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Extremal Trajectories in a Time-Optimal Problem on the Group of Motions of a Plane with Admissible Control in a Circular Sector

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Abstract—We consider a time-optimal problem for a car model that can move forward on a plane and turn with a given minimum turning radius. Trajectories of this system are applicable in image processing for the detection of salient lines. We prove the controllability and existence of optimal trajectories. Applying the necessary optimality condition given by the Pontryagin maximum principle, we derive a Hamiltonian system for the extremals. We provide qualitative analysis of the Hamiltonian system and obtain explicit expressions for the extremal controls and trajectories.

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1. INTRODUCTION

Consider a model of an idealized car moving on a plane (Fig. 1). The car has two parallel wheels equidistant from the axle of the wheelset. Both wheels have independent drives that can rotate so that the corresponding rolling of the wheels occurs without slipping. The configuration of the system is described by the triple $q = (x, y, \theta) \in \mathbb{M} = \mathbb{R}^2 \times S^1$, where $(x, y) \in \mathbb{R}^2$ is the central point and $\theta \in S^1$ is the orientation angle of the car. Thus, the configuration space $\mathbb{M}$ forms the Lie group of roto-translations (proper Euclidean motions of the plane) $\mathrm{SE}_2 \simeq \mathbb{M} = \mathbb{R}^2 \times S^1$.

The car has two controls: the linear speed u_1 and the angular speed u_2. The dynamics at an arbitrary configuration $q \in \mathrm{SE}_2$ is given by

$$\dot{q} = u_1 X_1(q) + u_2 X_2(q), \tag{1.1}$$

where X_i are left-invariant vector fields:

$$X_1(q) = (\cos\theta)\partial_x + (\sin\theta)\partial_y, \qquad X_2(q) = \partial_\theta, \qquad X_3(q) = (\sin\theta)\partial_x - (\cos\theta)\partial_y.$$

Various sets of admissible control parameters $U \ni (u_1, u_2)$ in the time-optimal problem (TOP) lead to different models (Fig. 2):

- if $u_1 = 1$ and $|u_2| \le \kappa$, $\kappa > 0$, then the TOP leads to the Dubins car [8];
- if $|u_1| = 1$ and $|u_2| \le \kappa$, $\kappa > 0$, then the TOP leads to the Reeds–Shepp car [16];
- if $u_1^2 + u_2^2 \le 1$, then the TOP leads to the model studied by Sachkov [18], with trajectories being sub-Riemannian length minimizers;
- if $u_1^2 + u_2^2 \le 1$ and $u_1 \ne 0$, then the TOP leads to the model studied by Berestovskii [5];
- if $u_1 \ge 0$ and $u_1^2 + u_2^2 \le 1$, then the TOP leads to the model of a car moving forward and turning on a spot, proposed and studied by Duits et al. [10];

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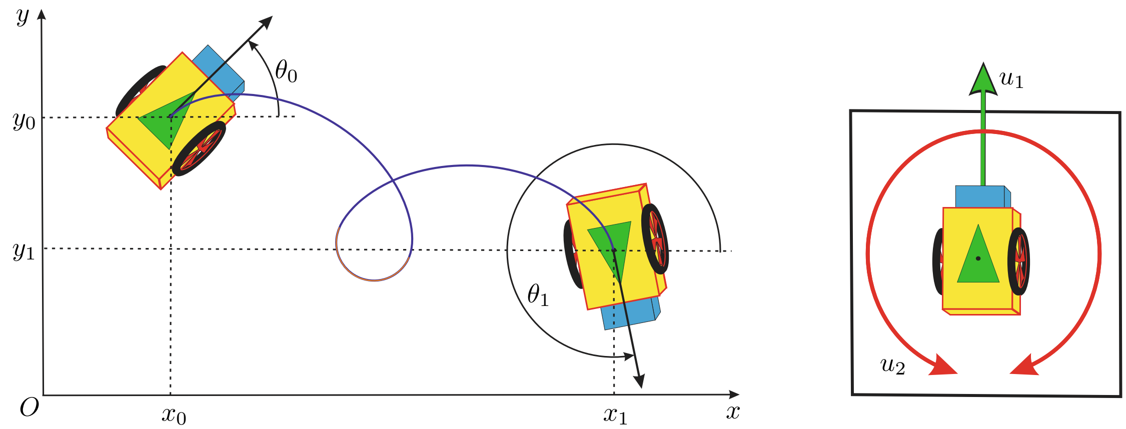

Fig. 1. A model of a vehicle (car) that can move forward and turn with a given minimum radius. The control u_1 is responsible for the forward movement, and the control u_2, for the turn.

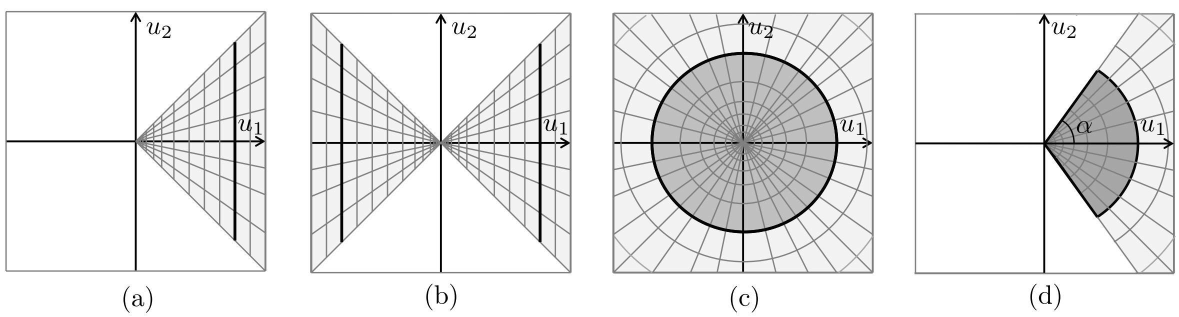

Fig. 2. Sets of admissible controls for various models of a car on a plane: (a) Dubins car [8], (b) Reeds–Shepp car [16], (c) sub-Riemannian model [18], and (d) our model.

- if $u_1 = r\cos\phi$ and $u_2 = r\sin\phi$, $0 \le r \le 1$, $|\phi| \le \alpha$, then the TOP leads to a general model of a car with control in a circular sector, which is studied in this paper. In our model, the car can move forward on a plane and turn with a given minimum turning radius. Such a limitation is natural in robotics.

System (1.1) is used in robotics to model a car-like robot. The system also arises in modeling the human visual system [15, 7, 9] and in image processing [14, 6, 4]. The model is relevant to the detection of salient lines in images and is aimed at solving the "cusp problem" in [4].

In the present paper, we study the time-optimal problem for system (1.1) with control in a circular sector. We generalize the results of [12, 13], where we performed a detailed analysis of the special case of admissible control in a half-disc, initially proposed by Duits et al. in [10]. We apply the Pontryagin maximum principle (PMP) and analyze the Hamiltonian system of the PMP. We provide a qualitative analysis of the dynamics and derive expressions for extremal controls and trajectories.

2. STATEMENT OF THE PROBLEM

For any $\alpha \in (0, \pi/2)$, consider the following control system:

$$\begin{cases} \dot{x} = u_1\cos\theta, & (x, y, \theta) = q \in \mathrm{SE}_2 = \mathbb{M}, \\ \dot{y} = u_1\sin\theta, & u_1^2 + u_2^2 \le 1, \\ \dot{\theta} = u_2, & |u_2| \le u_1\tan\alpha. \end{cases} \tag{2.1}$$

We study a time-optimal problem in which some boundary conditions $q_0, q_1 \in \mathbb{M}$ are fixed and it is required to find controls $u_1(t), u_2(t) \in L^\infty([0,T],\mathbb{R})$ such that the corresponding trajectory $\gamma\colon [0,T] \to \mathbb{M}$ transfers the system from the initial state q_0 to the final state q_1 in minimum time:

$$\gamma(0) = q_0, \qquad \gamma(T) = q_1, \qquad T \to \min. \tag{2.2}$$

Due to the invariance under the left SE_2 action, we set $q_0 = (0,0,0)$ without loss of generality.

3. EXISTENCE OF A SOLUTION

In this section, we prove that for any $q_1 \in \mathbb{M}$ there exists an optimal trajectory connecting q_0 to q_1. We also prove that system (2.1) is not small-time locally controllable.

Definition 1. Let $\mathcal{F} \subset \mathrm{Vec}(\mathbb{M})$, $q_0 \in \mathbb{M}$, and $t \geq 0$. Denote by $\mathcal{A}_{q_0,\leq t}(\mathcal{F})$ the attainable set of the system $\mathcal{F}$ from q_0 in time $\leq t$. The system $\mathcal{F}$ is called *small-time locally controllable at* q_0 if $\operatorname{int} \mathcal{A}_{q_0,\leq t}(\mathcal{F}) \ni q_0$ for all $t > 0$.

Definition 2. Let $\mathcal{F} \subset \mathrm{Vec}(\mathbb{M})$ and $q_0 \in \mathbb{M}$. Denote by $\mathcal{A}_{q_0}(\mathcal{F})$ the attainable set of the system $\mathcal{F}$ from q_0 (in any nonnegative time). The system $\mathcal{F}$ is called *globally controllable from* q_0 if $\mathcal{A}_{q_0}(\mathcal{F}) = \mathbb{M}$; it is called *globally controllable* if it is globally controllable from any point of $\mathbb{M}$.

For $0 < \alpha < \pi/2$ introduce the set of admissible controls

$$U = \big\{(u_1,u_2)\colon\ u_1 = r\cos\phi,\ u_2 = r\sin\phi,\ 0 \leq r \leq 1,\ |\phi| \leq \alpha\big\}.$$

Let $\mathcal{F} = \{u_1X_1 + u_2X_2\colon (u_1,u_2) \in U\} \subset \mathrm{Vec}(\mathbb{M})$.

Theorem 1. *For any $q \in \mathbb{M}$, the system $\mathcal{F}$ is not small-time locally controllable at q.*

Proof. Since $\mathcal{F}$ is left invariant, we restrict ourselves to the case $q = q_0 = (0,0,0)$. For any $u \in U$ we have $u_1 > 0$. Moreover, we have $\theta(0) = 0$ and

$$x(t) = \int\limits_0^t u_1(\tau)\cos\theta(\tau)\,\mathrm{d}\tau > 0 \qquad \text{for sufficiently small} \;\; t > 0.$$

Thus, $q_0 \notin \operatorname{int} \mathcal{A}_{q_0,\leq t}(\mathcal{F})$ and $\mathcal{F}$ is not small-time locally controllable at q_0. □

Theorem 2. *The system $\mathcal{F}$ is globally controllable.*

Proof. The global controllability of $\mathcal{F}$ follows from that of the smaller system $\widehat{\mathcal{F}} = \{X_1 + wX_2\colon |w| \leq \tan\alpha\}$. We have $\cos\alpha \cdot \widehat{\mathcal{F}} = \{\cos\alpha \cdot X_1 + vX_2\colon |v| \leq \sin\alpha\} \subset \mathcal{F}$. It remains to prove that $\widehat{\mathcal{F}}$ is globally controllable. To this end, we compute the Lie saturation $\mathrm{LS}(\widehat{\mathcal{F}})$ (see [17]) and show that $\mathrm{LS}(\widehat{\mathcal{F}}) = \mathrm{Lie}(X_1,X_2)$.

The vector field $X_1 + wX_2$, $w > 0$, has a periodic trajectory

$$\theta = \theta_0 + wt, \qquad x = x_0 + \frac{\sin(\theta_0 + wt) - \sin\theta_0}{w}, \qquad y = y_0 + \frac{\cos\theta_0 - \cos(\theta_0 + wt)}{w}.$$

Thus, $-(X_1 + wX_2) \in \mathrm{LS}(\widehat{\mathcal{F}})$, $|w| < \tan\alpha$. It follows that $-wX_2 = -(X_1 + wX_2) + X_1 \in \mathrm{LS}(\widehat{\mathcal{F}})$. Consequently, $\pm X_2 \in \mathrm{LS}(\widehat{\mathcal{F}})$, and so $\pm X_1 \in \mathrm{LS}(\widehat{\mathcal{F}})$. Since $\widehat{\mathcal{F}}$ has a full rank, we obtain $\mathrm{LS}(\widehat{\mathcal{F}}) \supset \mathrm{Lie}(X_1,X_2)$, which implies that $\widehat{\mathcal{F}}$ is globally controllable. Hence, $\mathcal{F}$ is globally controllable. □

Further, a question arises as to whether there exist optimal trajectories: does there always exist an admissible trajectory that connects the boundary conditions in minimum time? For our problem (2.1) the answer is positive. Since U is compact and convex and $\mathcal{F}$ is globally controllable, the existence of optimal trajectories is guaranteed by the Filippov theorem (see [2]).

4. PONTRYAGIN MAXIMUM PRINCIPLE

We apply a necessary optimality condition given by the PMP [1, 2]. The Pontryagin function reads

$$H_u = u_1(p_1\cos\theta + p_2\sin\theta) + u_2p_3, \qquad \text{where} \quad (p_1,p_2,p_3)\in T_q^*\mathbb{M}\simeq\mathbb{R}^3.$$

Let $(u(t),q(t))$, $t\in[0,T]$, be an optimal process. Then there exists a Lipschitz curve $p(t)$ for which the following relations hold:

(1) the Hamiltonian system

$$\dot p = -\frac{\partial H_u}{\partial q}, \qquad \dot q = \frac{\partial H_u}{\partial p};$$

(2) the maximum condition

$$H_{u(t)}(p(t),q(t)) = \max_{u\in U} H_u(p(t),q(t)) = H \in \{0,1\};$$

(3) the nontriviality condition $p_1^2+p_2^2+p_3^2\neq 0$.

The maximized Pontryagin function (Hamiltonian) H is a first integral of the Hamiltonian system. The case $H=0$ is said to be abnormal, and the case $H=1$, normal.

Let $h_i=\langle p,X_i\rangle$: $h_1=p_1\cos\theta+p_2\sin\theta$, $h_2=p_3$, and $h_3=p_1\sin\theta-p_2\cos\theta$.

The Pontryagin function reads $H_u=u_1h_1+u_2h_2$.

The Hamiltonian system is given by

$$\begin{cases}\dot x = u_1\cos\theta,\\ \dot y = u_1\sin\theta,\\ \dot\theta = u_2,\end{cases} \qquad \begin{cases}\dot h_1 = -u_2h_3,\\ \dot h_2 = u_1h_3,\\ \dot h_3 = u_2h_1.\end{cases} \tag{4.1}$$

The subsystem for the state variables x, y, θ is called the *horizontal* part, and the subsystem for the adjoint variables h_1, h_2, h_3 is called the *vertical* part of the Hamiltonian system. An extremal control is determined by the vertical part, while an extremal trajectory is a solution to the horizontal part.

Let $h_1=\rho\cos\psi$ and $h_2=\rho\sin\psi$, $\psi\in(-\pi,\pi]$, $\rho\geq 0$. The maximum condition implies the following:

- if $|\psi|\in(\pi/2+\alpha,\pi]$, then $H=0$ and $u_1=u_2=0$;
- if $\pm\psi=\pi/2+\alpha$, then $H=0$, $u_1=r\cos\alpha$, and $u_2=\pm r\sin\alpha$;
- if $\pm\psi\in(\alpha,\pi/2+\alpha)$, then $H=h_1\cos\alpha\pm h_2\sin\alpha$, $u_1=\cos\alpha$, and $u_2=\pm\sin\alpha$;
- if $|\psi|\leq\alpha$, then $H=\sqrt{h_1^2+h_2^2}$, $u_1=\cos\psi$, and $u_2=\sin\psi$;
- if $\rho=0$, then $H=0$ for any $(u_1,u_2)\in U$.

Since $H=0$ if and only if $|\psi|\in[\pi/2+\alpha,\pi]$ or $\rho=0$, the abnormal extremals are the following. The abnormal extremal controls are $u_1(t)=r(t)\cos\alpha$ and $u_2(t)=\pm r(t)\sin\alpha$, where $0\leq r(t)\leq 1$ and the sign $\pm$ switches every time when the angle θ increases by $\Delta\theta=\pm\pi$, except for the first and last arcs, on which the angle increment cannot be greater than π. Abnormal extremal trajectories correspond to the motion of a car along an arc of a circle of minimum possible radius. It is easy to show that if $r(t)<1$, then the trajectory is not optimal. Indeed, if $r(t)\in(0,1)$ for a positive measure time set, then one can reparametrize the trajectory with $\mathrm{d}s=r(t)\,\mathrm{d}t$. In this case, the trajectory itself remains the same, the new controls are also admissible and belong to the unit circle, but the new motion time S becomes strictly less than the original one:

$$S=\int_0^S \mathrm{d}s = \int_0^T r(t)\,\mathrm{d}t < \int_0^T \mathrm{d}t = T.$$

Next, we analyze the dynamics in the normal case $H = 1$, $|\psi| < \pi/2 + \alpha$.

The vertical part has first integrals: the Hamiltonian

$$1 = H = \begin{cases} h_1 \cos\alpha + |h_2| \sin\alpha & \text{for } \ \alpha + \dfrac{\pi}{2} > |\psi| > \alpha, \\ \sqrt{h_1^2 + h_2^2} & \text{for } \ |\psi| \le \alpha \end{cases} \tag{4.2}$$

and the Casimir function

$$E = \frac{h_1^2}{2} + \frac{h_3^2}{2};$$

i.e., E is constant along any solution of any left-invariant Hamiltonian system on SE_2.

To describe the phase portrait of the vertical part, we use the technique of convex trigonometry [11]. The polar set to U is

$$\begin{aligned} U^\circ &= \bigl\{(h_1, h_2) \in \mathbb{R}^{2*} \bigm| u_1 h_1 + u_2 h_2 \le 1 \ \forall (u_1, u_2) \in U\bigr\} \\ &= \left\{ (\underbrace{\rho\cos\psi}_{h_1}, \underbrace{\rho\sin\psi}_{h_2}) \;\middle|\; \begin{array}{ll} \rho \in [0,1] & \text{for } \ |\psi| \le \alpha, \\ h_1\cos\alpha + h_2\sin\alpha \le 1 & \text{for } \ \alpha < \psi < \alpha + \dfrac{\pi}{2}, \\ h_1\cos\alpha - h_2\sin\alpha \le 1 & \text{for } \ -\alpha - \dfrac{\pi}{2} < \psi < -\alpha \end{array} \right\}. \end{aligned}$$

The corresponding functions of convex trigonometry are

$$\cos_{U^\circ}\phi^\circ = \begin{cases} \cos\phi^\circ & \text{for } \ |\phi^\circ| \le \alpha, \\ \cos\alpha - (\phi^\circ - \alpha)\sin\alpha & \text{for } \ |\phi^\circ| > \alpha, \end{cases}$$

$$\sin_{U^\circ}\phi^\circ = \begin{cases} \sin\phi^\circ & \text{for } \ |\phi^\circ| \le \alpha, \\ (\sin\alpha + (\phi^\circ - \alpha)\cos\alpha)\operatorname{sgn}(\phi^\circ) & \text{for } \ |\phi^\circ| > \alpha. \end{cases}$$

Along the extremal trajectories we have

$$u_1 = \cos\phi, \qquad u_2 = \sin\phi, \qquad h_1 = \cos_{U^\circ}\phi^\circ, \qquad h_2 = \sin_{U^\circ}\phi^\circ.$$

Define $K(\phi^\circ) = (1/2)\cos^2_{U^\circ}\phi^\circ$. The Casimir function E can be regarded as a total energy integral (sum of potential and kinetic energy)

$$E = \frac{h_1^2}{2} + \frac{h_3^2}{2} = \frac{h_3^2}{2} + K(\phi^\circ)$$

of the conservative system with one degree of freedom (see [3])

$$\dot{\phi^\circ} = h_3, \qquad \dot{h}_3 = -K'(\phi^\circ). \tag{4.3}$$

Analyzing the phase portrait of system (4.3), we conclude that

- if $E = 0$, then $(\phi^\circ, h_3) \equiv (\pm(\alpha + \cot\alpha), 0)$ is a stable equilibrium;
- if $E \in (0, +\infty) \setminus \{1/2\}$, then the trajectory $(\phi^\circ, h_3)(t)$ is periodic;
- if $E = 1/2$, then either $(\phi^\circ, h_3) \equiv (0, 0)$ is an unstable equilibrium or $(\phi^\circ, h_3)(t)$ is a separatrix of the saddle point.

5. EXPLICIT EXPRESSION FOR THE EXTREMALS

In this section, we derive explicit formulas for the extremals. Since $u_1 > 0$, we can parametrize the trajectories of (2.1) by the arc-length of their projection to Oxy:

$$s(t) = \int_0^t \sqrt{\dot{x}^2(\tau) + \dot{y}^2(\tau)}\,\mathrm{d}\tau = \int_0^t u_1(\tau)\,\mathrm{d}\tau.$$

Rewriting (4.1) in the s-parameterization gives

$$\begin{cases} x' = \cos\theta, \\ y' = \sin\theta, \\ \theta' = u, \end{cases} \qquad \begin{cases} h_1' = -uh_3, \\ h_2' = h_3, \\ h_3' = uh_1, \end{cases} \tag{5.1}$$

where the prime stands for $\mathrm{d}/\mathrm{d}s$ and $u = u_2/u_1$.

In the previous section we showed that depending on the sign of $a = |\phi^\circ| - \alpha$, we have two different regimes. When a switches its sign, the dynamics switches from one regime to the other. Next, we consider each case separately.

Case $|\phi^\circ| > \alpha$. We have $u \equiv \pm\sin\alpha$, and the corresponding extremal trajectories are circular arcs on the plane Oxy:

$$x(s) = \frac{1}{u}\sin(us), \qquad y(s) = \frac{1}{u}(\cos(us) - 1), \qquad \theta(s) = us.$$

Case $|\phi^\circ| < \alpha$. The extremal trajectories are given by arcs of sub-Riemannian geodesics. Let $p_1 = h_1\cos\theta + h_3\sin\theta$, $p_2 = h_1\sin\theta - h_3\cos\theta$, and $p_3 = h_2$. The vertical part of the Hamiltonian system (5.1) takes the form

$$p_1'(s) = 0, \qquad p_2'(s) = 0, \qquad p_3'(s) = p_1\sin\theta(s) - p_2\cos\theta(s).$$

Note that the function $p_3(s)$ satisfies the equation

$$p_3''(s) = p_3(s), \qquad p_3'(0) = -p_2, \quad p_3(0) = p_{30},$$

which has the solution $p_3(s) = p_{30}\cosh s - p_2\sinh s$.

Let $P(s) = p_3'(s) = p_{30}\sinh s - p_2\cosh s$. Let $M = 1 + p_2^2 - p_{30}^2$. The equation for $\theta(s)$ is integrated as

$$\theta(s) = \int_0^s \frac{p_3(\sigma)}{\sqrt{1 - p_3^2(\sigma)}}\,\mathrm{d}\sigma = \arcsin\frac{P(s)}{\sqrt{M}} - \arcsin\frac{P(0)}{\sqrt{M}}.$$

Note that since $H = 1$, the following relation holds:

$$p_1x(s) + p_2y(s) = \int_0^s \sqrt{1 - p_3^2(\sigma)}\,\mathrm{d}\sigma = \int_0^s \sqrt{M - P^2(\sigma)}\,\mathrm{d}\sigma =: g(s). \tag{5.2}$$

Let us write the last integral in an explicit form:

$$g(s) = \begin{cases} -\mathbf{i}\sqrt{M}\left(E\left(\mathbf{i}(s-\alpha), \dfrac{M-1}{M}\right) - E\left(-\mathbf{i}\alpha, \dfrac{M-1}{M}\right)\right) & \text{for } M \neq 1,\ p_{30} \neq 0, \\ g_{\mathrm{c}}(s) & \text{for } M = 1,\ p_{30} \neq 0, \\ -\mathbf{i}E(\mathbf{i}s, -p_2^2) & \text{for } p_{30} = 0, \end{cases}$$

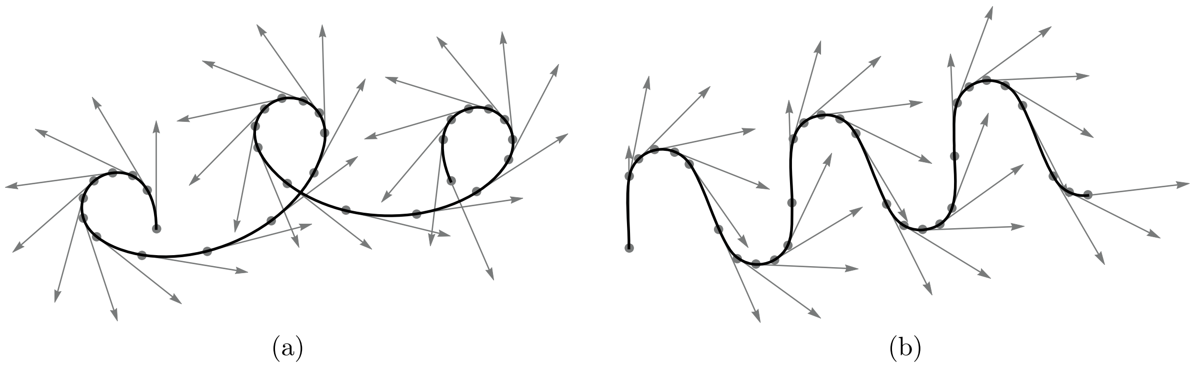

Fig. 3. Projection to the plane (x, y) of two different extremal trajectories for $\alpha = 3\pi/7$. The grey arrows indicate the orientation angle θ at the time instances $t \in \{0, 0.6, 1.2, \ldots, 18\}$: (a) for the initial data $h_1(0) = 10/17$, $h_2(0) = 3\sqrt{21}/17$, $h_3(0) = 1/2$; (b) for the initial data $h_1(0) = 15/17$, $h_2(0) = 8/17$, $h_3(0) = -12/5$.

where

$$E(\phi, m) = \int_0^\phi \sqrt{1 - m\sin^2\varphi}\,\mathrm{d}\varphi, \qquad \alpha = \operatorname{artanh}\frac{p_2}{p_{30}},$$

$$g_{\mathrm{c}}(s) = s_1\Bigl(\sqrt{1 - e^{2ss_1}p_{30}^2} - \operatorname{artanh}\sqrt{1 - e^{2s_1 s}p_{30}^2} + \operatorname{artanh} p_1 - p_1\Bigr), \qquad s_1 = -\operatorname{sgn}(p_2 p_{30}).$$

On the other hand, due to (4.1) we have $p_1 \sin\theta(s) - p_2\cos\theta(s) = P(s)$. Integrating the left- and right-hand sides of this equality, we obtain

$$p_1 y(s) - p_2 x(s) = \int_0^s P(\sigma)\,\mathrm{d}\sigma = p_3(s) - p_{30}. \tag{5.3}$$

Let $f(s) = p_3(s) - p_{30}$. Combining (5.2) with (5.3), we get

$$x(s) = \frac{1}{M}\bigl(p_1 g(s) - p_2 f(s)\bigr), \qquad y(s) = \frac{1}{M}\bigl(p_1 f(s) + p_2 g(s)\bigr).$$

Thus we obtained an explicit formula for the extremal trajectories. In Fig. 3 we show the plots of two such trajectories.

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