to (0, +\infty], \qquad \lambda = (\psi_0, c) \in C, \\ &c = 0 \quad \Rightarrow \quad \mathbf{t}(\lambda) = +\infty, \\ &c \neq 0,\ \psi_0 = 0 \quad \Rightarrow \quad \mathbf{t}(\lambda) = +\infty, \\ &c > 0,\ \psi_0 \neq 0 \quad \Rightarrow \quad \mathbf{t}(\lambda) = \frac{2kK}{l}, \\ &c < 0,\ \psi_0 \neq 0 \quad \Rightarrow \quad \mathbf{t}(\lambda) = \frac{kK}{l}. \end{aligned}$$

We prove (see Cor. 2) that $\mathbf{t}(\lambda)$ is the cut time for an extremal trajectory $\operatorname{Exp}(\lambda, t)$:

$$\mathbf{t}(\lambda) = t_{\text{cut}}(\lambda) := \sup\{t_1 > 0 \mid \operatorname{Exp}(\lambda, t) \text{ is optimal for } t \in [0, t_1]\}.$$

Theorem 5. *Let $\lambda \in C$, $t_1 \in (0, \mathbf{t}(\lambda))$. Then the trajectory $q(t) = \operatorname{Exp}(\lambda, t)$, $t \in [0, t_1]$, is optimal.*

Proof. By Theorem 4, there exists an optimal trajectory connecting q_0 and $q_1 := q(t_1)$. Since $q_1 \in \operatorname{int} \mathcal{A}_1$, this trajectory is a normal extremal trajectory. Since $q_1 \in M_+ \cup M_-$, by Theorem 2 there exists a unique arclength parametrized normal extremal trajectory connecting q_0 and q_1, hence it coincides with $q(t)$. □

Theorem 6. *Let $\lambda = (\psi_0, c) \in C$, $c > 0$, $\psi_0 \neq 0$, $t_1 = \mathbf{t}(\lambda)$. Then the trajectory $q(t) = \operatorname{Exp}(\lambda, t)$, $t \in [0, t_1]$, is optimal.*

Proof. Similarly to Th. 5 with the only distinction that now there are exactly two optimal trajectories corresponding to the covectors $(\pm\psi_0, c) \in C$, these trajectories are symmetric by virtue of the reflection (3.15). □

Proposition 8 together with Theorems 5 and 6 imply the following.

Corollary 2. *For any $\lambda \in C$ we have $t_{\mathrm{cut}}(\lambda) = \mathbf{t}(\lambda)$.*

Proposition 9. *Any abnormal extremal trajectory is optimal.*

Proof. Abnormal extremal trajectories are exactly trajectories belonging to the boundary of the attainable set $\mathcal{A}_1$. If $q(t)$, $t \in [0, t_1]$, is an abnormal trajectory, then, up to reparametrization, the corresponding control has the following form:

$$1)\ u(t) = \begin{cases} (\pm 1, 1), & t \in [0, \tau_1], \\ (\mp 1, 1), & t \in [\tau_1, t_1], \end{cases} \qquad \tau_1 \in [0, t_1], \text{ or}$$

$$2)\ u(t) = \begin{cases} (0, 1), & t \in [0, \tau_1], \\ (\pm 1, 1), & t \in [\tau_1, t_1], \end{cases} \qquad \tau_1 \in [0, t_1].$$

In the case 1) we have $l(q(\cdot)) = 0$. If $\widetilde{q}(t)$, $t \in [0, \tilde{t}_1]$, is a trajectory such that $\widetilde{q}(\tilde{t}_1) = q(t_1)$, then, up to reparametrization, $\widetilde{q}(t) \equiv q(t)$, thus $q(t)$ is optimal.

In the case 2) we have $l(q(\cdot)) = \tau_1$, and a similar argument shows that $q(t)$ is optimal. □

Theorems 5, 6 and Propos. 9 yield the following description of the optimal synthesis.

Theorem 7. (1) *Let $q_1 \in (\operatorname{int} \mathcal{A}_1) \setminus \Pi$. Then there exists a unique optimal trajectory $\operatorname{Exp}(\lambda, t)$, $t \in [0, t_1]$, where $(\lambda, t_1) = \operatorname{Exp}^{-1}(q_1)$.*

(2) *Let $q_1 \in (\operatorname{int} \mathcal{A}_1) \cap \Pi$. Then there exist exactly two optimal trajectories $\operatorname{Exp}(\lambda_i, t)$, $t \in [0, t_1]$, $i = 1, 2$, where $\{(\lambda_1, t_1), (\lambda_2, t_1)\} = \operatorname{Exp}^{-1}(q_1)$.*

(3) *Let $q_1 = (x_1, y_1, z_1) \in S_1 \cup S_1$, $\operatorname{sgn} x_1 = \pm 1$. Then there exists a unique optimal trajectory*

$$q(t) = \begin{cases} e^{t(\pm X_1 + X_2)}(q_0), & t \in [0, \tau_1], \\ e^{(t-\tau_1)(\mp X_1 + X_2)} \circ e^{\tau_1(\pm X_1 + X_2)}(q_0), & t \in [\tau_1, t_1], \end{cases}$$

$\tau_1 = (y_1 \pm x_1)/2$, $t_1 = y_1$.

(4) *Let* $q_1 = (x_1, y_1, z_1) \in S_3 \cup S_4$, $\operatorname{sgn} x_1 = \pm 1$. *Then there exists a unique optimal trajectory*

$$q(t) = \begin{cases} e^{tX_2}(q_0), & t \in [0, \tau_1], \\ e^{(t-\tau_1)(\pm X_1 + X_2)} \circ e^{\tau_1 X_2}(q_0), & t \in [\tau_1, t_1], \end{cases}$$

$\tau_1 = y_1 \pm x_1$, $t_1 = y_1$.

(5) *Let* $q_1 = (0, y_1, 0)$, $y_1 > 0$. *Then there exists a unique optimal trajectory* $q(t) = e^{tX_2}(q_0)$, $t \in [0, t_1]$, $t_1 = y_1$.

(6) *Let* $q_1 = (0, y_1, z_1)$, $z_1 = y_1^3/24 > 0$. *Then there exist exactly two optimal trajectories*

$$q^1(t) = \begin{cases} e^{t(X_1+X_2)}(q_0), & t \in [0, \tau_1], \\ e^{(t-\tau_1)(-X_1+X_2)} \circ e^{\tau_1(X_1+X_2)}(q_0), & t \in [\tau_1, t_1], \end{cases}$$

$$q^2(t) = \begin{cases} e^{t(-X_1+X_2)}(q_0), & t \in [0, \tau_1], \\ e^{(t-\tau_1)(X_1+X_2)} \circ e^{\tau_1(-X_1+X_2)}(q_0), & t \in [\tau_1, t_1], \end{cases}$$

$\tau_1 = t_1/2$, $t_1 = y_1$.

(7) *Let* $q_1 = (x_1, y_1, z_1)$, $\operatorname{sgn} x_1 = \pm 1$, $y_1 = \pm x_1$, $z_1 = y_1^3/6$. *Then there exists a unique optimal trajectory* $q(t) = e^{t(\pm X_1 + X_2)}(q_0)$, $t \in [0, t_1]$, $t_1 = y_1$.

Remark 4. *In Theorem* 7, *existence of exactly one (or two) optimal trajectories is understood up to reparametrization.*

3.7 Sub-Lorentzian distance

In this subsection we study the function $d(q) := d(q_0, q)$, $q \in M$.

Theorem 8. (1) *The distance* d *is real-analytic on* $(\operatorname{int} \mathcal{A}_1) \setminus \Pi$ *and continuous on* $\operatorname{int} \mathcal{A}_1$.

(2) *The distance* d *has discontinuity of the first kind at each point of* $S_3 \cup S_4$.

(3) *The restriction* $d|_\Pi$ *is real-analytic on the set* $\{x = 0,\ z \in (0, y^3/24),\ y > 0\}$ *and discontinuous of the first kind on the set* $\{x = 0,\ z = y^3/24,\ y > 0\}$.

(4) *The distance* d *is homogeneous of order* 1 *w.r.t. dilations* (3.16):

$$d(\delta_\alpha(q)) = \alpha\delta(q), \qquad \delta_\alpha(x, y, z) = (\alpha x, \alpha y, \alpha^3 z), \qquad \alpha > 0, \quad q = (x, y, z) \in M.$$

Proof. (1) Follows from Th. 2 and Propos. 7.

(2) Take any $q_1 = (x_1, y_1, z_1) \in S_3 \cup S_4$, then $d(q_1) = y_1 > 0$ by item (4) of Th. 7. On the other hand, $q_1 \in \partial\mathcal{A}_1$, thus there are points $q \in M \setminus \mathcal{A}_1$ arbitrarily close to q_1. Since $d(q) = 0$, the distance d has discontinuity of the first kind at q_1.

(3) The restriction $d|_\Pi$ is real-analytic on the set $\{x = 0,\ z \in (0, y^3/24),\ y > 0\}$ by Propos. 7.

Let $q_1 = (x_1, y_1, z_1) \in \mathcal{A}_1$, $x_1 = z_1 = 0$, $y_1 > 0$. Then $q_1 \in \partial\mathcal{A}_1$, and the distance d has discontinuity of the first kind at q_1 similarly to item (2).

(4) is obvious in view of symmetry (3.16). □

Let $q = (0, y, z) \in \Pi$, then $d(q) = y^3 d(0, 1, z/y^3) = y^3 d(0, 1, \eta + 1/24)$ since $\eta|_\Pi = z/y^3 - 1/24$. We plot the function $\eta \mapsto d(0, 1, \eta + 1/24)$ in Fig. 11.

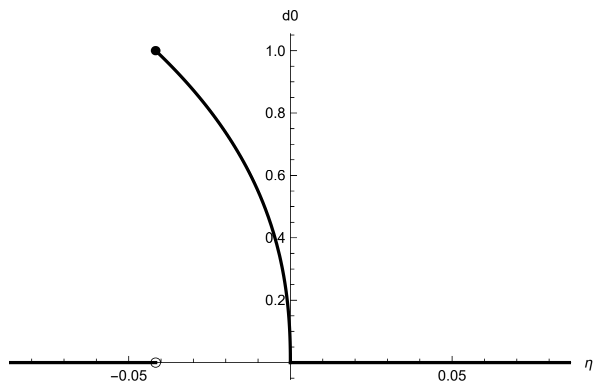

Figure 11: Plot of $\eta \mapsto d(0, 1, \eta + 1/24)$

3.8 Sub-Lorentzian sphere

By virtue of dilations (3.16), the Lorentzian spheres

$$S(R) = \{q \in M \mid d(q) = R\}, \qquad R > 0,$$

satisfy the relation $S(R) = \delta_R(S(1))$, thus we describe only the unit sphere $S := S(1)$.

Theorem 9. (1) *The set $S \setminus (\Pi \cup \partial \mathcal{A}_1)$ is a real-analytic manifold.*

(2) *The sphere S is nonsmooth and Lipschitz at points of $S \cap \Pi$.*

(3) *The intersection $S \cap \Pi$ is defined parametrically for $k \in [0, 1)$:*

$$y = \frac{4E(k) - 2(1-k^2)K(k)}{2(1-k^2)K(k)} = 1 + k^2 + O(k^4), \qquad k \to 0,$$
$$z = \frac{2(1-k^2)K(k) - 2(1+k^2)E(k)}{12(1-k^2)^3K^3(k)} = \frac{k^2}{\pi^2} + O(k^4), \qquad k \to 0.$$

In particular, the intersection $S \cap \Pi$ is semi-analytic.

Proof. (1) Follows from Th. 2.

(2) The hemi-spheres $S_\pm = S \cap \{\pm x \geq 0\}$ have at points of $S \cap \Pi$ the normal vectors $n_\pm = \pm(g(k), *, *)$, where the function $g(k)$ is given by (3.25) and is negative for $k \in (0, 1)$. So the sphere S has a transverse self-intersection at points of $S \cap \Pi$. .

(3) The parametrization of $S \cap \Pi$ is obtained from the parametrization of the exponential mapping in the case 2.2 of Subsec. 3.3.2 for $\tau = 2K$, $t = 1$. □

Remark 5. *One of the most important results of the paper [3] is non-subanalyticity of the sub-Riemannian sphere in the Martinet flat case. Its proof relies on non-semianalyticity of the intersection of the sub-Riemannian sphere with the Martinet surface. Item (3) of the preceding theorem states that such an intersection is semi-analytic in the sub-Lorentzian case. We leave the question of subanalyticity of the sub-Lorentzian sphere in the first flat problem on the Martinet distribution open since we cannot conclude on subanalyticity (or its lack) at points of the boundary of the sphere S.*

Denote $S_n = S \cap \operatorname{int} \mathcal{A}_1$, $S_a = S \cap \partial \mathcal{A}_1$.

Remark 6. *The set S_n (resp. S_a) is filled with the endpoints of optimal normal (resp. abnormal) trajectories of length 1 starting at q_0.*

Lemma 1. *We have* $\operatorname{cl}(S_n) \supset S_a$.

Proof. Take any point $q \in S_a$ and choose any neighbourhood $q \in O \subset M$. By Proposition 8.1 [7], the distance d is upper semicontinuous on $\mathcal{A}_1$, thus there exists a point $q_- \in O \cap \operatorname{int} \mathcal{A}_1$ such that $d(q_-) \le d(q) = 1$. Further, for small $t > 0$ there exists a point

$$q_+ = e^{t(u_1 X_1 + u_2 X_2)}(q) \in O \cap \operatorname{int} \mathcal{A}_1, \qquad (u_1, u_2) \in U_1.$$

Then $d(q_+) \ge 1$. Since $O \cap \operatorname{int} \mathcal{A}_1$ is arcwise connected and $d|_{\operatorname{int} \mathcal{A}_1}$ is continuous, there exists a point $\bar{q} \in O \cap \operatorname{int} \mathcal{A}_1$ such that $d(\bar{q}) = 1$. Then $\bar{q} \in S_n \cap O$. Hence $\operatorname{cl}(S_n) \supset S_a$. $\square$

Proposition 10. *The sphere S is homeomorphic to the closed half-plane $\mathbb{R}^2_+ := \{(a, b) \in \mathbb{R}^2 \mid b \ge 0\}$.*

Proof. Denote the group of dilations (3.16) as $G = \{\delta_\alpha \mid \alpha > 0\}$ and consider the projection

$$p : \overset{\circ}{M} \to \overset{\circ}{M}/G, \qquad \overset{\circ}{M} = M \setminus \{q_0\}. \tag{3.26}$$

If $y \ne 0$ (which is the case on S), then projection (3.26) is given in coordinates as

$$p \ : \ q = (x, y, z) \mapsto \sigma = (\xi, \eta),$$

where ξ, η are defined in (3.11). The mapping (3.26) is smooth.

The image of the sphere S under the action of the projection p is given as follows:

$$p(S) \ : \ \begin{cases} \varphi_3(\xi) \le \eta < \varphi_1(\xi), & 0 < \xi < 1, \\ \varphi_4(\xi) \le \eta < \varphi_2(\xi), & -1 < \xi < 0, \end{cases}$$

see Fig. 19. It is obvious that $p(S)$ is homeomorphic to $\mathbb{R}^2_+$. Let us show that $p : S \to p(S)$ is a homeomorphism.

First, the mapping $p : S \to p(S)$ is a bijection since the sphere S intersects with each orbit of G at not more than one point.

Second, the mapping $p : S \to p(S)$ is continuous as a restriction of a smooth mapping (3.26).

It remains to prove that the inverse mapping $p^{-1} : p(S) \to S$ is continuous. Continuity of $p^{-1}|_{p(S_n)}$ and $p^{-1}|_{p(S_a)}$ is obvious. Let

$$p(S_n) \ni \sigma_k \underset{k \to \infty}{\to} \bar{\sigma} \in p(S_a),$$

we show that $p^{-1}(\sigma_k) =: q_k \underset{k \to \infty}{\to} \bar{q} := p^{-1}(\bar{\sigma})$. It follows from Propos. 3 that $q_k \underset{k \to \infty}{\to} \widehat{q} \in S_a$, where $p(\widehat{q}) = \bar{\sigma}$. Since orbits of G are one-dimensional, it remains to prove that $d(\widehat{q}) = 1$.

By Lemma 1, there exists a sequence $S_n \ni \widetilde{q}_k \underset{k \to \infty}{\to} \bar{q}$. Thus $\widetilde{\sigma}_k := p(\widetilde{q}_k) \underset{k \to \infty}{\to} \bar{\sigma}$. Denote by ρ and $\widehat{\rho}$ the Euclidean distances in M and $\overset{\circ}{M}/G$ respectively. Then $\widehat{\rho}(\sigma_k, \widetilde{\sigma}_k) \underset{k \to \infty}{\to} 0$, thus $\rho(q_k, \widetilde{q}_k) \underset{k \to \infty}{\to} 0$, whence $\widehat{q} = \bar{q}$.

Thus $p : S \to p(S)$ is a homeomorphism, and $S \simeq \mathbb{R}^2_+$. $\square$

Corollary 3. (1) *The sphere S is a topological manifold with boundary S_a, homeomorphic to the closed half-plane $\mathbb{R}^2_+$.*

(2) *The sphere S is a stratified space with real-analytic strata $S_n^{\pm} := S_n \cap \{\operatorname{sgn} x = \pm 1\}$, $S_n^0 := S \cap \Pi$, $S_a^{\pm} := S_a \cap \{\operatorname{sgn} x = \pm 1\}$, $S_a^0 := S_a \cap \Pi = \{(0, 1, 0)\}$. Wherein there are diffeomorphisms $S_n^{\pm} \to \mathbb{R}^2$; $S_n^0, S_a^{\pm} \to \mathbb{R}$.*

(3) *Under a homeomorphic embedding of the sphere S into the half-plane $\mathbb{R}^2_+ = \{(a, b) \in \mathbb{R}^2 \mid b \geq 0\}$ the indicated strata are mapped as follows: $S_n^{\pm} \to \{\operatorname{sgn} a = \pm 1, b > 0\}$, $S_n^0 \to \{a = 0, b > 0\}$, $S_a^{\pm} \to \{\operatorname{sgn} a = \pm 1, b = 0\}$, $S_a^0 \to \{a = b = 0\}$.*

See Fig. 19.

Remark 7. *There is a numerical evidence that the sphere S is a piecewise smooth manifold with boundary, with a stratification shown in Fig.* 19.

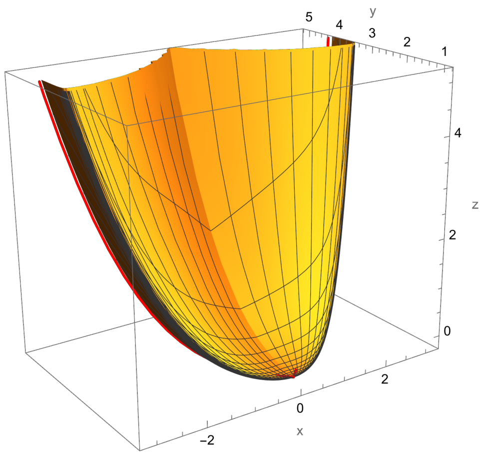

Figure 12: The sphere S

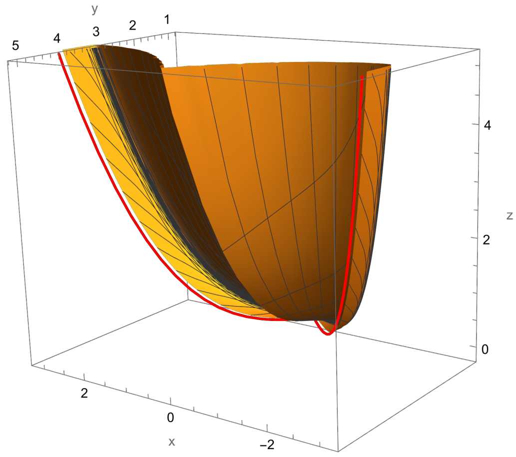

Figure 13: The sphere S

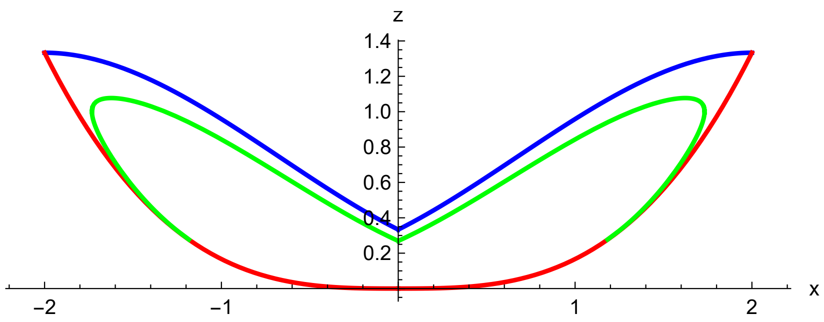

Figure 14: Intersection of S and $\partial\mathcal{A}_1$ with $\{y = 2\}$

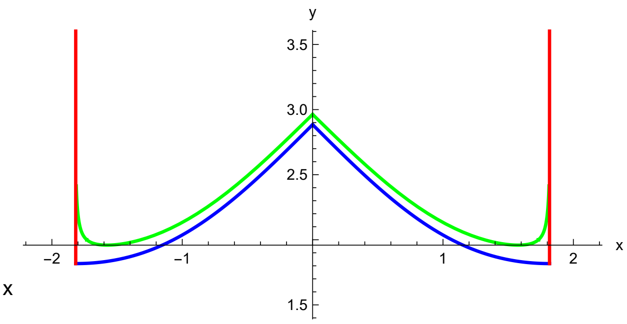

Figure 15: Intersection of S and $\partial\mathcal{A}_1$ with $\{z = 1\}$

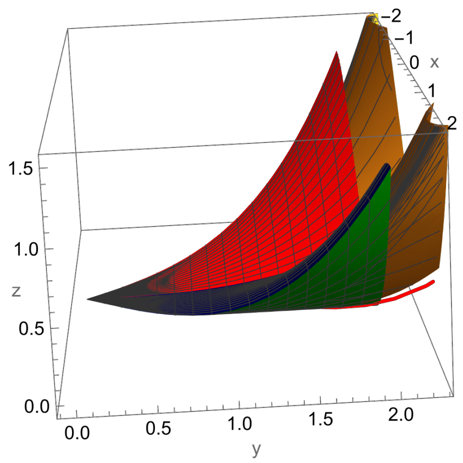

Figure 16: S inside of $\partial\mathcal{A}_1$

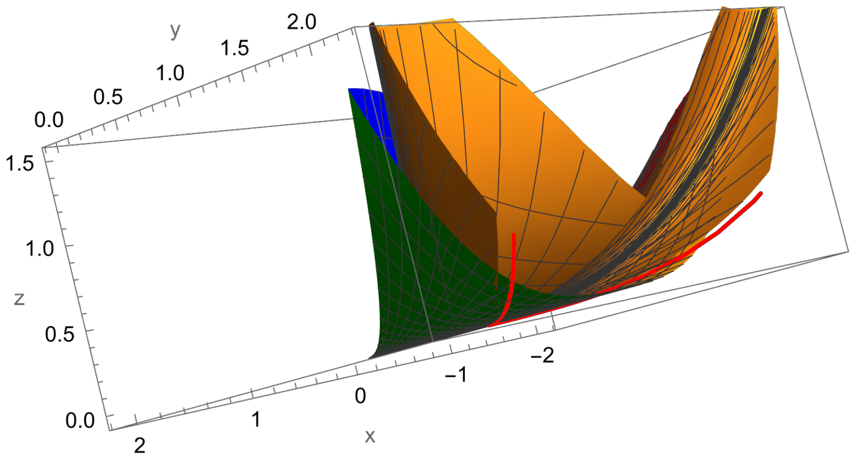

Figure 17: S inside of $\partial\mathcal{A}_1$

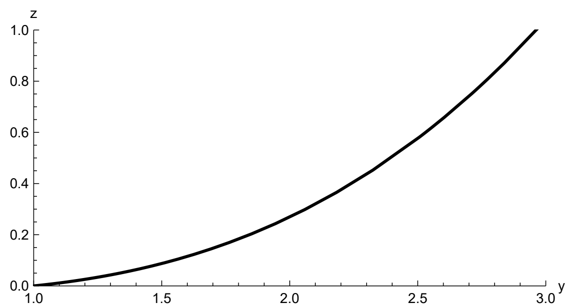

Figure 18: Intersection of S with Π

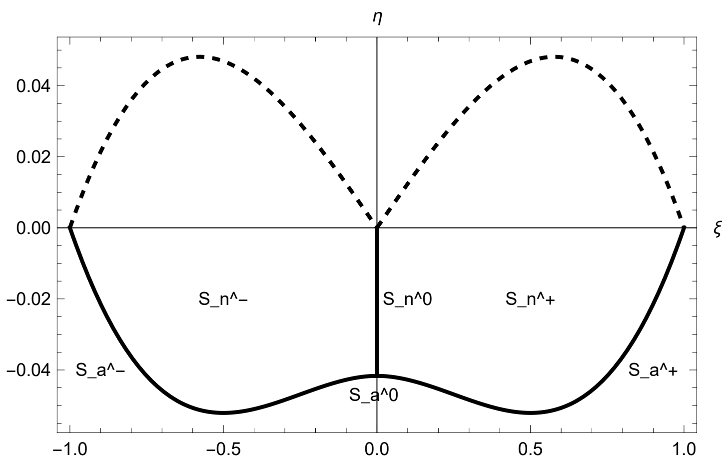

Figure 19: Stratification of $p(S)$

Proposition 11. *The sphere S is closed.*

Proof. Since $S = S_n \cup S_a$, $\mathrm{cl}(S_n) \supset S_a$ and $\mathrm{cl}(S_a) = S_a$, we have to prove that $\mathrm{cl}(S_n) \subset S$. Take any sequence $S_n \ni q_k \underset{k\to\infty}{\to} \bar{q}$. Let $\sigma_k = p(q_k)$ and $\bar{\sigma} = p(\bar{q}) \in \overset{\circ}{M}/G$. We may consider only the case $\bar{\sigma} \in p(S_a)$. If $\bar{\sigma} = (\bar{\xi}, \bar{\eta})$ is on the upper part of the boundary of $p(S_n)$, i.e., $\bar{\eta} = \varphi_1(\bar{\xi})$, $\bar{\xi} \in (0,1)$, or $\bar{\eta} = \varphi_2(\bar{\xi})$, $\bar{\xi} \in (0,1)$, then it follows from the proof of Propos. 2 that $q_k \underset{k\to\infty}{\to} \infty$, which is impossible. If $\bar{\sigma} = (\bar{\xi}, \bar{\eta})$ is on the lower part of the boundary of $p(S_n)$, i.e., $\bar{\eta} = \varphi_3(\bar{\xi})$, $\bar{\xi} \in (0,1)$, or $\bar{\eta} = \varphi_4(\bar{\xi})$, $\bar{\xi} \in (0,1)$, then $\bar{q} \in \partial\mathcal{A}_1$. A similar argument to that used in the proof of Propos. 10 shows that $d(\bar{q}) = 1$. So the claim of this proposition follows. □

Proposition 12. *The restriction $d|_{\mathcal{A}_1}$ is continuous.*

Proof. Let $\mathcal{A}_1 \ni q_k \underset{k\to\infty}{\to} \bar{q} \in \mathcal{A}_1$, we have to prove that $d(q_k) \underset{k\to\infty}{\to} d(\bar{q})$.

If $\bar{q} \in \mathrm{int}\,\mathcal{A}_1$, then the claim follows by Th. 8.

Let $\bar{q} \in \partial\mathcal{A}_1$. Since d is upper semicontinuous (Propos. 8.1 [7]), we have $\limsup_{k\to\infty} d(q_k) \leq d(\bar{q})$. In order to show that $d(\bar{q}) \leq \liminf_{k\to\infty} d(q_k)$, we assume by contradiction that $d(\bar{q}) > \liminf_{k\to\infty} d(q_k)$. In other words, there exists a subsequence $\{q_{k_m}\}$ such that $d(\bar{q}) > \lim_{m\to\infty} d(q_{k_m})$. By virtue of dilation (3.16), we can construct a sequence $q^m \in S$ converging to a point $\widehat{q} \in \partial\mathcal{A}_1$ with $d(\widehat{q}) > 1$. This contradicts the closedness of S. Thus $d(q_k) \underset{k\to\infty}{\to} d(\bar{q})$, and the statement follows. □

4 The second problem

In this section we consider a flat sub-Lorentzian problem on the Martinet distribution whose future cone has trivial intersection with the tangent plane to the Martinet surface Π. It is natural that this problem is more simple than the first problem considered in the previous section. All proofs for the second problem are completely similar or more simple than for the first one, so we skip them.

4.1 Problem statement

The second sub-Lorentzian problem on the Martinet distribution is stated as the following optimal control problem:

$$\dot{q} = u_1 X_1 + u_2 X_2, \qquad q \in M, \tag{4.1}$$

$$u = (u_1, u_2) \in U_2 = \{u_1 \geq |u_2|\}, \tag{4.2}$$

$$q(0) = q_0 = (0,0,0), \qquad q(t_1) = q_1, \tag{4.3}$$

$$l = \int_0^{t_1} \sqrt{u_1^2 - u_2^2} dt \to \max. \tag{4.4}$$

See Fig. 20.

4.2 Invariant set

By PMP, the boundary of the attainable set $\mathcal{A}_2$ of system (4.1), (4.2) from the point q_0 for arbitrary nonnegative time consists of lightlike trajectories corresponding to piecewise constant controls with values $u = (1, \pm 1)$ and up to one switching. These trajectories fill the boundary of the set

$$\mathcal{B}_2 = \{(x, y, z) \in M \mid x \geq |y|, \ z^1(x, y) \leq z \leq z^2(x, y)\},$$

where $z^1(x, y) = ((x + y)^3 - 4x^3)/24$, $z^2(x, y) = (4x^3 - (x - y)^3)/24$. See Fig. 21.

Proposition 13. *The set $\mathcal{B}_2$ is an invariant domain of system* (4.1), (4.2). *Moreover, $\mathcal{A}_2 \subset \mathcal{B}_2$.*

Proof. Similarly to Propos. 1. □

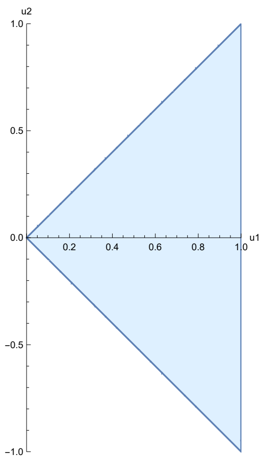

Figure 20: The set U_2

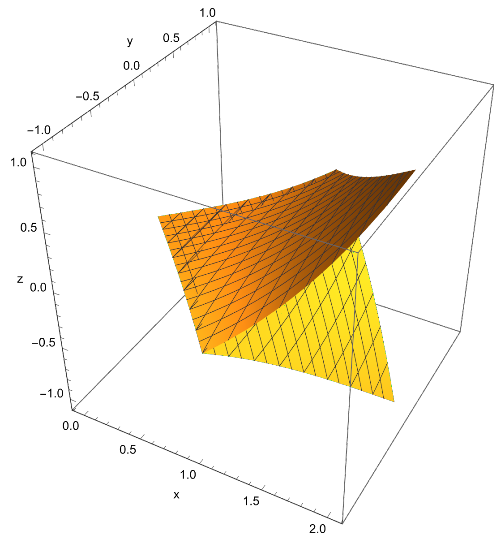

Figure 21: The set $\mathcal{B}_2$

4.3 Extremal trajectories

4.3.1 Abnormal extremal trajectories

Abnormal trajectories, up to time reparametrization, correspond to controls $u = (1, \pm 1)$ with up to one switching.

4.3.2 Normal extremals

Normal extremals satisfy the Hamiltonian system with the Hamiltonian $H = (-h_1^2 + h_2^2)/2$, $h_1 < -|h_2|$:

$$\dot{h}_1 = -h_2 h_3 x, \quad \dot{h}_2 = -h_1 h_3 x, \quad \dot{h}_3 = 0, \quad \dot{x} = -h_1, \quad \dot{y} = h_2, \quad \dot{z} = h_2 x^2/2. \tag{4.5}$$

We can choose arclength parametrization on normal extremal trajectories and thus assume that $H \equiv -1/2$. In the coordinates $h_1 = -\cosh \psi$, $h_2 = \sinh \psi$, $h_3 = c$; $\psi, c \in \mathbb{R}$, the Hamiltonian system (4.5) reads as

$$\dot{\psi} = cx, \qquad \dot{c} = 0, \tag{4.6}$$

$$\dot{x} = \cosh \psi, \qquad \dot{y} = \sinh \psi, \qquad \dot{z} = \frac{x^2}{2} \sinh \psi. \tag{4.7}$$

This system has a first integral $E = \frac{cx^2}{2} - \sinh\psi \in \mathbb{R}$.

Solutions to this system with the initial condition $\psi(0) = \psi_0$, $x(0) = y(0) = z(0) = 0$ are as follows.

1) If $c = 0$, then

$$\psi \equiv \psi_0, \qquad x = t\cosh\psi_0, \qquad y = t\sinh\psi_0, \qquad z = t^3/6\cosh^2\psi_0\sinh\psi_0.$$

2) Let $c \neq 0$. Denote $k = \sqrt{\frac{1}{2}\left(1 + \frac{E}{\sqrt{1+E^2}}\right)} \in (0,1)$, $l = \sqrt{|c|}$, $\pm 1 = \operatorname{sgn} c$, $æ = \sqrt{\frac{\sqrt{1+E^2}}{2}}$. Then

$$\sinh\psi = 2æ^2\frac{1 - k^2(1 + \operatorname{sn}^4\tau)}{\operatorname{cn}^2\tau},$$
$$x = 2æ\frac{\operatorname{dn}\tau\operatorname{sn}\tau}{l\operatorname{cn}\tau},$$
$$y = \pm\frac{2æ}{l}\left(\frac{\tau}{4k^2æ^4} - \operatorname{E}(\tau) + \frac{\operatorname{dn}\tau\operatorname{sn}\tau}{l\operatorname{cn}\tau}\right),$$
$$z = \pm\frac{æ}{3l^3}\left(\left(\frac{\tau}{3k^2æ^2} - 4E\operatorname{E}(\tau)\right)\operatorname{cn}^3\tau - \frac{1}{4æ^2k^2}\operatorname{dn}\tau\operatorname{sn}\tau + 2E\operatorname{cn}^2\tau\operatorname{dn}\tau\operatorname{sn}\tau + 4æ^2k^2\operatorname{cn}^4\tau\operatorname{dn}\tau\operatorname{sn}\tau\right),$$
$$\tau = ælt \in [0, K(k)).$$

4.4 Exponential mapping

Formulas of Subsec. 4.3.2 parametrize the exponential mapping

$$\operatorname{Exp} : N \to M, \qquad \operatorname{Exp}(\lambda, t) = \pi \circ e^{t\vec{H}}(\lambda),$$
$$N = \{(\lambda, t) \in C \times \mathbb{R}_+ \mid t \in (0, +\infty) \text{ for } c = 0; \quad t \in (0, +K/(læ)) \text{ for } c \neq 0\},$$
$$C = T^*_{q_0}M \cap \{H = -1/2, \ h_1 < 0\}.$$

Proposition 14. *The inclusion* $\operatorname{Exp}(N) \subset \operatorname{int}\mathcal{B}_2$ *holds. Moreover, the mapping* $\operatorname{Exp} : N \to \operatorname{int}\mathcal{B}_2$ *is a real-analytic diffeomorphism.*

Proof. A similar argument to that used in the proof of Th. 2 proves the claim. □

4.5 Attainable set and existence of optimal trajectories

Theorem 10. *We have* $\mathcal{A}_2 = \mathcal{B}_2$.

Proof. A similar argument to that used in the proof of Th. 3 proves the required equality. □

Theorem 11. *Let points* $q_0, q_1 \in M$ *satisfy the inclusion* $q_1 \in \mathcal{A}_2$. *Then there exists an optimal trajectory in problem* (4.1)–(4.4).

Proof. A similar argument to that used in the proof of Th. 4 proves the claim. □

4.6 Optimality of extremal trajectories

Define the following function:

$$\begin{aligned}
&\mathbf{t} : C \to (0, +\infty], \qquad \lambda = (\psi_0, c) \in C, \\
&c = 0 \quad \Rightarrow \quad \mathbf{t}(\lambda) = +\infty, \\
&c \neq 0 \quad \Rightarrow \quad \mathbf{t}(\lambda) = \frac{K}{læ}.
\end{aligned}$$

Theorem 12. *Let $\lambda \in C$, $t_1 \in (0, \mathbf{t}(\lambda))$. Then the trajectory $q(t) = \mathrm{Exp}(\lambda, t)$, $t \in [0, t_1]$, is optimal.*

Proof. A similar argument to that used in the proof of Th. 5 proves the claim. □

Corollary 4. *For any $\lambda \in C$ we have $t_{\mathrm{cut}}(\lambda) = \mathbf{t}(\lambda)$.*

Proof. Let $\lambda \in C$. By virtue of Th. 12, $t_{\mathrm{cut}}(\lambda) \geq \mathbf{t}(\lambda)$. On the other hand, the extremal trajectory $\mathrm{Exp}(\lambda, t)$ is defined only for $t \in [0, \mathbf{t}(\lambda))$, thus $t_{\mathrm{cut}}(\lambda) = \mathbf{t}(\lambda)$. □

Proposition 15. *Any abnormal extremal trajectory is optimal.*

Proof. A similar argument to that used in the proof of Propos. 9 proves the claim. □

Theorem 13. (1) *Let $q_1 \in \mathrm{int}\, \mathcal{A}_2$. Then there exists a unique optimal trajectory $\mathrm{Exp}(\lambda, t)$, $t \in [0, t_1]$, where $(\lambda, t_1) = \mathrm{Exp}^{-1}(q_1)$.*

(2) *Let $q_1 = (x_1, y_1, z_1) \in \partial \mathcal{A}_2$, $-y_1 < x_1 \leq y_1$, $z_1 = z^2(x_1, y_1)$. Then there exists a unique optimal trajectory corresponding to a control*

$$u(t) = \begin{cases} (1, 1), & t \in [0, \tau_1], \\ (1, -1), & t \in [\tau_1, t_1], \end{cases}$$

$$q(t) = \begin{cases} e^{t(X_1 + X_2)}(q_0), & t \in [0, \tau_1], \\ e^{(t-\tau_1)(X_1 - X_2)} \circ e^{\tau_1(X_1 + X_2)}(q_0), & t \in [\tau_1, t_1], \end{cases}$$

$$\tau_1 = (y_1 + x_1)/2, \quad t_1 = x_1.$$

(3) *Let $q_1 = (x_1, y_1, z_1) \in \partial \mathcal{A}_2$, $-y_1 \leq x_1 < y_1$, $z_1 = z^1(x_1, y_1)$. Then there exists a unique optimal trajectory corresponding to a control*

$$u(t) = \begin{cases} (1, -1), & t \in [0, \tau_1], \\ (1, 1), & t \in [\tau_1, t_1], \end{cases}$$

$$q(t) = \begin{cases} e^{t(X_1 - X_2)}(q_0), & t \in [0, \tau_1], \\ e^{(t-\tau_1)(X_1 + X_2)} \circ e^{\tau_1(X_1 - X_2)}(q_0), & t \in [\tau_1, t_1], \end{cases}$$

$$\tau_1 = (y_1 + x_1)/2, \quad t_1 = x_1.$$

Proof. A similar argument to that used in the proof of Th. 7 proves the claim. □

4.7 Sub-Lorentzian distance

Theorem 14. *The distance $d(q) = d(q_0, q)$ is real-analytic on* $\operatorname{int} \mathcal{A}_2$ *and continuous on* M*.*

Proof. A similar argument to that used in the proof of Th. 8 proves the claim. □

4.8 Sub-Lorentzian sphere

Theorem 15. *The sphere S is a real-analytic manifold diffeomorphic to $\mathbb{R}^2$ parametrized as follows: $S = \{\text{Exp}(\lambda, 1) \mid \lambda \in C\}$.*

Proof. A similar argument to that used in the proof of Th. 9 proves the claim. □

See the plot of two sub-Lorentzian spheres $S(R_1)$, $S(R_2)$ inside the attainable set $\mathcal{A}_2$ in Fig. 22.

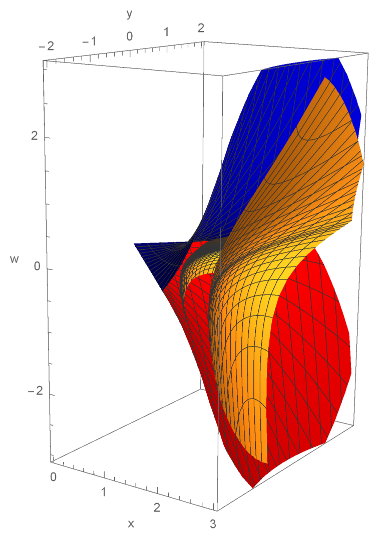

Figure 22: Spheres inside the attainable set for the second problem

5 Conclusion

The first problem is fundamentally different from the second one by the following properties of the optimal synthesis:

- some optimal trajectories change causal type,
- extremal trajectories have cut points on the Martinet surface Π,
- the optimal synthesis is two-valued on Π,
- the sub-Lorentzian distance is nonsmooth on Π and suffers a discontinuity of the first kind at some points of the boundary of the attainable set $\partial\mathcal{A}_1$,

- the sub-Lorentzian sphere S is a manifold with boundary.

These features are associated with non-trivial intersection of the attainable set $\mathcal{A}_1$ (the causal future of the initial point q_0) and the Martinet surface Π for the first problem.

The optimal synthesis in the second problem is qualitatively the same as in the sub-Lorentzian problem on the Heisenberg group [22].

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List of Figures

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