Lorentzian problem on 2-dimensional de Sitter space*

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Abstract

The article considers the Lorentzian optimal control problem on the two-dimensional de Sitter space. Normal and abnormal optimal trajectories are studied using the Pontryagin maximum principle. Attainable sets, spheres and distance in the Lorentzian metric are computed. Killing vector fields and isometries are described.

Keywords: Lorentzian geometry, de Sitter space, optimal control
MSC2010: 53C50, 49K15

1 Lorentzian geometry

A Lorentzian structure on a smooth manifold M is a nondegenerate quadratic form g of index 1. Let us recall some basic definitions of Lorentzian geometry [1]. A vector $v \in T_qM$, $q \in M$, is called timelike if $g(v) < 0$, spacelike if $g(v) > 0$ or $v = 0$, lightlike (or null) if $g(v) = 0$ and $v \neq 0$, and nonspacelike if $g(v) \leq 0$. A Lipschitzian curve in M is called timelike if it has timelike velocity vector a.e.; spacelike, lightlike and nonspacelike curves are defined similarly.

A time orientation X_0 is an arbitrary timelike vector field in M. A nonspacelike vector $v \in T_qM$ is future directed if $g(v, X_0(q)) < 0$, and past directed if $g(v, X_0(q)) > 0$.

A future directed timelike curve $q(t)$, $t \in [0, t_1]$, is called arclength parametrized if $g(\dot{q}(t), \dot{q}(t)) \equiv -1$. Any future directed timelike curve can be parametrized by arclength, similarly to Riemannian geometry.

The Lorentzian length of a nonspacelike curve $\gamma \in \mathrm{Lip}([0, t_1], M)$ is $l(\gamma) = \displaystyle\int_0^{t_1} |g(\dot{\gamma}, \dot{\gamma})|^{1/2} dt$. For points $q_0, q_1 \in M$ denote by $\Omega_{q_0q_1}$ the set of all future directed nonspacelike curves in M that connect q_0 to q_1. In the case $\Omega_{q_0q_1} \neq \emptyset$ define the Lorentzian distance (time separation function) from the point q_0 to the point q_1 as

$$d(q_0, q_1) = \sup\{l(\gamma) \mid \gamma \in \Omega_{q_0q_1}\}. \tag{1}$$

And if $\Omega_{q_0q_1} = \emptyset$, then by definition $d(q_0, q_1) = 0$.

A future directed nonspacelike curve γ is called a Lorentzian length maximizer if it realizes the supremum in (1) between its endpoints $\gamma(0) = q_0$, $\gamma(t_1) = q_1$.

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The causal future of a point $q_0 \in M$ is the set $J^+_{q_0}$ of points $q_1 \in M$ for which there exists a future directed nonspacelike curve γ that connects q_0 and q_1. The causal past $J^-_{q_0}$ is defined analogously in terms of past directed nonspacelike curves.

Let $q_0 \in M$, $q_1 \in J^+_{q_0}$. The search for Lorentzian length maximizers that connect q_0 with q_1 reduces to the search for future directed nonspacelike curves γ that solve the problem

$$l(\gamma) \to \max, \qquad \gamma(0) = q_0, \quad \gamma(t_1) = q_1. \tag{2}$$

A set of vector fields $X_1, \dots, X_n \in \mathrm{Vec}(M)$, $n = \dim M$, is an orthonormal frame for a Lorentzian structure g if for all $q \in M$

$$\begin{aligned} &g_q(X_1, X_1) = -1, \qquad g_q(X_i, X_i) = 1, \quad i = 2, \dots, n, \\ &g_q(X_i, X_j) = 0, \quad i \neq j. \end{aligned}$$

Assume that time orientation is defined by a timelike vector field $X \in \mathrm{Vec}(M)$ for which $g(X, X_1) < 0$ (e.g., $X = X_1$). Then the Lorentzian problem for the Lorentzian structure with the orthonormal frame $X_1, \dots, X_n$ is stated as the following optimal control problem [5]:

$$\begin{aligned} &\dot q = \sum_{i=1}^n u_i X_i(q), \qquad q \in M, \\ &u \in U = \left\{ (u_1, \dots, u_n) \in \mathbb{R}^n \mid u_1 \geq \sqrt{u_2^2 + \dots + u_n^2} \right\}, \\ &q(0) = q_0, \qquad q(t_1) = q_1, \\ &l(q(\cdot)) = \int_0^{t_1} \sqrt{u_1^2 - u_2^2 - \dots - u_n^2}\, dt \to \max. \end{aligned}$$

2 Statement of the Lorentzian problem

Consider the space $\mathbb{R}^3_1 = \{x = (x_1, x_2, x_3) \mid x_i \in \mathbb{R}\}$ endowed with the Lorentzian metric $g = -dx_1^2 + dx_2^2 + dx_3^2$, the 3D Minkowski space. The 2-dimensional de Sitter space [1–3] is the one-sheet hyperboloid

$$M = S^2_1 = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3_1 \mid -x_1^2 + x_2^2 + x_3^3 = 1\}$$

with the Lorentzian metric $\widetilde g = g|_{S^2_1}$. In this work we describe Lorentzian length maximizers, distance, and spheres on the Lorentzian space $M = S^2_1$.

In the coordinates $x_1 = \sinh\theta$, $x_2 = \cos\varphi \cosh\theta$, $x_3 = \sin\varphi\cosh\theta$ we have

$$M = \{(\theta, \varphi) \mid \theta \in \mathbb{R},\ \varphi \in \mathbb{R}/(2\pi\mathbb{Z})\} \cong \mathbb{R}_\theta \times S^1_\varphi.$$

The vector fields

$$X_1 = \frac{\partial}{\partial\theta}, \qquad X_2 = \frac{1}{\cosh\theta}\frac{\partial}{\partial\varphi}$$

form an orthonormal frame of the Lorentzian structure $\widetilde g$, i.e.,

$$\widetilde g(X_2, X_2) = -\widetilde g(X_1, X_1) = 1, \qquad \widetilde g(X_1, X_2) = 0.$$

Thus Lorentzian length maximizers are solutions to the optimal control problem

$$\dot q = u_1 X_1(q) + u_2 X_2(q), \tag{3}$$

$$q \in M, \qquad u \in U = \{(u_1, u_2) \in \mathbb{R}^2 | u_1 \geq |u_2|\}, \tag{4}$$

$$q(0) = q_0, \qquad q(t_1) = q_1, \tag{5}$$

$$l = \int_0^{t_1} \sqrt{u_1^2 - u_2^2} dt \to \max. \tag{6}$$

3 Attainable set

Let $q_0 \in M$. The attainable sets $J^+_{q_0}$, $J^-_{q_0}$ are bounded by lightlike trajectories with controls $u_1 = \pm u_2$ starting from the point $q_0 = (\theta_0, \varphi_0)$:

$$\theta(t) = \theta_0 + t,$$
$$\varphi(t) = \varphi_0 \pm \arctan(\sinh(\theta_0 + t)).$$

Thus the attainable sets are given as

$$J^+_{q_0} = \{q = (\theta, \varphi) \in M \mid \theta_0 \le \theta,\ \varphi_0 - |\arctan(\sinh(\theta))| \le \varphi \le \varphi_0 + |\arctan(\sinh(\theta))|\}, \tag{7}$$
$$J^-_{q_0} = \{q = (\theta, \varphi) \in M \mid \theta \le \theta_0,\ \varphi_0 - |\arctan(\sinh(\theta))| \le \varphi \le \varphi_0 + |\arctan(\sinh(\theta))|\}. \tag{8}$$

Boundaries of the attainable sets $J^+_{q_0}$, $J^-_{q_1}$ are shown in Fig. 1, and the boundary of the attainable set $J^+_{q_0}$ from the point $q_0 = (0, 0)$ is shown in Fig. 2, both in cylindrical coordinates (θ, φ).

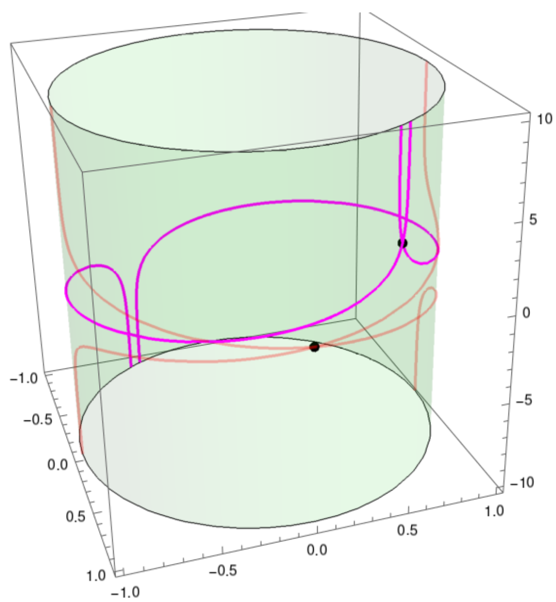

Figure 1: Attainable sets $J^+_{q_0}$, $J^-_{q_1}$

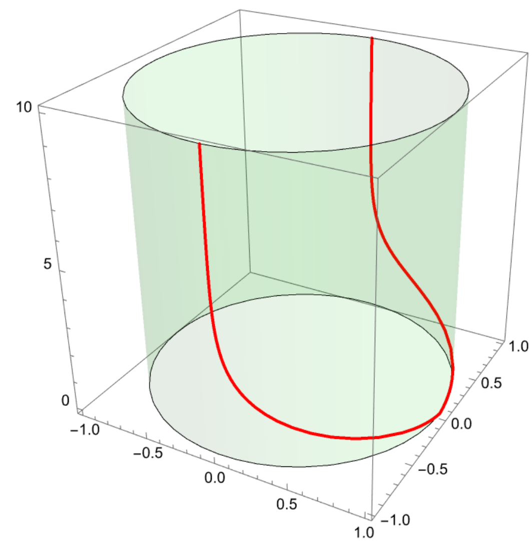

Figure 2: Attainable set $J^+_{(0,0)}$

4 Existence of optimal trajectories

Along with problem (3)–(6), consider an equivalent problem with u_1 normalized to 1:

$$\dot{q} = X_1(q) + u_2 X_2(q), \tag{9}$$
$$q \in M, \qquad |u_2| \le 1, \tag{10}$$
$$q(0) = q_0, \qquad q(t_1) = q_1, \tag{11}$$
$$l = \int_0^{t_1} \sqrt{1 - u_2^2}\, dt \to \max. \tag{12}$$

Sufficient conditions for the existence of optimal trajectories for (3)–(6) are given by the following statement.

Theorem 4.1 ([8]). *Let the following conditions for problem (9)–(12) hold:*

(1) $q_1 \in J^+_{q_0}$,

(2) *the set* $J^+_{q_0} \cap J^-_{q_1}$ *is compact,*

(3) $T(q_0, q_1) < +\infty$.

Then there is an optimal trajectory in problem (3)–(6).

Here

$$T(q_0, q_1) := \sup\{t_1 > 0 \mid \exists \text{ trajectory } q(t) \text{ of } (9), (10), \ t \in [0, t_1] : \ q(0) = q_0, \ q(t_1) = q_1\}. \tag{13}$$

Proposition 4.1. *Let $q_0 = (\theta_0, \varphi_0), q_1 = (\theta_1, \varphi_1) \in M$, and let $q_1 \in J^+_{q_0}$. Then there is an optimal trajectory in problem* (3)–(6).

Proof. First, the set $J^+_{q_0} \cap J^-_{q_1}$ is closed, see (7), (8). Second, for any point $q = (\theta, \varphi) \in J^+_{q_0} \cap J^-_{q_1}$ we have $\theta_0 \leq \theta \leq \theta_1$, so this set is bounded. Consequently, this set is compact.

Moreover, since $\dot\theta = 1$ for system (9), (10), then $T(q_0, q_1) = \theta_1 - \theta_0 < +\infty$.

All hypotheses of Th. 4.1 hold, thus an optimal trajectory exists. □

We assume in Sections 5–8 that $q_0 = (0, 0)$, and return to the general case $q_0 \in M$ in Section 11.

5 Pontryagin maximum principle

We apply the Pontryagin maximum principle [4–6] to problem (3)–(6).

The Hamiltonian of Pontryagin maximum principle has the form

$$h^\nu_u(\lambda) = u_1 h_1 + u_2 h_2 - \nu\sqrt{u_1^2 - u_2^2}, \quad \lambda \in T^*M, \quad u \in U, \quad \nu \in \{-1, 0\},$$

where $h_i(\lambda) = \langle \lambda, X_i \rangle$, $i = 1, 2$, so that $h_1 = \lambda_1$ and $h_2 = \lambda_2 / \cosh\theta$, where $\lambda = (\lambda_1, \lambda_2)$ in canonical coordinates on the cotangent bundle T^*M.

5.1 Abnormal extremals

Let $\nu = 0$. Abnormal extremals λ_t and controls $u(t)$ have the following form:

1. if $|h_2| < -h_1$, then $u_1 = u_2 = 0$ and the trajectory is trivial: $q(t) \equiv q_0$,

2. if $|h_2| = -h_1$, then $u_2 = \operatorname{sgn}(h_2) u_1$. Lightlike abnormal trajectories

$$\theta(t) = t, \qquad \varphi(t) = \pm \arctan(\sinh t)$$

form the boundary of the attainable set $J^+_{q_0}$.

5.2 Normal extremals

Let $\nu = -1$. Then the maximality condition of the Pontryagin maximum principle has the form

$$h^{-1}_u(\lambda) = u_1 h_1 + u_2 h_2 + \sqrt{u_1^2 - u_2^2} \to \max_{u \in U}.$$

It is easy to see that for $h_1 \geq -|h_2|$ this maximum is attained only for the trivial control $u = 0$. Consider the remaining case $h_1 < -|h_2|$. If $u_1 = |u_2| = \pm u_2$ then $h^{-1}_u = u_1(h_1 \pm h_2) \leq 0$ attains maximum in the trivial case $u_1 = 0$. Thus we consider further the case $u \in \operatorname{int} U = \{(u_1, u_2) \in \mathbb{R}^2 \mid u_1 > |u_2|\}$ and introduce hyperbolic coordinates k, ψ in the set $\operatorname{int} U$:

$$u_1 = k\cosh\psi, \quad u_2 = k\sinh\psi, \qquad \psi \in \mathbb{R}, \quad k > 0.$$

Introduce also hyperbolic coordinates on the set $\{h_1 < -|h_2|\}$:

$$h_1 = -\rho\cosh\alpha, \quad h_2 = \rho\sinh\alpha, \qquad \alpha \in \mathbb{R}, \quad \rho > 0.$$

Then it is easy to see that $h^{-1}_u = k(1 - \rho\cosh(\psi - \alpha))$ takes maximum for $\psi = \alpha$ in the case $\rho = 1$ only. Passing to arclength parametrization ($k \equiv 1$) we get the normal extremals and controls

$$h_2 = u_2 = \sinh\psi, \qquad h_1 = -u_1 = -\cosh\psi.$$

Then the Hamiltonian system of the Pontryagin maximum principle reads

$$\dot{\psi} = -\tanh\theta \sinh\psi, \tag{14}$$

$$\dot{\theta} = \cosh\psi, \tag{15}$$

$$\dot{\varphi} = \frac{\sinh\psi}{\cosh\theta}. \tag{16}$$

Equations (14) and (15) can be easily solved by separation of variables and then equation (16) can be solved by direct integration. Equations (14) and (15) have the first integral $C = \sinh\psi \cosh\theta \in \mathbb{R}$. Then

$$\varphi = \arctan\left(C \tanh t\right), \tag{17}$$

$$\theta = \frac{1}{2}\operatorname{arccosh}\left((1+C^2)\cosh 2t - C^2\right) = \operatorname{arcsinh}\left(\sqrt{C^2+1}\sinh t\right), \tag{18}$$

$$\psi = \operatorname{arcsinh} \frac{C\sqrt{2}}{\sqrt{(1+C^2)\cosh 2t + (1-C^2)}}. \tag{19}$$

Remark. Formulas (17), (18) imply that extremal trajectories $(x_1, x_2, x_3)(t)$ are intersections of the hyperboloid S^1_2 with planes in $\mathbb{R}^3_1$ passing through the origin [2]. In detail:

- abnormal trajectories are intersections of the hyperboloid S^1_2 with the planes $x_1 = \pm x_3$ in $\mathbb{R}^3_1$,
- normal trajectories are intersections of the hyperboloid S^1_2 with planes $x_1 = \pm x_3\sqrt{\frac{C^2}{1+C^2}}$ in $\mathbb{R}^3_1$.

Normal extremal trajectories in cylindrical coordinates φ, θ with respect to the abnormal trajectory (red) are shown in Fig. 3.

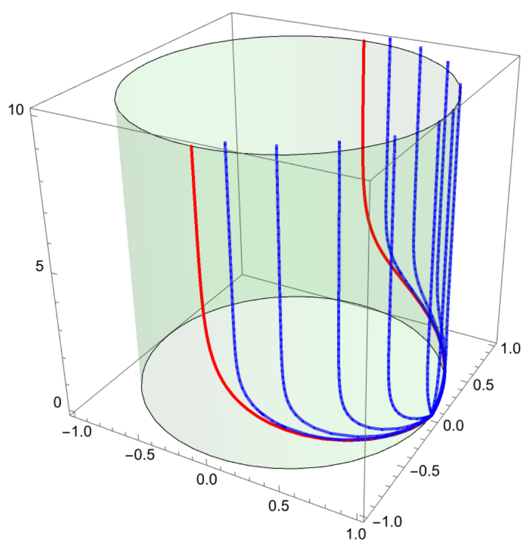

Figure 3: Normal extremal trajectories $(\varphi, \theta)(t)$

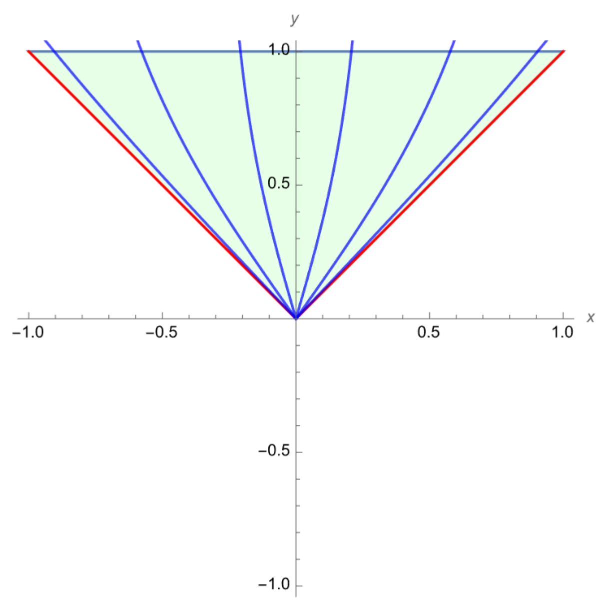

Figure 4: Normal extremal trajectories $(X, Y)(t)$

In the coordinates

$$X = \tan\varphi, \qquad Y = \sinh\theta,$$

Eqs. (17), (18) read

$$X = C\tanh t, \qquad Y = \sqrt{1+C^2}\sinh t. \tag{20}$$

Normal optimal trajectories in coordinates X, Y with respect to the abnormal trajectory (red) are shown in Fig. 4.

6 Optimality of extremal trajectories

We prove that all extremal trajectories are optimal by the following theorem.

Theorem 6.1 (Hadamard [7,9]). *Let $F : X \to Y$ be a smooth mapping between smooth manifolds for which the following conditions hold:*

(1) $\dim X = \dim Y$,

(2) *X, Y are connected, and Y is simply connected,*

(3) *F is nondegenerate,*

(4) *F is proper (preimage of a compact set is compact).*

Then F is a diffeomorphism, thus a bijection.

Proposition 6.1. *The mapping*

$$\begin{aligned} &\mathrm{Exp} : N \to G, \qquad (C,t) \mapsto (X,Y), \\ &N = \{(C,t) \mid C \in \mathbb{R},\ t \in \mathbb{R}_+\}, \\ &G = \{(X,Y) \in \mathbb{R}^2 \mid Y \geq |X|\} \end{aligned} \tag{21}$$

given by (20) *is a diffeomorphism.*

Proof. Let us check the hypotheses of Theorem 6.1 for mapping (21).

The following properties are obvious:

- $\mathrm{Exp}(N) \subset G$,
- N and G are diffeomorphic to $\mathbb{R} \times \mathbb{R}_+$, thus connected and simply connected,
- $\mathrm{Exp}|_N$ is nondegenerate:

$$\det \frac{\partial(X,Y)}{\partial(C,t)} = \det \begin{pmatrix} \tanh t & \frac{C}{\cosh^2 t} \\ \frac{C \sinh t}{\sqrt{1+C^2}} & \sqrt{1+C^2}\cosh t \end{pmatrix} = \frac{\sinh t(\cosh^2 t + C^2 \sinh^2 t)}{\cosh^2 t\sqrt{1+C^2}} > 0 \text{ for } t > 0.$$

Let us check that $\mathrm{Exp} : N \to G$ is proper. Consider the following increasing exhausting sequences of compacts in N and G:

$$N_n = \left\{\nu \in N | \frac{1}{n} \leq t \leq n, |C| \leq n\right\}, \qquad N_n \subset N_{n+1} \subset N = \cup_{n=1}^{\infty} N_n,$$
$$G_n = \{g \in G | Y \geq |X| + \frac{1}{n}, Y \leq n\}, \qquad G_n \subset G_{n+1} \subset G = \cup_{n=1}^{\infty} G_n.$$

Then it is obvious that

$$\nu = (C,t) \to \partial N \iff \left[\begin{array}{l} t \to 0, \\ t \to +\infty, \\ |C| \to +\infty, \end{array}\right. \qquad g = (X,Y) \to \partial N \iff \left[\begin{array}{l} X^2 + Y^2 \to +\infty, \\ \dfrac{Y}{X} \to 1+, \\ X^2 + Y^2 \to 0. \end{array}\right.$$

Let $\nu = (C,t) \to \partial N$, we prove that $g = (X,Y) \to \partial G$:

1. If $t \to +\infty$, then $Y \to +\infty$ for any C by (20).
2. If $t \to 0+$ and $|C| \to +\infty$, then $\frac{Y}{X} = \sqrt{1 + \frac{1}{C^2}}\cosh t \to 1+$.
3. If $t \to 0+$ and $|C|$ is bounded, then $X^2 + Y^2 \to 0+$ by (20).

4. If t is bounded and $|C| \to +\infty$, then $X^2 + Y^2 \to +\infty$ by (20).

So the mapping $\mathrm{Exp} : N \to G$ is proper.

Summing up, by Hadamard's theorem 6.1, the mapping $\mathrm{Exp} : N \to G$ is a diffeomorphism. $\square$

Theorem 6.2. *All extremal trajectories (both normal and abnormal) in problem (3)–(6) are optimal.*

Proof. In the normal case optimal trajectories exist and trajectories corresponding to the Pontryagin maximum principle are unique (by Propos. 6.1), thus all normal extremal trajectories are optimal. They connect q_0 to interior points of $J^+_{q_0}$.

Abnormal trajectories are lightlike thus they have zero Lorentzian length, so they are optimal trajectories connecting q_0 to boundary points of $J^+_{q_0}$. Notice that unlike normal trajectories the abnormal ones are not unique due to different possible parametrizations. $\square$

7 Optimal synthesis

Let $q_1 = (\theta_1, \varphi_1) \in J^+_{q_0}$. We present the optimal trajectory $q(t)$, $t \in [0, t_1]$, of problem (3)–(6) that connects q_0 and q_1.

If $q_1 \in \partial J^+_{q_0}$, then

$$q(t) = (\theta, \varphi) = (t, \operatorname{sgn} \varphi_1 \arctan(\sinh t)), \qquad t_1 = \theta_1.$$

If $q_1 \in \operatorname{int} J^+_{q_0}$, then $q(t) = \mathrm{Exp}(C, t)$, $t \in [0, t_1]$, where

$$C = \frac{\cosh \theta_1 \tan \varphi_1}{\sqrt{\sinh^2 \theta_1 - \tan^2 \varphi_1}} = \sqrt{\frac{1 + Y_1^2}{Y_1^2 - X_1^2}} X_1,$$

$$t_1 = \operatorname{arcsinh} \sqrt{\frac{\sinh^2 \theta_1 - \tan^2 \varphi_1}{1 + \tan^2 \varphi_1}} = \operatorname{arccosh}(\cosh \theta_1 \cos \varphi_1) = \operatorname{arccosh} \sqrt{\frac{Y_1^2 + 1}{X_1^2 + 1}},$$

$$X_1 = \tan \varphi_1, \qquad Y_1 = \sinh \theta_1.$$

Examples of optimal trajectories are shown in Fig. 5.

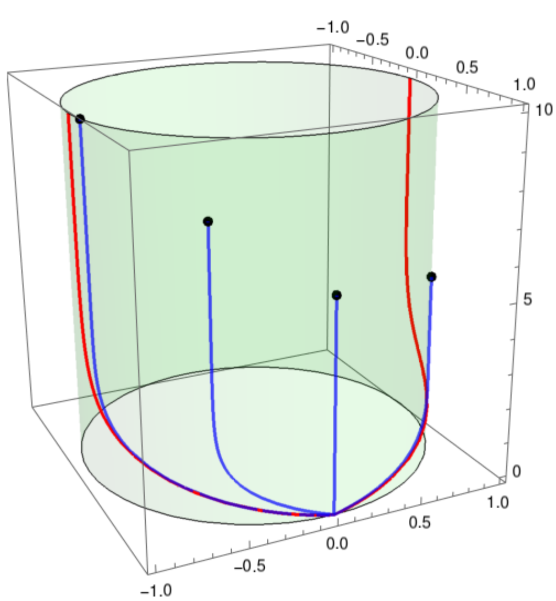

Figure 5: Examples of optimal trajectories

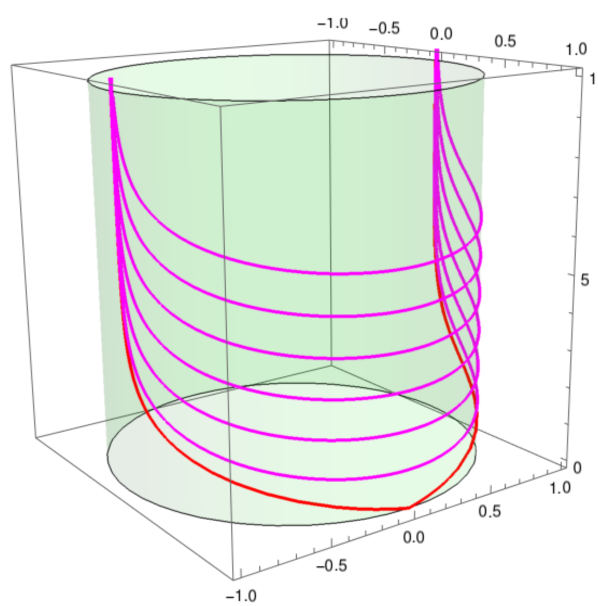

Figure 6: Lorentzian spheres

8 Lorentzian distance and spheres

Lorentzian distance is expressed as follows:

$$d(q_0, q) = \begin{cases} \operatorname{arccosh}(\cosh\theta\cos\varphi) = \operatorname{arccosh}\sqrt{\frac{Y^2+1}{X^2+1}}, & q \in J^+_{q_0}, \\ 0, & q \in M \setminus J^+_{q_0}. \end{cases}$$

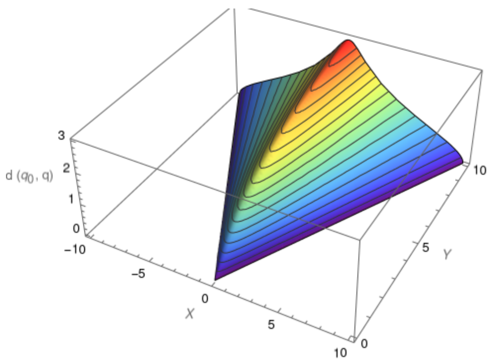

Figure 7: Lorentzian distance in coordinates (X, Y) with isolines (spheres)

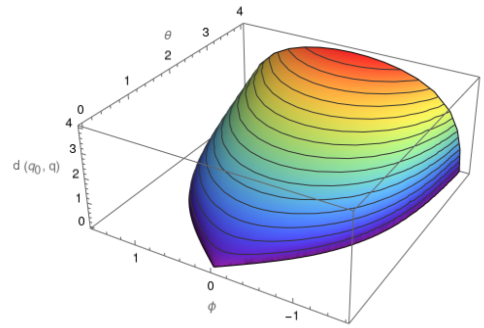

Figure 8: Lorentzian distance in coordinates (θ, φ) with isolines (spheres)

Proposition 8.1. *The Lorentzian distance $q \mapsto d(q_0, q)$ is real analytic on $M \setminus \partial J^+_{q_0}$ and Hölder continuous with exponent $\frac{1}{2}$ at the points of $\partial J^+_{q_0} \setminus q_0$.*

Proof. Lorentzian distance $q \mapsto d(q_0, q)$ is real analytic on $M \setminus \partial J^+_{q_0}$ as composition of real analytic functions: $\cosh, \cos$ and $\operatorname{arccosh} x$, $x > 1$ since $\partial J^+_{q_0}$ is excluded.

Let $q \in M \setminus \partial J^+_{q_0}$, $q' \in \partial J^+_{q_0}$. We can assume that $q \in M \setminus J^+_{q_0}$. Then Hölder continuity with exponent $\frac{1}{2}$ at the points of $\partial J^+_{q_0} \setminus q_0$ reads as follows:

$$d(q_0, q) - d(q_0, q') = d(q_0, q) = \operatorname{arccosh}\sqrt{\frac{Y^2+1}{X^2+1}} \leq C\sqrt{||q - q'||} = C\sqrt{\frac{Y - X}{\sqrt{2}}}$$

for some $C > 0$. Then the Taylor expansion of the function cosh yields that

$$d(q_0, q) - d(q_0, q') = \leq \sqrt{2\sqrt{2}}\sqrt{||q - q'||}.$$

□

Plots of the function $q \mapsto d(q_0, q)$ in coordinates (X, Y) and (θ, φ) are given in Figs. 7 and 8.

Lorentzian spheres $S(r) = \{q \in M \mid d(q_0, q) = r\}$, $r > 0$, are shown in Fig. 6.

Proposition 8.2. *Lorentzian spheres $S(r)$, $r > 0$, are real analytic noncompact curves tending to asymptotes $Y = |X| \cosh r$ at infinity. They are graphs of the functions*

$$\theta = \operatorname{arccosh}\frac{\cosh r}{\cos\varphi}, \qquad \varphi \in (-\pi/2, \pi/2). \tag{22}$$

Proof. Let us find asymptote $aX + b$ for a sphere of radius $r > 0$:

$$a = \frac{\sqrt{(X^2+1)\cosh r - 1}}{X} \to \pm\cosh r, \qquad X \to \pm\infty, \tag{23}$$

$$b = \sqrt{(X^2+1)\cosh r - 1} - aX = \frac{(X^2+1)\cosh r - X^2\cosh r}{\sqrt{(X^2+1)\cosh r - 1} + X\cosh r} \to 0, \qquad X \to \pm\infty, \tag{24}$$

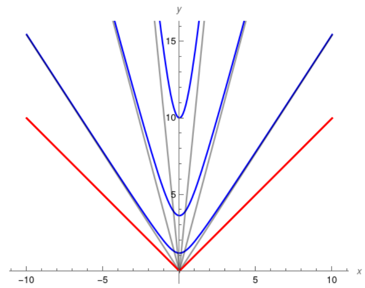

Figure 9: Lorentzian spheres with asymptotes in coordinates (X, Y)

thus

$$Y = |X| \cosh r + o(1), \qquad X \to \infty. \tag{25}$$

Note: $\partial J^+_{q_0}$ is a zero-radius sphere:

$$Y = |X| \cosh 0 = |X|. \tag{26}$$

□

9 Optimal trajectories in $\mathbb{R}^3_1$

Optimal trajectories represent the intersection of a single-strip hyperboloid with a plane passing through the origin:

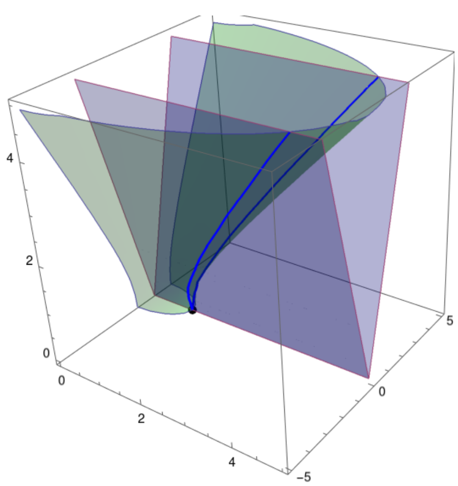

Figure 10: Normal trajectories: intersection with plane $x_1 = \pm x_3 \sqrt{\frac{C^2}{1+C^2}}$

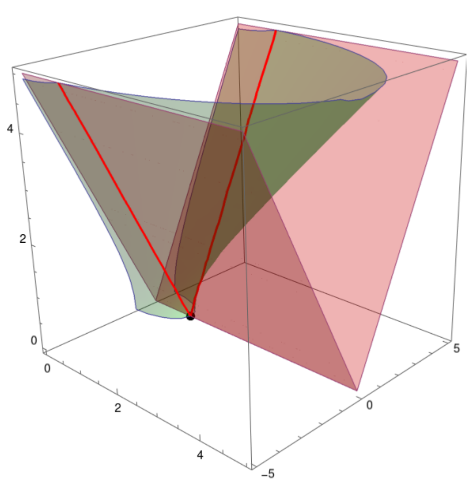

Figure 11: Abnormal trajectories: intersection with plane $x_1 = \pm x_3$

10 Killing vector fields

10.1 Definitions and general facts

We recall some necessary facts of Lorentzian (in fact, pseudo-Riemannian geometry) [3].

A vector field X on a Lorentzian manifold (M, g) is called a Killing vector field if $L_X g = 0$.

Proposition 10.1 ([3], Propos. 23)**.** *A vector field X is Killing iff the mappings ψ_t of its local flow satisfy $\psi_t^* g = g$, where $\psi_t : M \mapsto M$ is the shift of M along X by time t.*

Corollary 10.1. *A vector field X is Killing iff $d(q_1, q_2) = d(\psi_t(q_1), \psi_t(q_2))$ for all $q_1, q_2 \in M$ and all t for which the right-hand side is defined.*

Proposition 10.2 ([3], Propos. 25)**.** *A vector field X is Killing iff*

$$Xg(V, W) = g([X, V], W) + g(V, [X, W]), \qquad V, W \in \mathrm{Vec}(M). \tag{27}$$

Denote by $i(M)$ the set of Killing vector fields on a Lorentzian manifold M. The set $i(M)$ is a Lie algebra over $\mathbb{R}$ w.r.t. Lie bracket of vector fields.

Lemma 10.1 ([3], Lemma 28)**.** *The Lie algebra $i(M)$ on a connected Lorentzian manifold M, $\dim M = n$, has dimension at most $\frac{n(n+1)}{2}$.*

Remark. Let M be a connected Lorentzian manifold of dimension n. Then $\dim i(M) = \frac{n(n+1)}{2}$ iff M has constant curvature (Exercises 14, 15 *[3]*).

Denote by $I(M)$ the set of all isometries of a Lorentzian manifold M.

Theorem 10.1 ([3], Theorem 32)**.** *$I(M)$ is a Lie group.*

Denote by $ci(M)$ the set of all complete Killing vector fields on M.

Proposition 10.3 ([3], Propos. 33)**.** (1) *$ci(M)$ is a Lie subalgebra of $i(M)$.*

(2) *There is a Lie anti-isomorphism between the Lie algebra of the Lie group $I(M)$ and the Lie algebra $ci(M)$.*

Denote by $I_0(M)$ the connected component of the identity in the Lie group $I(M)$.

10.2 Killing vector fields and isometries of 2D de Sitter space

Let us compute the Lie algebra of Killing vector fields for the Lorentzian structure on the 2D de Sitter space.

Let $Y \in \mathrm{Vec}(M)$ be a Killing vector field. We decompose $Y = a_1 X_1 + a_2 X_2$ with unknown functions $a_i \in C^\infty(M)$, write down Eq. (27) with $X = Y$ and V, W equal to X_1, X_2, and derive the following differential equations for a_i:

$$\begin{aligned}
\frac{\partial a_1}{\partial \varphi} &= -a_2 \sinh\theta + \frac{\partial a_2}{\partial \theta}\cosh\theta, \\
\frac{\partial a_1}{\partial \theta} &= 0, \\
\frac{\partial a_2}{\partial \varphi} &= -a_1 \sinh\theta.
\end{aligned}$$

This system of PDEs gives the following independent Killing vector fields:

$$\begin{aligned}
Y_1 &= \sin\varphi X_1 + \cos\varphi \sinh\theta X_2 = \sin\varphi \frac{\partial}{\partial\theta} + \cos\varphi \tanh\theta \frac{\partial}{\partial\varphi}, \\
Y_2 &= \cos\varphi X_1 - \sin\varphi \sinh\theta X_2 = \cos\varphi \frac{\partial}{\partial\theta} - \sin\varphi \tanh\theta \frac{\partial}{\partial\varphi}, \\
Y_3 &= \cosh\theta X_2 = \frac{\partial}{\partial\varphi}.
\end{aligned}$$

The Lie brackets in the Lie algebra $i(M) = \mathrm{span}(Y_1, Y_2, Y_3)$ are the following ones:

$$[Y_3, Y_1] = Y_2, \qquad [Y_1, Y_2] = -Y_3, \qquad [Y_2, Y_3] = Y_1,$$

thus $i(M) \cong \mathfrak{sl}(2)$ [10].

10.3 Phase flows of the Killing fields Y_1, Y_2, Y_3

10.3.1 Phase flow of Y_1

The system of ODEs determined by the vector field Y_1

$$\dot{\theta} = \sin\varphi, \qquad \dot{\varphi} = \cos\varphi \tanh\theta \tag{28}$$

has the first integral $C = \cos\varphi\cosh\theta \in \mathbb{R}$. Let $s_1 = \operatorname{sgn}\cos\varphi_0$, $s_2 = \operatorname{sgn}\sin\varphi_0$, $s_3 = \operatorname{sgn}\theta_0$, $n = [(\varphi_0 + \pi/2)/(2\pi)]$. Then system (28) has the following solutions with the initial conditions $\theta(0) = \theta_0$, $\varphi(0) = \varphi_0$ depending on the value of C.

1) If $C = 0$ then $\varphi(t) \equiv \varphi_0$, $\theta(t) = \theta_0 + s_2 t$.

2) Let $C^2 = 1$.

   2.1) If $\sin\varphi_0 = 0$ then $\varphi(t) \equiv \varphi_0$, $\theta(t) \equiv \theta_0$.

   2.2) If $\sin\varphi_0 \neq 0$ then

$$\theta(t) = \operatorname{arsinh}(\sinh\theta_0 \exp(s_2 s_3 t)),$$
$$\varphi(t) = \begin{cases} s_2 s_3 \arcsin(\tanh\theta(t)) + 2\pi n & \text{if } s_1 = 1, \\ \pi - s_2 s_3 \arcsin(\tanh\theta(t)) + 2\pi n & \text{if } s_1 = -1. \end{cases} \tag{29}$$

3) If $C^2 \in (0, 1)$ then

$$\theta(t) = \operatorname{arsinh}(\sqrt{1 - C^2}\sinh\tau), \qquad \tau = s_2 t + \operatorname{arsinh}(\sinh\theta_0/\sqrt{1 - C^2}),$$
$$\varphi(t) = \begin{cases} s_2 \arcsin\sqrt{1 - C^2/\cosh^2\theta(t)} + 2\pi n & \text{if } s_1 = 1, \\ \pi - s_2 \arcsin\sqrt{1 - C^2/\cosh^2\theta(t)} + 2\pi n & \text{if } s_1 = -1. \end{cases} \tag{30}$$

4) If $C^2 > 1$ then

$$\varphi(t) = \begin{cases} \arcsin x + 2\pi n & \text{if } s_1 = 1, \\ \pi - \arcsin x + 2\pi n & \text{if } s_1 = -1, \end{cases} \tag{31}$$
$$\theta(t) = s_3 \operatorname{arcosh}(C/\cos\varphi(t)),$$

   where $x = (|s| - C^2 + 1)/(2\sqrt{s})$, $s = (e^\tau s_+ + s_-)/(1 + e^\tau)$, $s_\pm = C^2 + 1 \pm 2|C|$, $\tau = s_1 s_3 (s_+ - s_-)t/(2C) + \ln((s_0 - s_-)/(s_+ - s_0))$, $s_0 = r^2$, $r = \sqrt{x_0^2 + C^2 - 1} + x_0$, $x_0 = \sin\varphi_0$.

Formulas (29)–(31) parametrize explicitly the phase flow $e^{tY_1} : (\theta_0, \varphi_0) \mapsto (\theta(t), \varphi(t))$, $t \in \mathbb{R}$. Notice that in fact these formulas parametrize this flow on $\mathbb{R}^2_{\theta,\varphi}$. In order to get the flow on $M = \mathbb{R}_\theta \times S^1_\varphi$, the variable φ should be understood modulo 2π.

10.3.2 Phase flow of Y_2

The system of ODEs determined by the vector field Y_2

$$\dot{\widetilde{\theta}} = \cos\widetilde{\varphi}, \qquad \dot{\widetilde{\varphi}} = -\sin\widetilde{\varphi}\tanh\widetilde{\theta}$$

is obtained from system (28) by the change of variables $\Phi : (\theta, \varphi) \to (\widetilde{\theta}, \widetilde{\varphi}) = (\theta, \varphi - \frac{\pi}{2})$, thus it has the flow $\exp(Y_2 t) = \Phi \circ \operatorname{Exp}(Y_1 t)$.

10.3.3 Phase flow of Y_3

We obviously have $\exp(Y_3 t) : (\theta_0, \varphi_0) \mapsto (\theta_0, \varphi_0 + t)$.

Remark. Since the vector fields Y_i, $i = 1, 2, 3$, are complete, then $ci(M) = i(M) = \operatorname{span}(Y_1, Y_2, Y_3)$.

By Proposition 10.1, all flows $\exp(Y_i t)$, $i = 1, 2, 3$, are isometries of M, thus $I_0(M)$ is the Lie group generated by these flows.

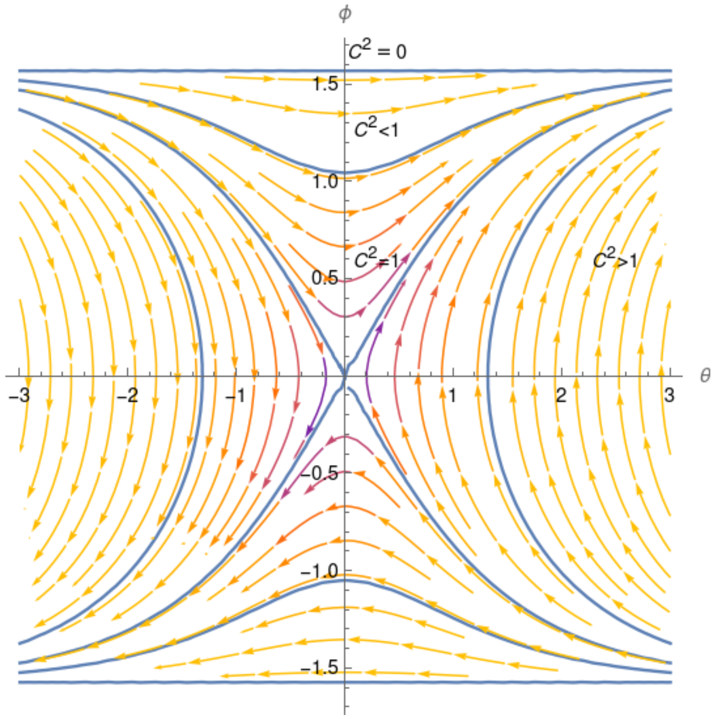

Figure 12: Phase portrait of the vector field Y_1 for $\varphi \in [-\pi/2, \pi/2]$

11 General Lorentzian problem on the de Sitter plane

In this section we consider problem (3)–(6) for general points $q_0, q_1 \in M$ by applying the group of isometries $I(M)$ to the special case $q_0 = (0,0)$ considered in Sections 5–8.

Let $q = (\theta, \varphi) \in M$. Define the mapping $F_q = \exp(\theta Y_3) \circ \exp(\varphi Y_2) \in I_0(M)$.

Lemma 11.1. (1) *We have* $F_q(0,0) = q$ *for any* $q \in M$.

(2) *The Lie group* $I_0(M)$ *acts transitively on* M.

Proof. (1) follows from the explicit form of the flows of the vector fields Y_i described in Subsec. 10.3.

(2) follows from item (1). □

Theorem 11.1. (1) *Let* $q_0, q_1 \in M$ *be such that* $q_1 \in J^+_{q_0}$. *Then there exists a Lorentzian length maximizer* $q(t)$ *connecting* q_0 *to* q_1*, and it is given as* $q(t) = F_{q_0} \circ \gamma(t)$*, where* $\gamma(t)$ *is the Lorentzian length maximizer connecting the point* $(0,0)$ *to the point* $F^{-1}_{q_0}(q_1)$.

(2) *For any points* $q_0, q_1 \in M$ *we have* $d(q_0, q_1) = d((0,0), F^{-1}_{q_0}(q_1))$.

Proof. (1) If $\delta(t)$ is a Lorentzian length maximizer and $g \in I(M)$ then $g \circ \delta(t)$ is a Lorentzian length maximizer by Proposition 10.1.

(2) follows by Corollary 10.1. □

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