TIME MINIMIZATION PROBLEM ON THE ROTO-TRANSLATION GROUP WITH ADMISSIBLE CONTROL IN A CIRCULAR SECTOR

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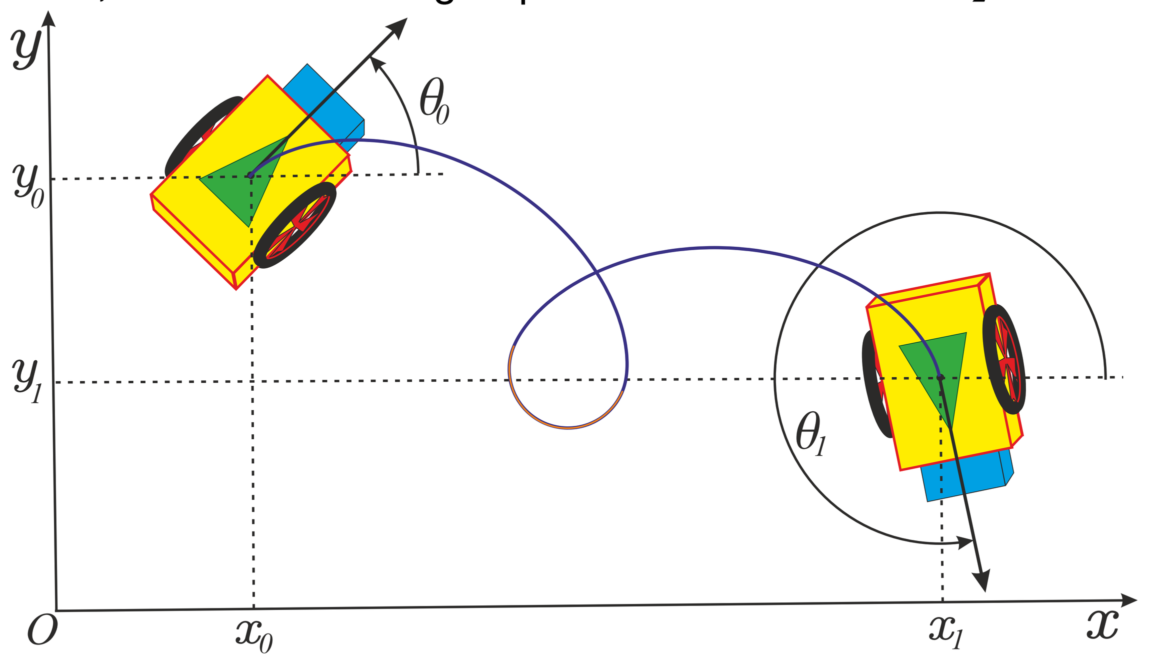
Summary

We study a time minimization problem for a car model on a plane. The problem is a modification of a well-known sub-Riemannian problem in the roto-translation group, where the set of admissible controls is a circular sector. We prove controllability and existence of optimal trajectories. Then, we apply Pontryagin maximum principle, a necessary optimality condition. We provide a qualitative analysis of the Hamiltonian system and obtain an explicit expression for the extremal controls and trajectories. We partially study optimality of extremals.

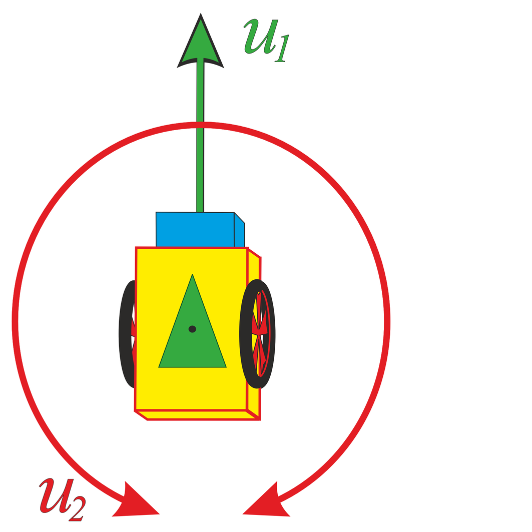
Introduction

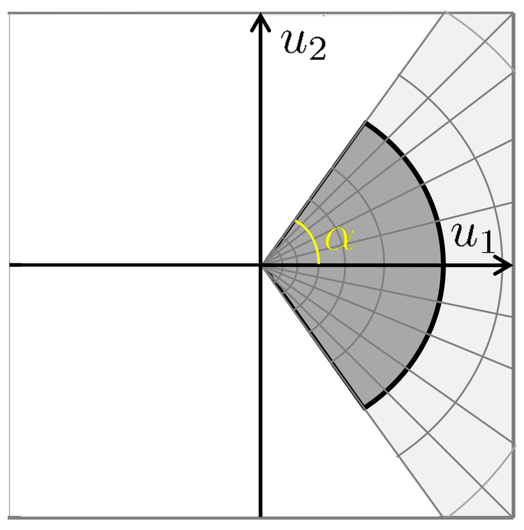
We study a time minimization problem for a car model that can move forward on a plane and turn with a given minimum turning radius. The car has two parallel wheels, equidistant from the axle of the wheelset. Both wheels have independent drives that can rotate so that the corresponding rolling of the wheels occurs without slipping. The configuration of the system is described by $q = (x, y, \theta) \in \mathbb{M} = \mathbb{R}^2 \times S^1$, where $(x, y) \in \mathbb{R}^2$ is the central point, and $\theta \in S^1$ is the orientation angle of the car. Thus, $\mathbb{M}$ forms the Lie group of roto-translations $\mathrm{SE}_2 \simeq \mathbb{M}$.



The car has two controls: the linear speed u_1 and the angular speed u_2. Dynamics at an arbitrary configuration q is given by $\dot{q} = u_1 X_1(q) + u_2 X_2(q)$, where X_i are left-invariant vector fields:

$$X_1(q) = \cos\theta\,\partial_x + \sin\theta\,\partial_y,\ X_2(q) = \partial_\theta,\ X_3(q) = \sin\theta\,\partial_x - \cos\theta\,\partial_y.$$

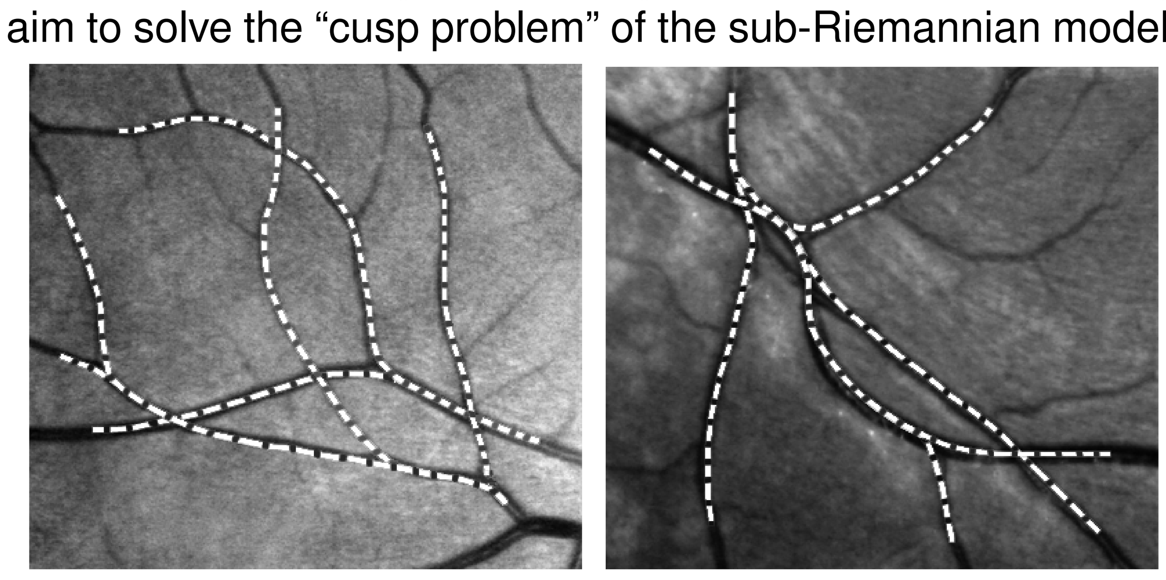




Various sets of admissible controls $U \ni (u_1, u_2)$ lead to different models. The time minimization problem for

- $u_1 = 1,\ |u_2| \leq \kappa,\ \kappa > 0$ leads to Dubins car;
- $|u_1| = 1,\ |u_2| \leq \kappa,\ \kappa > 0$ leads to Reeds-Shepp car;
- $u_1^2 + u_2^2 \leq 1$ leads to the model whose trajectories are sub-Riemannian length minimizes, studied by Sachkov;
- $u_1^2 + u_2^2 \leq 1,\ u_1 \neq 0$ studied by Berestovskii;
- $u_1 \geq 0,\ u_1^2 + u_2^2 \leq 1$ leads to the model of a car moving forward and turning in place, studied by Duits;
- $u_1 = r\cos\phi,\ u_2 = r\sin\phi,\ 0 \leq r \leq 1,\ |\phi| \leq \alpha$ leads to a model with control in a sector, studied in this work.

The problem is a modification of a well-known sub-Riemannian problem. The problem is of interest in geometric control theory as a model example, in which the set of admissible controls contains zero on the boundary. The trajectories of this system are applicable in image processing to detect salient lines. They aim to solve the "cusp problem" of the sub-Riemannian model.



Statement of the Problem

Consider the following control system:

$$\begin{cases} \dot{x} = u_1 \cos\theta, & (x, y, \theta) = q \in \mathrm{SE}_2 = \mathbb{M}, \\ \dot{y} = u_1 \sin\theta, & u_1 = r\cos\phi,\ u_2 = r\sin\phi, \\ \dot{\theta} = u_2, & 0 \leq r \leq 1,\ |\phi| \leq \alpha, 0 < \alpha \leq \frac{\pi}{2}. \end{cases}$$

For given boundary conditions $q_0, q_1 \in \mathbb{M}$, we aim to find the controls $u_1(t), u_2(t) \in L^\infty([0,T], \mathbb{R})$, such that the corresponding trajectory $\gamma : [0,T] \to \mathbb{M}$ transfers the system from the initial state q_0 to the final state q_1 by the minimal time:

$$\gamma(0) = q_0, \quad \gamma(T) = q_1, \qquad T \to \min.$$

Due to invariance under SE_2 action w.l.o.g. we set $q_0 = (0, 0, 0)$.

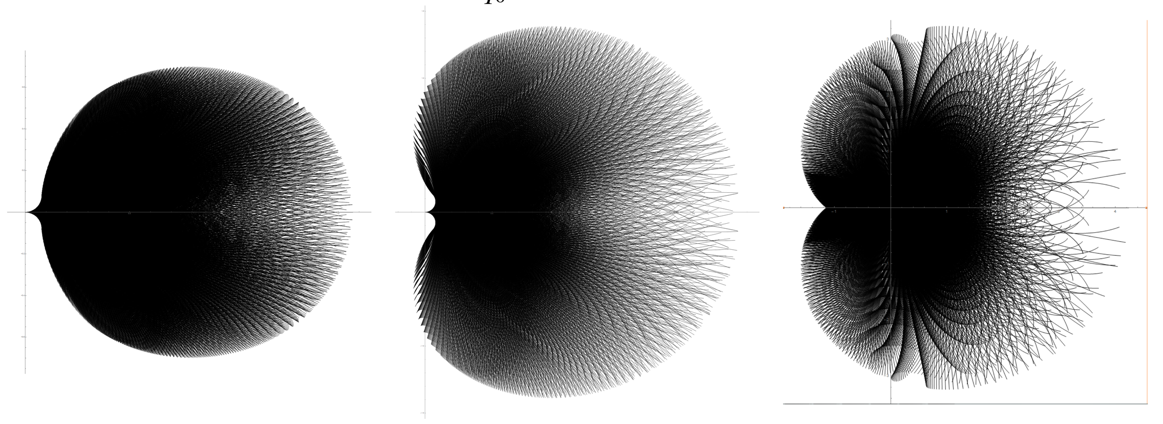
Existence of the Solution

By Lie saturation method we prove that there exists a trajectory connecting any q_0 and q_1. Further, a question of existence of optimal trajectories arises: does there always exist an admissible trajectory that connects the boundary conditions by minimal time? For our problem, due to compactness and convexity of U and global controllability existence of optimal trajectories is guaranteed by the Filippov theorem.

Theorem. In the time minimization problem on the roto-translation group with admissible control in a circular sector with a convex central angle, there always exists an optimal trajectory that transfers the system from an arbitrary given initial configuration to an arbitrary given final configuration.

We also prove that the system is not small time controllable, i.e. the attainable set $\mathcal{A}^t_{q_0}$ from q_0 by time $\leq t$ may not contain q_0 in its interior: $\exists t > 0 \,:\, q_0 \notin \mathrm{int}\,\mathcal{A}^t_{q_0}$.

Theorem. Let $T = \frac{2\pi}{\sin\alpha}$. Then for any $\epsilon > 0$ and any $q_0 \in \mathbb{M}$ the system is locally controllable at the point q_0 for time not greater than $T + \epsilon$, i.e., $q_0 \in \mathrm{int}\,\mathcal{A}^{T+\epsilon}_{q_0}$.



Pontryagin Maximum Principle

A necessary optimality condition is given by PMP.
Denote $h_i = \langle \lambda, X_i \rangle$, $\lambda \in T^*\mathbb{M}$.
The Pontryagin function reads as $H_u = u_1 h_1 + u_2 h_2$.
The Hamiltonian system is given by

$$\begin{cases} \dot{x} = u_1 \cos\theta, \\ \dot{y} = u_1 \sin\theta, \\ \dot{\theta} = u_2, \end{cases} \qquad \begin{cases} \dot{h}_1 = -u_2 h_3, \\ \dot{h}_2 = u_1 h_3, \\ \dot{h}_3 = u_2 h_1. \end{cases}$$

The subsystem for state variables x, y, θ is called the *horizontal* part, and the subsystem for adjoint variables h_1, h_2, h_3 is called the *vertical* part of the Hamiltonian system. An extremal control is determined by the vertical part, while an extremal trajectory is a solution to the horizontal part.

The maximum condition reads as $H = \max\limits_{u \in U} H_u$.

The nontriviality condition implies that if $h_1 = h_2 = 0$ then the corresponding extremal is trivial, i.e., it is a fixed point.

Let $h_1 = \rho\cos\psi$, $h_2 = \rho\sin\psi$, $\psi \in (-\pi, \pi]$, $\rho > 0$.

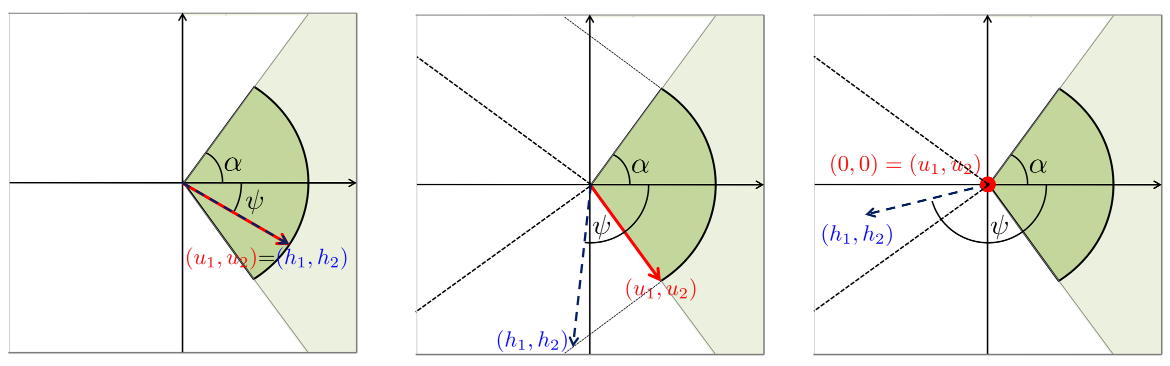
The maximum condition implies the following.

For $|\psi| \in (\frac{\pi}{2} + \alpha, \pi]$: $H = 0$, $u_1 = u_2 = 0$.

For $\pm\psi = \frac{\pi}{2} + \alpha$: $H = 0$, $u_1 = r\cos\alpha$, $u_2 = \pm r \sin\alpha$.

For $\pm\psi \in (\alpha, \frac{\pi}{2} + \alpha)$: $H = h_1\cos\alpha \pm h_2 \sin\alpha$, $u_1 = \cos\alpha$, $u_2 = \pm\sin\alpha$.

For $|\psi| \leq \alpha$: $H = \sqrt{h_1^2 + h_2^2}$, $\quad u_1 = \cos\psi, u_2 = \sin\psi$.



First Integrals

The vertical part has the first integrals:
the Hamiltonian (the maximized Pontryagin function)

$$H = \begin{cases} \sqrt{h_1^2 + h_2^2}, & \text{for } |\psi| \leq \alpha, \\ h_1 \cos\alpha + |h_2| \sin\alpha, & \text{for } \alpha < |\psi| < \alpha + \frac{\pi}{2}, \\ 0, & \text{for } |\psi| \geq \alpha + \frac{\pi}{2}. \end{cases}$$

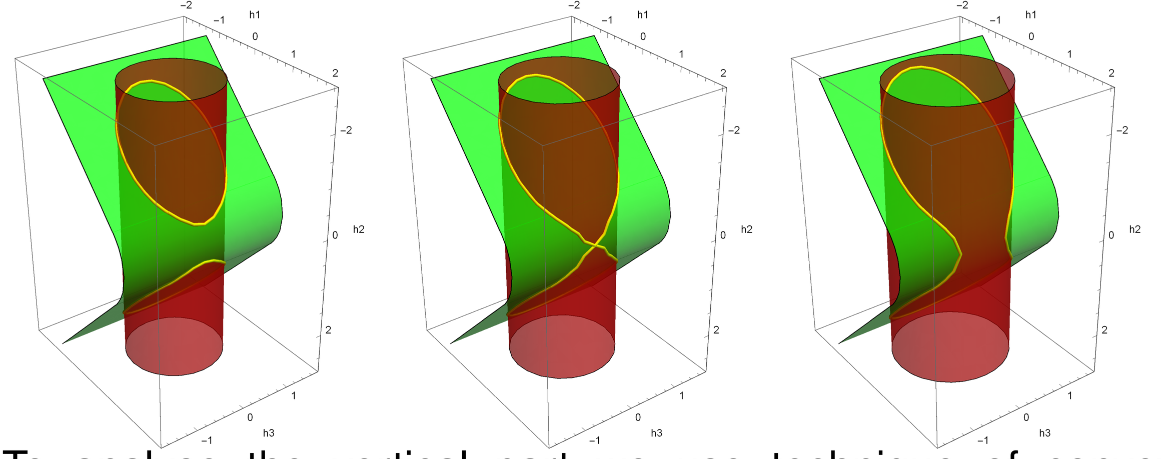
and the Casimir $E = \frac{h_1^2}{2} + \frac{h_3^2}{2}$.
The case $H = 0$ is called abnormal; $H = 1$ is called normal.

Abnormal Extremals

Abnormal extremal controls are given by $u_1(t) = r(t)\cos\alpha$, $u_2(t) = \pm r(t) \sin\alpha$, where $0 \leq r(t) \leq 1$ and the sign $\pm$ is fixed. It corresponds to motion of a car along an arc of a circle of minimal possible radius. It is easy to show that if $r(t) < 1$ then the trajectory is not optimal. The trajectories for $r(t) \equiv 1$ are not strictly abnormal, i.e. they are also normal extremals.

Normal Extremals

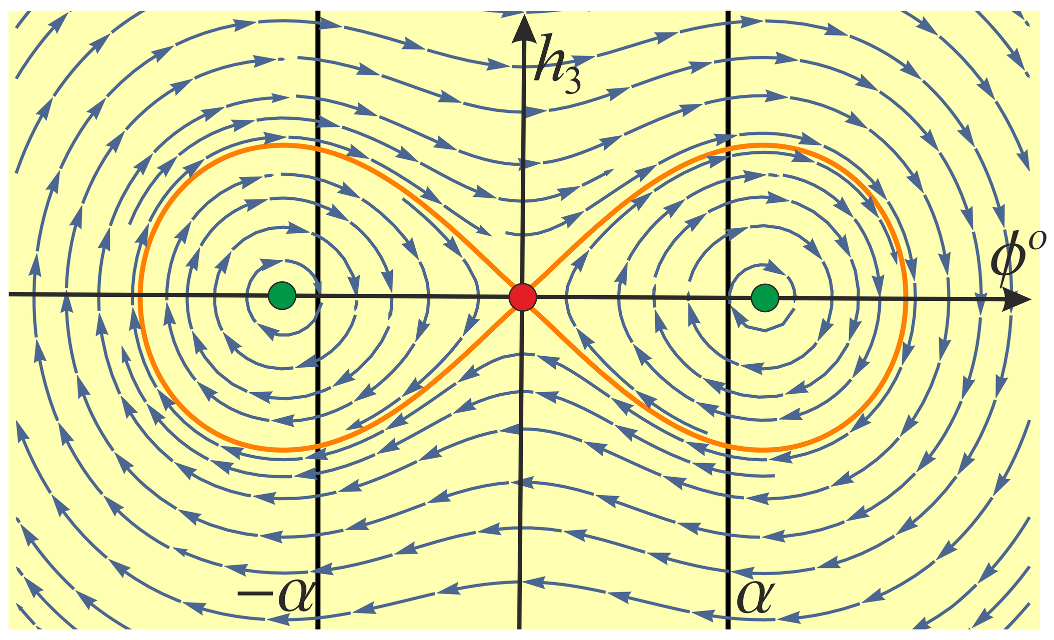
The level surface of the Hamiltonian $H = 1$ consists of two half-planes glued with a segment of the cylinder, and the level surface of the Casimir $E \geq 0$ is a cylinder.



To analyse the vertical part we use technique of convex trigonometry. Along the extremal trajectories we have

$$u_1 = \cos\phi,\ u_2 = \sin\phi,\ h_1 = \cos_{U^o}\phi^o,\ h_2 = \sin_{U^o}\phi^o,$$

where U^o is the polar set to U, $\cos_{U^o}$ and $\sin_{U^o}$ are the functions of convex trigonometry. Denote $K(\phi^o) = \frac{1}{2}\cos^2_{U^o}\phi^o$. The vertical part is reduced to the system $\dot{\phi}^o = h_3$, $\dot{h}_3 = K'(\phi^o)$.



Analyzing the phase portrait we conclude:

- $E = 0 \Rightarrow (\phi^o, h_3) \equiv (\pm(\alpha + \cot\alpha), 0)$ is stable equilibrium;
- $E \in (0, \frac{1}{2}) \cup (\frac{1}{2}, +\infty) \Rightarrow$ the trajectory $(\phi^o, h_3)(t)$ is periodic;
- $E = \frac{1}{2} \Rightarrow$ either $(\phi^o, h_3) \equiv (0, 0)$ is unstable equilibrium or $(\phi^o, h_3)(t)$ is a separatrix.

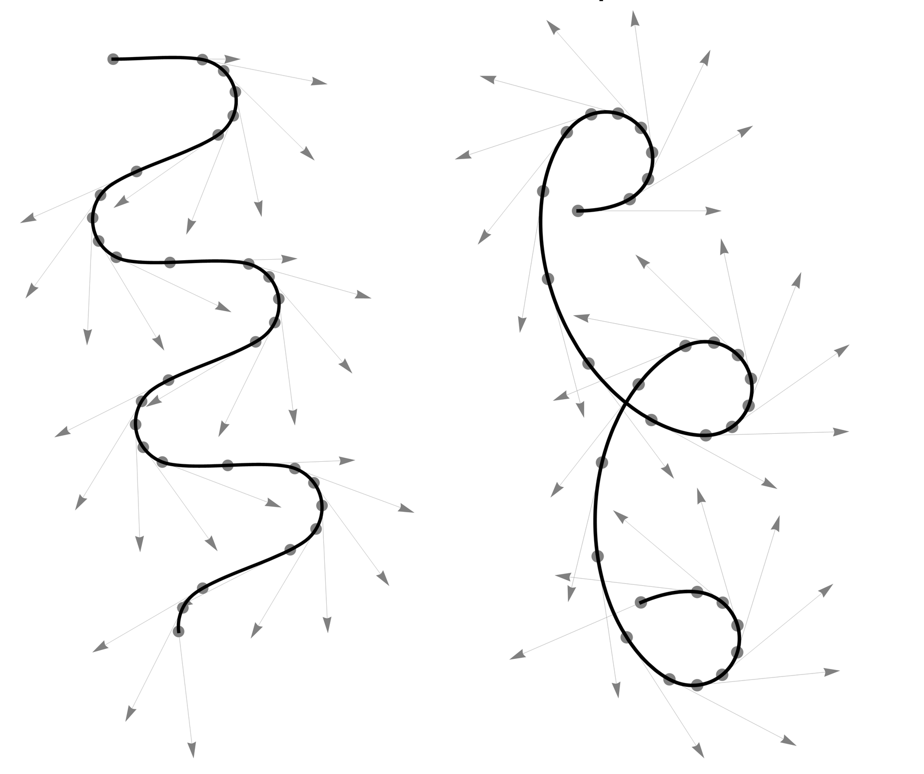
Explicit Expression of Extremals

We introduce rectifying coordinates for the vertical part and derive explicit expression for the extremals.

- For stable equilibrium, the corresponding extremal trajectory is given by a segment of a straight line. It is optimal up to infinity.
- For periodic adjoint trajectories, depending on the sign of $a = \mathrm{sign}(|\phi^o| - \alpha) = \mathrm{sign}(h_1(t) - \cos\alpha)$, we have two different dynamics. When a switches, the dynamics switch from one to another. The corresponding solution to the horizontal part is given by consequent concatenation of motions of a car along an arc of the circle and an arc of the sub-Riemannian geodesic.
- For unstable equilibrium, the corresponding extremal trajectory is given by motion of a car along the circle.
- For separatrix, the extremal trajectory is given by concatenation of maximum three intervals (possibly zero length): motion along a segment of the tractrix, an arc of the circle, a segment of the tractrix.

For $\alpha < \frac{\pi}{2}$ we also derive explicit parametrization of the extremals by the arc-length in the plane Oxy. It is always possible for nontrivial trajectories since $u_1 > 0$.

For $\alpha = \frac{\pi}{2}$, projection to Oxy of the extremals coincide with the projection of sub-Riemannian geodesics, while the dynamics of θ differs: cusp points are replaced by in-place rotations. An optimal motion can not have internal in-place rotations.



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