A CORTICAL-INSPIRED CONTOUR COMPLETION MODEL BASED ON CONTOUR ORIENTATION AND THICKNESS

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SUMMARY

An extended four-dimensional version of the traditional Petitot–Citti–Sarti model on contour completion in the visual cortex is examined. The neural configuration space is considered as the group of similarity transformations, denoted as $M = \mathrm{SIM}(2)$. The left-invariant subbundle of the tangent bundle models possible directions for establishing neural communication. The sub-Riemannian distance is proportional to the energy expended in interneuron activation between two excited border neurons. According to the model, the damaged image contours are restored via sub-Riemannian geodesics in the space M of positions, orientations and thicknesses (scales). We study the geodesic problem in M using geometric control theory techniques. We prove the existence of a minimal geodesic between arbitrary specified boundary conditions. We apply the Pontryagin maximum principle and derive the geodesic equations. In the special cases, we find explicit solutions. In the general case, we provide a qualitative analysis. Finally, we support our model with a simulation of the association field.

INTRODUCTION

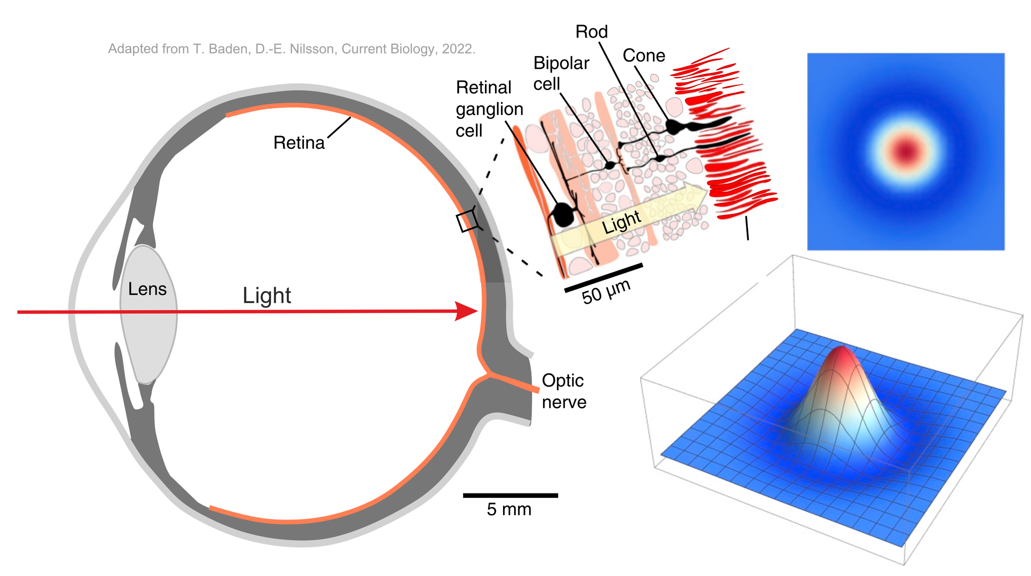
A mathematical description of the functioning of the human body is a pressing problem in the modern world. The specification of cerebration and neuron operation of the human visual system is of particular interest. The visual cortex consists of billions of neural cells. Neurons are connected in a complex network, which is extremely difficult to analyze due to the huge number of elements and even more connections between them. The direct simulation approach to modeling such systems faces inevitable obstacles. However, there are some fundamental principles that are used in network configuration, e.g., the principle of minimum energy spent on establishing communication between two excited neurons of the network.
A promising direction for studying such complex systems is to understand such principles and propose simple mathematical models based on these principles.
The visual system has a multilayered organization. The complete mechanism of the visual signal processing is not fully understood, however, there is a profound understanding of its early stages and the corresponding mathematical models [1].
The the visual signal (the image on the retinal plane)

$$F:\mathbb{R}^2\to\mathbb{R}^+ : (x,y)\mapsto F(x,y)$$

processing is modeled as the action (convolution)

$$(F*K)(x,y)=\int_{\mathbb{R}^2} F(x,y)K(x-\xi,y-\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta$$

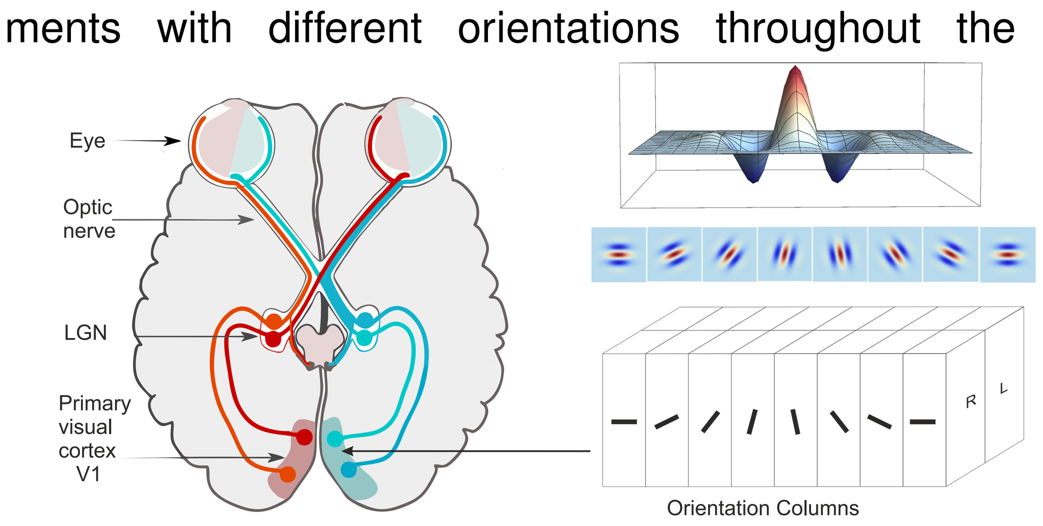
of an appropriate filter $K:\mathbb{R}^2\to\mathbb{R}$ on the image F. An appropriate filter is a function with a graph similar to the cells' receptive field in a given layer.



The first layer is the eye retina, where the image processing is carried out via the information accumulated in light-sensitive receptors, bipolar, and ganglion cells. Such information includes the spatial coordinates of the image. The receptive field of the retinal cells is well approximated by the Laplacian of Gaussian (LoG) filter specified by a scale parameter $\kappa > 0$:

$$LoG(x,y)=\frac{1}{\pi\kappa^4}\left(1-\frac{x^2+y^2}{2\kappa^2}\right)e^{-\frac{x^2+y^2}{2\kappa^2}},\qquad \kappa=e^\sigma,\sigma\in\mathbb{R}.$$

After the retina, the visual signal passes through LGN cells of the thalamus and arrives in the visual cortex. Hubel and Wiesel [2] understood the principles of the primary visual cortex V1 processing. They discovered the ability of V1 cells to detect contour segments with different orientations throughout the image.



Mathematically, the operation of V1 simple cells can be modeled as lifting a two-dimensional input image into an expanded space $\mathrm{SE}(2)$ of positions and orientations. The receptive fields of the V1 neurons are well approximated by the Gabor filters

$$G_{(\theta,\sigma)}(x,y)=e^{-\left(x_\theta^2+y_\theta^2\right)}\cos y_\theta,$$

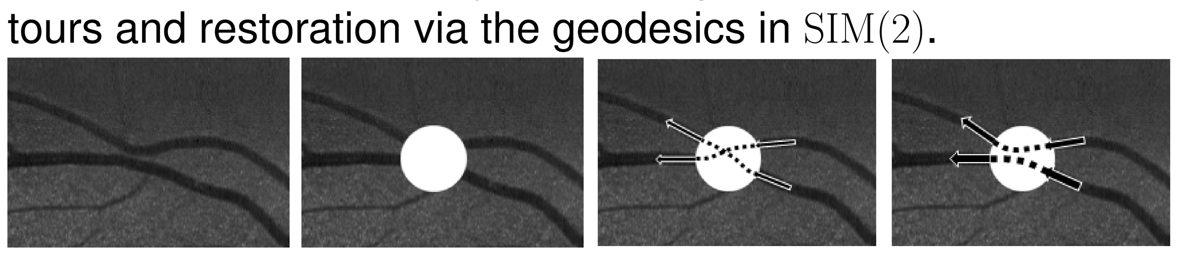
where $x_\theta=e^{-\sigma}(x\cos\theta+y\sin\theta)$, $y_\theta=e^{-\sigma}(-x\sin\theta+y\cos\theta)$.
The classic model of Petitot [3], Citti, and Sarti [4] states that the visual system performs contour completion (restoration of a corrupted or partially hidden from observation contour) by finding a sub-Riemannian length minimizer in $\mathrm{SE}(2)$ between two configurations on the boundary of the damaged area.

ACCOUNTING FOR THICKNESS

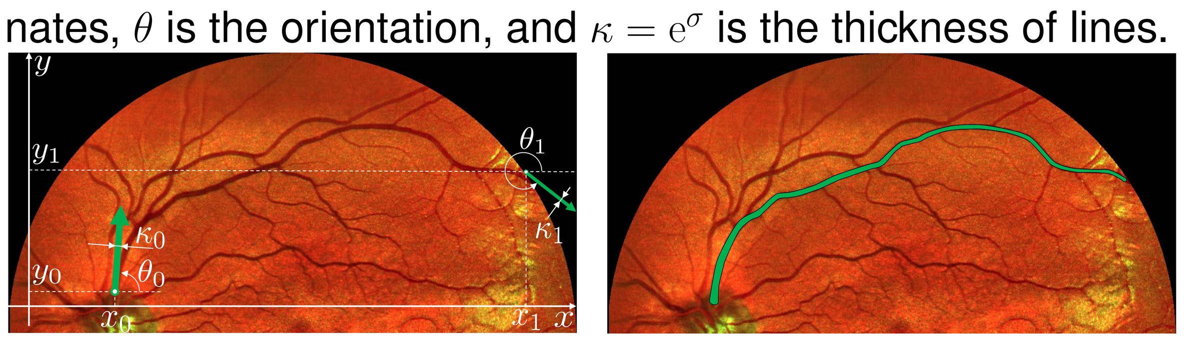
Neurophysiological studies show that spatial hypercolumns in V1 also accumulate secondary information about the visible image, such as contour thickness (scale) and other features. In [5] the classic Petitot-Citti-Sarti model has been extended by taking into account the variable parameter σ. According to the model, the contour completion mechanism by V1 is invariant under parallel translations, rotations, and scaling of the image on the retina. Such transformations constitute the group of orientation-preserving similarity transformations on the plane

$$\mathrm{SIM}(2)=\left\{q=\begin{pmatrix} e^\sigma\cos\theta & -e^\sigma\sin\theta & x\\ e^\sigma\sin\theta & e^\sigma\cos\theta & y\\ 0&0&1\end{pmatrix}\;\middle|\;\begin{matrix}(x,y)\in\mathbb{R}^2,\\ \theta\in S^1,\sigma\in\mathbb{R}\end{matrix}\right\}.$$

This extension is intended for image processing tasks to restore damaged image contours. The below figure shows the original and the corrupted image. Recovering contours via the classic model sometimes leads to the wrong result. Such problem cases are avoided by accounting for the thickness of contours and restoration via the geodesics in $\mathrm{SIM}(2)$.



The extended model is also motivated by application to the problem of finding salient lines in images. The below figure illustrates finding blood vessels (salient lines) in the photography of the human retina. Specifications: x, y are spatial coordinates, θ is the orientation, and $\kappa=e^\sigma$ is the thickness of lines.



STATEMENT OF THE PROBLEM

We formulate our contour completion model as the optimal control problem. Consider the following control system:

$$\begin{cases}\dot x=u_1e^\sigma\cos\theta,\\ \dot y=u_1e^\sigma\sin\theta,\\ \dot\theta=u_3,\\ \dot\sigma=u_4,\end{cases}\qquad \begin{aligned}&(x,y,\theta,\sigma)=q\in\mathrm{SIM}(2),\\ &(u_1,u_3,u_4)\in U,\\ &U=\left\{(u_1,u_3,u_4)\in\mathbb{R}^3\;\middle|\;u_1^2+u_3^2+u_4^2\le 1\right\}.\end{aligned}$$

For given boundary conditions $q_0,\ q_1\in\mathrm{SIM}(2)$, we aim to find the controls $u_1(t),\ u_3(t),\ u_4(t)\in L^\infty([0,T],\mathbb{R})$, such that the corresponding trajectory $q:[0,T]\to\mathrm{SIM}(2)$ satisfies

$$q(0)=q_0,\qquad q(T)=q_1,\qquad T\to\min.$$

Due to invariance under $\mathrm{SIM}(2)$ action we set $q_0=(0,0,0)$.

EXISTENCE OF THE SOLUTION

The system is symmetric with respect to the controls and it satisfies Hormander condition. By Chow–Rashevsky theorem, these two conditions plus connectedness of $\mathrm{SIM}(2)$ guarantee complete controllability. Existence of an optimal admissible trajectory is ensured by Filippov's theorem.
Theorem. A solution to the optimal control problems exists for any boundary condition.

PONTRYAGIN MAXIMUM PRINCIPLE

Denote by X_i the left-invariant vector fields

$$X_1(q)=e^\sigma\left(\cos\theta\frac{\partial}{\partial x}+\sin\theta\frac{\partial}{\partial y}\right),\qquad X_3(q)=\frac{\partial}{\partial\theta},$$
$$X_2(q)=e^\sigma\left(-\sin\theta\frac{\partial}{\partial x}+\cos\theta\frac{\partial}{\partial x}\right),\qquad X_4(q)=\frac{\partial}{\partial\sigma}.$$

Denote $h_i=\langle\lambda,X_i\rangle$, $\lambda\in T^*\,\mathrm{SIM}(2)$. The Pontryagin function is

$$H_u=u_1h_1+u_3h_3+u_4h_4.$$

The Hamiltonian system is given by

$$\begin{cases}\dot x=u_1e^\sigma\cos\theta,\\ \dot y=u_1e^\sigma\sin\theta,\\ \dot\theta=u_3,\\ \dot\sigma=u_4,\end{cases}\qquad\begin{cases}\dot h_1=u_3h_2+u_4h_1,\\ \dot h_2=-u_3h_1+u_4h_2,\\ \dot h_3=-u_1h_2,\\ \dot h_4=-u_1h_1.\end{cases}$$

PMP states that H_u is maximum $H=\max\limits_{u\in U}H_u$ on optimal control.

ABNORMAL EXTREMALS $H=0$

The Hamiltonian H is a first integral the Hamiltonian system. Without loss of generality there are to distinct cases $H=0$ (abnormal case) and $H=1$ (normal case).
Theorem. Abnormal optimal trajectories have the following form: $x(t)=y(t)=\theta(t)=0,\quad \sigma(t)=\pm t$.

NORMAL EXTREMALS $H=1$

In the normal case we have $H=h_1^2+h_3^2+h_4^2=1$.
The Hamiltonian system takes the form

$$\begin{cases}\dot x=h_1e^\sigma\cos\theta,\\ \dot y=h_1e^\sigma\sin\theta,\\ \dot\theta=h_3,\\ \dot\sigma=h_4,\end{cases}\qquad\begin{cases}\dot h_1=h_3h_2+h_4h_1,\\ \dot h_2=-h_3h_1+h_4h_2,\\ \dot h_3=-h_1h_2,\\ \dot h_4=-h_1^2.\end{cases}$$

This system has the following independent first integrals:

$$H,\ g_1=e^{-\sigma}\left(h_1\cos\theta-h_2\sin\theta\right),\ g_2=e^{-\sigma}\left(h_2\cos\theta+h_1\sin\theta\right).$$

The question of Liouville integrability remains open.
Consider the Poisson bivector $P=(P_{ij})$ with the components $P_{ij}=\{h_i,h_j\}$. We have $\det P=(h_1^2+h_2^2)^2$; thus $\operatorname{rank}P=0$ if $h_1^2+h_2^2=0$, and $\operatorname{rank}P=4$ otherwise.
In the case $h_1^2+h_2^2=0$, the coadjoint orbit is zero dimensional and we have the explicit expression for the extremals.
Theorem. For the initial covector values $h_{10}=h_{20}=0$ normal extremal trajectories have the following form:

$$x(t)=y(t)=0,\quad\theta(t)=h_{30}\,t,\quad\sigma(t)=h_{40}\,t.$$

They are optimal on a time interval $t\in[0,\frac{\pi}{h_{30}}]$, when $h_{30}\neq 0$; and up to infinity, when $h_{30}=0$.

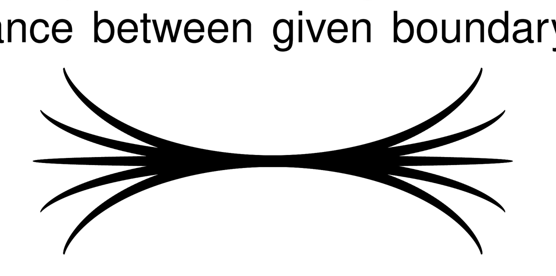
In the general case $h_1^2+h_2^2>0$, the coadjoint orbit is four dimensional. We performed a qualitative analysis of the Hamitonian system leading to the following theorem.
Theorem. Any solution to the vertical part corresponding to the initial covector $h_{10}^2+h_{20}^2>0$, $h_{40}<0$ has the following asymptotic behavior:

$$\lim_{t\to\infty}h_1(t)=0,\quad\lim_{t\to\infty}h_2(t)=0,\quad\lim_{t\to\infty}h_3(t)=h_{31},\quad\lim_{t\to\infty}h_4(t)=h_{41}.$$

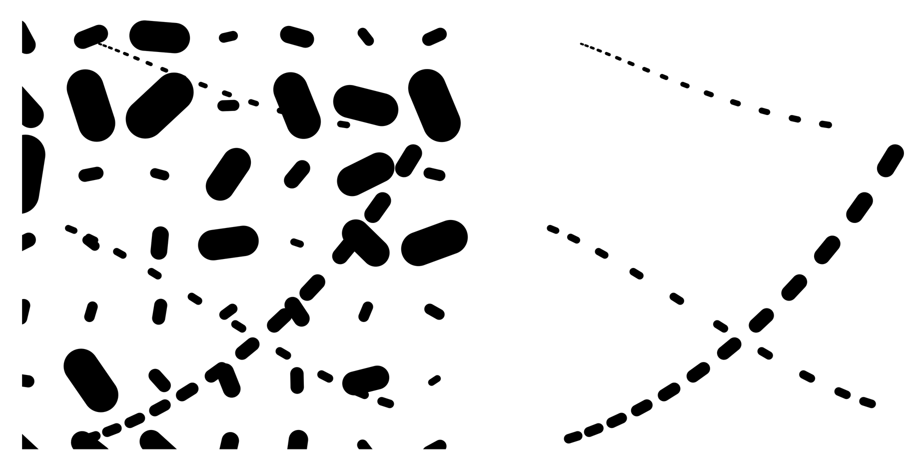
Note that the condition $h_{40}<0$ is technical, and we use it in the proof. Based on the numerical experiments, we formulate the conjecture that the limiting behavior holds for all $h_{40}\in\mathbb{R}$.

MODELING OF ASSOCIATION FIELD

Psychophysicists investigated the problem of contour completion (integration) by the human visual system. Gestalt laws have been proposed for several phenomena of visual perception. Among them, the law of good continuation plays a central role in perceptual completion. The principle of good continuation has resulted in the notion of association field, which describes the set of possible subjective contours starting from a given initial configuration. The role of the scale in the contour integration process was also noticed.
We provide a simulation of the association field by sub-Riemannian geodesics in $\mathrm{SIM}(2)$. A remarkable property of this model is that the further spatial propagation of the present geodesics does not appear with growing time, which corresponds to our conjecture. This gives a natural bound for the spatial distance between given boundary configurations.



We provide another simulation showing that the sub-Riemannian distance in $\mathrm{SIM}(2)$ can be used as a criterion for perceptual grouping of the patterns with different positions, orientations, and sizes.



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