Lorentzian distance on the Lobachevsky plane*

Yu.L. Sachkov
Program Systems Institute
Russian Academy of Sciences
Pereslavl-Zalessky, Russian Federation

RUDN University
6 Miklukho-Maklaya St, Moscow, 117198, Russian Federation

yusachkov@gmail.com

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Abstract

Left-invariant Lorentzian structures on the 2D solvable non-Abelian Lie group are studied. Sectional curvature, attainable sets, Lorentzian length maximizers, distance, spheres, and infinitesimal isometries are described.

Contents

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1 Introduction

Lorentzian geometry is the mathematical foundation of the theory of General Relativity [4–6]. It differs from the Riemannian one in that here information can only propagate along curves with velocity vectors from some sharp cone. Here, the natural problem is to find the curves that maximize the length-type functional along admissible curves. Therefore, an important problem is to describe the Lorentzian length maximizers for all pairs of points where the second point is reachable from the first one along an admissible curve. As far as we know, this problem has been fully investigated only in the simplest case of a left-invariant Lorentzian structure in $\mathbb{R}^n$, for the Minkowski space $\mathbb{R}^n_1$ [4].

This paper presents a description of Lorentzian length maximizers, distances and spheres for the next natural case — for left-invariant Lorentzian structures on a unique connected simply connected non-Abelian two-dimensional Lie group. These results are obtained by methods of geometric control theory [1, 2]. Curiously, in these problems, the Lorentzian length maximizers do not exist for some reachable pairs of points, and the Lorentzian distance can be infinite at some points. In these problems, all extremal trajectories (satisfying the Pontryagin maximum principle) are optimal, that is, there are neither conjugate points nor cut points. Optimal trajectories are parametrized by elementary functions, as are spheres and distances.

This work has the following structure. In Sec. 2 we recall necessary basic definitions of Lorentzian geometry. In Sec. 3 we describe the group of proper affine mappings of the real line $\mathrm{Aff}_+(\mathbb{R})$ which bears the left-invariant Lorentzian problems stated in Sec. 4. We show in Sec. 5 that these problems have constant curvature K, thus are locally isometric to model Lorentzian spaces of constant curvature (2D Minkowski space for $K = 0$, de Sitter space for $K > 0$, anti-de Sitter space for $K < 0$). In Sec. 6 we describe positive and negative time attainable sets (causal futures and pasts) of the corresponding control systems. Section 7 is devoted to the study of existence of Lorentzian length maximizers. In Sec. 8 we apply the Pontryagin maximum principle to the problems studied and parametrize geodesics. In Sec. 9 we prove that in fact all geodesics are optimal, and construct explicitly optimal synthesis. On the basis of these results in Sec. 10 we describe Lorentzian distance and spheres. In Sec. 11 we describe Lie algebras of infinitesimal isometries (Killing vector fields) and the connected component of identity of the Lie groups of isometries for the problems considered. Moreover, in the case $K = 0$ we construct explicitly an isometric embedding of $\mathrm{Aff}_+(\mathbb{R})$ into a half-plane of the 2D Minkowski space. Finally, in Sec. 12 we specialize the results obtained to three model problems P_1, P_2, P_3.

2 Lorentzian geometry

Let us recall some basic definitions of Lorentzian geometry [4, 5]. A Lorentzian metric on a smooth manifold M is a nondegenerate quadratic form g of index 1. A vector $v \in T_qM$, $q \in M$, is called timelike if $g(v) < 0$, spacelike if $g(v) > 0$ or $v = 0$, lightlike (or null) if $g(v) = 0$ and $v \neq 0$, and nonspacelike if $g(v) \leq 0$. A Lipschitzian curve in M is called timelike if it has timelike velocity vector a.e.; spacelike, lightlike and nonspacelike curves are defined similarly.

A time orientation X_0 is an arbitrary timelike vector field in M. A nonspacelike vector $v \in T_qM$ is future directed if $g(v, X_0(q)) < 0$, and past directed if $g(v, X_0(q)) > 0$.

A future directed timelike curve $q(t)$, $t \in [0, t_1]$, is called arclength parametrized if $g(\dot{q}(t), \dot{q}(t)) \equiv -1$. Any future directed timelike curve can be parametrized by arclength, similarly to Riemannian geometry.

The Lorentzian length of a nonspacelike curve $\gamma \in \mathrm{Lip}([0, t_1], M)$ is $l(\gamma) = \displaystyle\int_0^{t_1} |g(\dot{\gamma}, \dot{\gamma})|^{1/2} dt$. For points $q_0, q_1 \in M$ denote by $\Omega_{q_0q_1}$ the set of all future directed nonspacelike curves in M that connect q_0 to q_1. In the case $\Omega_{q_0q_1} \neq \emptyset$ define the Lorentzian distance (time separation function) from the point q_0 to the point q_1 as

$$d(q_0, q_1) = \sup\{l(\gamma) \mid \gamma \in \Omega_{q_0q_1}\}. \tag{2.1}$$

And if $\Omega_{q_0q_1} = \emptyset$, then by definition $d(q_0, q_1) = 0$. A future directed nonspacelike curve γ is called a Lorentzian length maximizer if it realizes the supremum in (2.1) between its endpoints $\gamma(0) = q_0$, $\gamma(t_1) = q_1$.

The causal future of a point $q_0 \in M$ is the set $J^+(q_0)$ of points $q_1 \in M$ for which there exists a future directed nonspacelike curve γ that connects q_0 and q_1. The causal past $J^-(q_0)$ is defined analogously in terms of past directed nonspacelike curves. The chronological future $I^+(q_0)$ and chronological past $I^-(q_0)$ of a point $q_0 \in M$ are defined similarly via future directed and past directed timelike curves γ.

Let $q_0 \in M$, $q_1 \in J^+(q_0)$. The search for Lorentzian length maximizers that connect q_0 with q_1 reduces to the search for future directed nonspacelike curves γ that solve the problem

$$l(\gamma) \to \max, \qquad \gamma(0) = q_0, \quad \gamma(t_1) = q_1. \tag{2.2}$$

A set of vector fields $X_1, \dots, X_n \in \mathrm{Vec}(M)$, $n = \dim M$, is an orthonormal frame for a Lorentzian structure g if for all $q \in M$

$$\begin{aligned} &g_q(X_1, X_1) = -1, \qquad g_q(X_i, X_i) = 1, \quad i = 2, \dots, n, \\ &g_q(X_i, X_j) = 0, \quad i \neq j. \end{aligned}$$

Assume that time orientation is defined by a timelike vector field $X \in \mathrm{Vec}(M)$ for which $g(X, X_1) < 0$ (e.g., $X = X_1$). Then the Lorentzian problem for the Lorentzian structure with the orthonormal frame $X_1, \dots, X_n$ is stated as the following optimal control problem:

$$\begin{aligned} &\dot{q} = \sum_{i=1}^n u_i X_i(q), \qquad q \in M, \\ &u \in U = \left\{(u_1, \dots, u_n) \in \mathbb{R}^n \mid u_1 \geq \sqrt{u_2^2 + \dots + u_n^2}\right\}, \\ &q(0) = q_0, \qquad q(t_1) = q_1, \\ &l(q(\cdot)) = \int_0^{t_1} \sqrt{u_1^2 - u_2^2 - \dots - u_n^2}\, dt \to \max. \end{aligned}$$

Remark 1. *The Lorentzian length is preserved under monotone Lipschitzian time reparametrizations $t(s)$, $s \in [0, s_1]$. Thus if $q(t)$, $t \in [0, t_1]$, is a Lorentzian length maximizer, then so is any its reparametrization $q(t(s))$, $s \in [0, s_1]$.*

In this paper we choose primarily the following parametrization of trajectories: the arclength parametrization ($u_1^2 - u_2^2 - \dots - u_n^2 \equiv 1$) for timelike trajectories, and the parametrization with $u_1(t) \equiv 1$ for future directed lightlike trajectories. Another reasonable choice is to set $u_1(t) \equiv 1$ for all future directed nonspacelike trajectories.

Remark 2. *In Lorentzian geometry, only nonspacelike curves have a physical meaning since according to the Relativity Theory information cannot move with a speed greater than the speed of light* [4–6]*. By this reason, in Lorentzian geometry typically only nonspacelike curves are studied.*

Geometrically, spacelike curves may well be considered. For 2-dimensional Lorentzian manifolds there is not much geometric difference between timelike and spacelike curves since the first ones are obtained from the second ones by a change of Lorentzian form $g \mapsto -g$, or, equivalently, by a change of controls $(u_1, u_2) \mapsto (u_2, u_1)$. Although, for Lorentzian manifolds of dimension greater than 2 the spacelike cone is nonconvex, so the optimization problem of finding the longest spacelike curve is not well-defined (optimal trajectories do not exist).

Notice also that curves $q(\cdot)$ of variable causality ($\mathrm{sgn}\, g(\dot{q}) \neq \mathrm{const}$) cannot be optimal: it is easy to show that the causal character of extremal trajectories is preserved.

Remark 3. *The Lorentzian distance is defined by maximization* (2.1)*, not by minimization as in Riemannian geometry. In Lorentzian geometry, the distance means physically the space-time interval between events in a space-time* [4–6]*. On the other hand, the minimum of Lorentzian length is always zero (by virtue of lightlike trajectories), so the minimization problem here is not interesting.*

Notice also that the Lorentzian distance d is not a distance (metric) in the sense of metric spaces since d is not symmetric and satisfies the reverse *triangle inequality.*

Example 1. The simplest example of Lorentzian geometry is the Minkowski space [4]. In the 2D case it is defined as $\mathbb{R}^2_1 = \mathbb{R}^2_{xy}$, $g = -dx^2 + dy^2$. The Lorentzian length maximizers are straight line segments along which $g \leq 0$, the Lorentzian distance is

$$d((x_0, y_0), (x_1, y_1)) = \begin{cases} \sqrt{(x_1 - x_0)^2 - (y_1 - y_0)^2} & \text{for } (x_1 - x_0)^2 - (y_1 - y_0)^2 > 0, \\ 0 & \text{for } (x_1 - x_0)^2 - (y_1 - y_0)^2 \leq 0, \end{cases}$$

and positive radius Lorentzian spheres are arcs of hyperbolas with asymptotes parallel to lightlike curves $x = \pm y$. See Fig. 1.

This example has the following generalizations and variations, see [12], Sec. 5.2. Let $\mathbb{R}^n_s$, $0 \leq s \leq n$, denote the vector space $\mathbb{R}^n = \{(x_1, \dots, x_n)\}$ with the quadratic form $g^n_s = -\sum_{i=1}^s dx_i^2 + \sum_{j=s+1}^n dx_j^2$.

Example 2. Let $n \geq 2$. The Minkowski space $\mathbb{R}^n_1$ is a Lorentzian manifold with the Lorentzian form g^n_1. It has constant curvature $K = 0$ ([12], Th. 2.4.3).

Example 3. Let $n \geq 2$, and let $r > 0$. The de Sitter space is the Lorentzian manifold

$$\mathbb{S}^n_1 = \left\{x = (x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1}_1 \mid -x_1^2 + x_2^2 + \dots + x_{n+1}^2 = r^2\right\}$$

with the Lorentzian form $g = g^{n+1}_1\big|_{\mathbb{S}^n_1}$. The space $\mathbb{S}^n_1$ has constant curvature $K = \frac{1}{r^2}$ ([12], Th. 2.4.4).

Consider the Lorentzian manifold

$$\mathbb{H}^n_1 = \left\{x = (x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1}_2 \mid -x_1^2 - x_2^2 + x_3^2 + \dots + x_{n+1}^2 = -r^2\right\}$$

with the Lorentzian form $g = g^{n+1}_2\big|_{\mathbb{H}^n_1}$. The universal covering $\widetilde{\mathbb{H}^n_1}$ of $\mathbb{H}^n_1$ is called anti-de Sitter space. The spaces $\mathbb{H}^n_1$ and $\widetilde{\mathbb{H}^n_1}$ have constant curvature $K = -\frac{1}{r^2}$ ([12], Th. 2.4.4).

Let M_j be a Lorentzian manifold with Lorentzian distance d_j, $j = 1, 2$. A mapping $i : M_1 \to M_2$ is called an isometry if $d_1(q, p) = d_2(i(q), i(p))$ for all $q, p \in M_1$.

Example 4. The group of isometries of the Minkowski plane $\mathbb{R}^2_1$ is generated by translations, hyperbolic rotations e^{tX}, $X = y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$, and reflections $(x, y) \mapsto (x, -y)$.

Remark 4. *A recent generalization of Lorentzian geometry is the sub-Lorentzian geometry which studies vector distributions endowed with a Lorentzian form, see e.g.* [16–18].

3 Lobachevsky plane

Proper affine functions on the line are mappings of the form

$$a \mapsto y \cdot a + x, \qquad a \in \mathbb{R}, \quad y > 0, \quad x \in \mathbb{R}. \tag{3.1}$$

Consider the group of such functions $G = \text{Aff}_+(\mathbb{R}) = \{(x, y) \in \mathbb{R}^2 \mid y > 0\}$ with the group product induced by composition of functions (3.1):

$$(x_2, y_2) \cdot (x_1, y_1) = (x_2 + y_2 x_1, y_2 y_1), \qquad (x_i, y_i) \in G$$

and the identity element $\text{Id} = (0, 1) \in G$. This group is a semi-direct product $\text{Aff}_+(\mathbb{R}) = \mathbb{R}_+ \rtimes \mathbb{R}$.

G is a two-dimensional Lie group, connected and simply connected. The vector fields $X_1 = y\frac{\partial}{\partial x}$, $X_2 = y\frac{\partial}{\partial y}$ form a left-invariant frame on G, thus the Lie algebra of G is $\mathfrak{g} = \operatorname{span}(X_1, X_2)$. In view of the Lie bracket $[X_2, X_1] = X_1$, $\mathfrak{g}$ and G are solvable and non-Abelian. In fact, $\mathfrak{g}$ is a unique solvable non-Abelian two-dimensional Lie algebra [7].

One-parameter subgroups in G are rays (or straight lines if $u_2 = 0$)

$$u_1(y-1) = u_2 x, \qquad (u_1, u_2) \neq (0,0), \quad (x,y) \in G,$$

with the parametrization

$$x = \frac{u_1}{u_2}(e^{u_2 t} - 1), \quad y = e^{u_2 t}, \qquad u_2 \neq 0, \tag{3.2}$$

$$x = u_1 t, \quad y = 1, \qquad u_2 = 0, \tag{3.3}$$

see Fig. 2. Formulas (3.2), (3.3) for $t = 1$ describe the exponential mapping

$$\exp : \mathfrak{g} \to G, \qquad u_1 X_1 + u_2 X_2 \mapsto (x,y)(1). \tag{3.4}$$

Notice that left translations of one-parameter subgroups in G are also rays (or straight lines if $u_2 = 0$) since left translations in G are compositions of homotheties with parallel translations in $\mathbb{R}^2_{x,y}$.

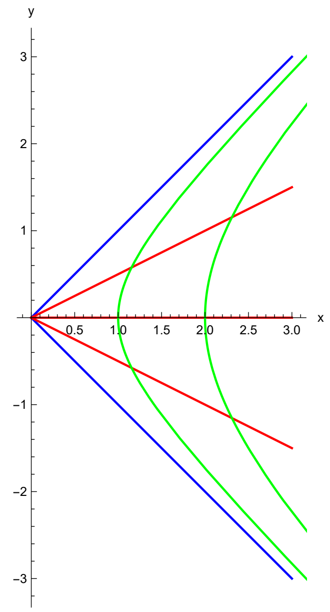

Figure 1: 2D Minkowski space

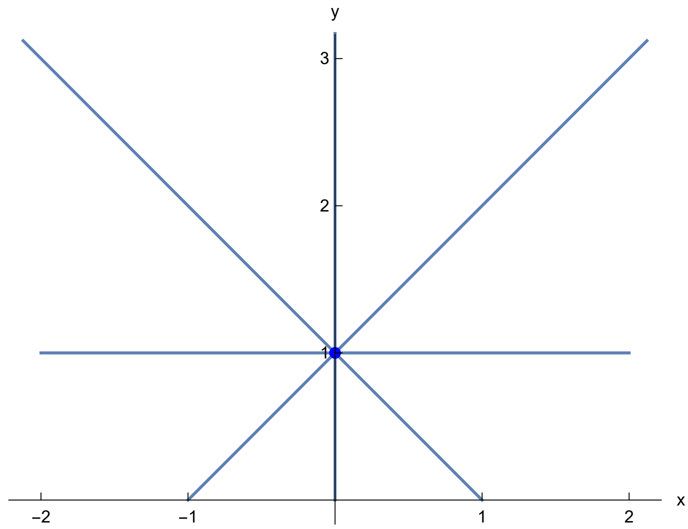

Figure 2: One-parameter subgroups in G

Remark 5. *Riemannian geometry on* $\mathrm{Aff}_+(\mathbb{R})$ *with the orthonormal frame* X_1, X_2 *is the Lobachevsky (Gauss, Bolyai) non-Euclidean geometry (in Poincaré's model in the upper half-plane)* [8,9].

4 Left-invariant Lorentzian problems on the Lobachevsky plane

In this work we consider left-invariant Lorentzian problems on the Lie group $G = \mathrm{Aff}_+(\mathbb{R})$. Such a problem is specified by an index 1 quadratic form g on the Lie algebra $\mathfrak{g}$ and a timelike time orientation vector field $X_0 \in \mathfrak{g}$.

A Lipschitzian curve $q : [0, t_1] \to G$ is a Lorentzian length maximizer that connects the point Id to a point $q_1 \in G$ iff it is a solution to the following optimal control problem:

$$g(\dot q(t)) \leq 0, \qquad \bar g(\dot q(t), X_0(q(t)) < 0, \tag{4.1}$$

$$q(0) = \mathrm{Id}, \qquad q(t_1) = q_1, \tag{4.2}$$

$$l = \int_0^{t_1} |g(\dot q(t))|^{1/2} dt \to \max, \tag{4.3}$$

where $\bar{g}$ is the bilinear form on $\mathfrak{g}$ corresponding to the quadratic form g.

Let us decompose a vector $\mathfrak{g} \ni v = u_1X_1 + u_2X_2$, then the Lorentzian form g and the bilinear form $\bar{g}$ are represented as $g(v) = g(u_1, u_2)$, $\bar{g}(v^1, v^2) = \bar{g}(v_1^1, v_2^2; v_1^2, v_2^2)$, where $v^i = v_1^iX_1 + v_2^iX_2$. Let $X_0 = v_1^0X_1 + v_2^0X_2$, and denote the linear form $g_0(u_1, u_2) = \bar{g}(v_1^0, v_2^0; u_1, u_2)$. Then the Lorentzian problem (4.1)–(4.3) reads as

$$\dot{q}(t) = u_1X_1 + u_2X_2, \qquad q \in G, \quad u = (u_1, u_2) \in U, \tag{4.4}$$
$$U = \{u \in \mathbb{R}^2 \mid g(u) \leq 0, \ g_0(u) < 0\}, \tag{4.5}$$
$$q(0) = \mathrm{Id}, \qquad q(t_1) = q_1, \tag{4.6}$$
$$l = \int_0^{t_1} |g(u)|^{1/2} dt \to \max . \tag{4.7}$$

The Lorentzian quadratic form can be decomposed as a sum of squares

$$g(u) = -(au_1 + bu_2)^2 + (cu_1 + du_2)^2, \tag{4.8}$$
$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2, \mathbb{R}). \tag{4.9}$$

Notice that the matrix A in (4.9) is not unique: it is determined up to the symmetries

$$\varepsilon_1 \ : \ (a, b, c, d) \mapsto (-a, -b, c, d), \qquad \qquad \varepsilon_2 \ : \ (a, b, c, d) \mapsto (a, b, -c, -d).$$

The inequality $g_0|_U = au_1 + bu_2 < 0$ fixes signs of a and b, thus eliminating the reflection ε_1. If we further assume that $|A| > 0$ in (4.9), then the signs of c and d become fixed, thus ε_2 is eliminated. Summing up, we have the following.

Lemma 1. *The space of left-invariant Lorentzian problems* (4.4)–(4.7) *is parametrized by matrices*

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_+(2, \mathbb{R}) = \{A \in \mathrm{GL}(2, \mathbb{R}) \mid |A| > 0\}.$$

Given a problem (4.4)–(4.7) determined by a matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_+(2, \mathbb{R})$, introduce new controls

$$v_1 = au_1 + bu_2, \qquad v_2 = cu_1 + du_2,$$

or, equivalently,

$$u_1 = \alpha v_1 + \beta v_2, \qquad u_2 = \gamma v_1 + \delta v_2, \qquad \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = A^{-1}.$$

Introduce further the vector fields

$$Y_1 = \alpha X_1 + \gamma X_2, \qquad Y_2 = \beta X_1 + \delta X_2.$$

Then the problem (4.4)–(4.7) reads as

$$\dot{q} = v_1Y_1 + v_2Y_2, \qquad q \in G, \tag{4.10}$$
$$g = -v_1^2 + v_2^2 \leq 0, \qquad g_0 = -v_1 < 0, \tag{4.11}$$
$$q(0) = \mathrm{Id}, \qquad q(t_1) = q_1, \tag{4.12}$$
$$l = \int_0^{t_1} \sqrt{v_1^2 - v_2^2} dt \to \max . \tag{4.13}$$

The Lorentzian form factorizes as

$$g = l_1 l_2, \qquad l_1(u_1, u_2) = (c - a)u_1 + (d - b)u_2, \qquad l_2(u_1, u_2) = (c + a)u_1 + (d + b)u_2.$$

Introduce the corresponding functions on G:

$$\lambda_1(x, y) = \operatorname{grad} l_1 \cdot \begin{pmatrix} x \\ y - 1 \end{pmatrix} = (c - a)x + (d - b)(y - 1),$$
$$\lambda_2(x, y) = \operatorname{grad} l_2 \cdot \begin{pmatrix} x \\ y - 1 \end{pmatrix} = (c + a)x + (d + b)(y - 1).$$

Remark 6. *By virtue of the change of variables* $(u_1, u_2) \mapsto (-u_1, -u_2)$*,* $A \mapsto -A$*,* $t \mapsto -t$*, we can get*

$$a \geq 0 \text{ or, equivalently, } \delta \geq 0, \tag{4.14}$$

which we assume in the sequel.

Example 5. As typical examples of Lorentzian problems (4.10)–(4.13), we consider in Sec. 12 the following model problems P_i, $i = 1, 2, 3$:

P_1: $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $U = \{u = (u_1, u_2) \in \mathbb{R}^2 \mid -u_1^2 + u_2^2 \leq 0,\ -u_1 \leq 0\}$, $g = -u_1^2 + u_2^2$, $g_0 = -u_1$,

P_2: $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $U = \{u = (u_1, u_2) \in \mathbb{R}^2 \mid -u_2^2 + u_1^2 \leq 0,\ -u_2 \leq 0\}$, $g = -u_2^2 + u_1^2$, $g_0 = -u_2$,

P_3: $A = \begin{pmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{pmatrix}$, $U = \{u = (u_1, u_2) \in \mathbb{R}^2 \mid u_1 \geq 0,\ u_2 \geq 0\}$, $g = -u_1 u_2$, $g_0 = -(u_1 + u_2)/2$.

See the sets of control parameters U for these problems resp. in Figs. 3–5.

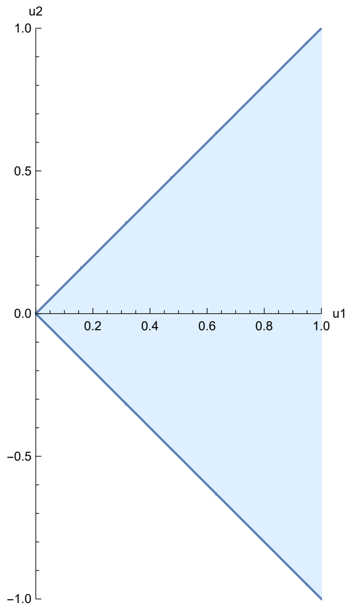

Figure 3: The set U for the problem P_1

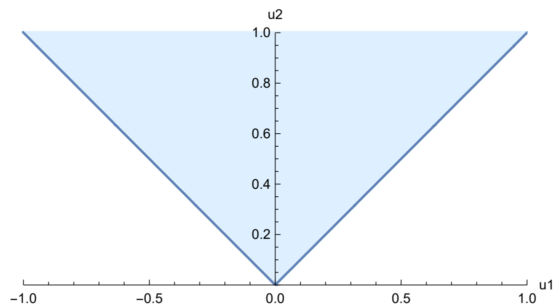

Figure 4: The set U for the problem P_2

We denote $J^+ = J^+(\mathrm{Id})$ and $d(q) = d(\mathrm{Id}, q)$, $q \in G$.

5 Curvature

In this section we show that each left-invariant Lorentzian structure on the group $G = \mathrm{Aff}_+(\mathbb{R})$ has constant sectional curvature K, thus it is locally isometric to the 2D Minkowski space (if $K = 0$), to a 2D de Sitter space (if $K > 0$), or to a 2D anti-de Sitter space (if $K < 0$).

5.1 Levi-Civita connection and sectional curvature of Lorentzian manifolds

Here we recall some standard facts of Lorentzian (in fact, pseudo-Riemannian) geometry, following [4, 12, 13].

A connection D on a smooth manifold M is a mapping $D : (\mathrm{Vec}(M))^2 \to \mathrm{Vec}(M)$ such that

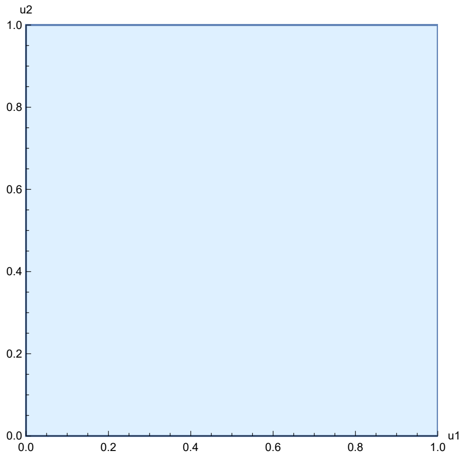

Figure 5: The set U for the problem P_3

(1) $D_V W$ is $C^\infty(M)$-linear in V,

(2) $D_V W$ is $\mathbb{R}$-linear in W,

(3) $D_V(fW) = (Vf)W + fD_V W$ for $f \in C^\infty(M)$.

The vector field $D_V W$ is called the covariant derivative of W w.r.t. V for the connection D.

Theorem 1 ([13], Th. 11)**.** *On a Lorentzian manifold (M, g) there is a unique connection D such that*

(4) $[V, W] = D_V W - D_W V$, *and*

(5) $Xg(V, W) = g(D_X V, W) + g(V, D_X W)$,

for all $X, V, W \in \mathrm{Vec}(M)$. D is called the Levi-Civita connection on M, and is characterized by the Koszul formula

$$2g(D_V W, X) = Vg(W, X) + Wg(X, V) - Xg(V, W) - g(V, [W, X]) + g(W, [X, V]) + g(X, [V, W]).$$

Let (M, g) be a Lorentzian manifold with Levi-Civita connection D. The mapping $R : (C^\infty(M))^3 \to C^\infty(M)$ given by $R_{XY}Z = D_{[X,Y]}Z - [D_X, D_Y]Z$ is called the Riemannian curvature tensor of (M, g).

Let $q \in M$, and let P be a 2D plane in T_qM. For vectors $v, w \in T_qM$, define $Q(v, w) = g(v, v)g(w, w) - (g(v, w))^2$. A plane P is called nondegenerate if $Q(v, w) \neq 0$ for some (hence every) basis v, w for P.

Lemma 2 ([13], Lemma 39)**.** *Let $P \subset T_qM$ be a nondegenerate plane. The number*

$$K(q, P) = \frac{g(R_{vw}v, w)}{Q(v, w)} \tag{5.1}$$

is independent of the choice of basis v, w in P, and is called the sectional curvature of the plane section P.

A Lorentzian manifold which has the same sectional curvature on all nondegenerate sections is said to have constant curvature.

Theorem 2 ([12], Theorem 2.4.1)**.** *Let (M, g) be a Lorentzian manifold of dimension $n \geq 2$, and let $K \in \mathbb{R}$. Then the following conditions are equivalent:*

(1) *M has constant curvature K,*

(2) *for any $q \in M$ there exists a neighbourhood of q isometric to an open subset of de Sitter space $\mathbb{S}^n_1$ for $K > 0$, Minkowski space $\mathbb{R}^n_1$ for $K = 0$, anti-de Sitter space $\widetilde{\mathbb{H}^n_1}$ for $K < 0$.*

5.2 Sectional curvature of $\mathrm{Aff}_+(\mathbb{R})$

In this subsection we compute Levi-Civita connection and sectional curvature of left-invariant Lorentzian structures on the group $G = \mathrm{Aff}_+(\mathbb{R})$.

Theorem 3. *Levi-Civita connection D of a left-invariant Lorentzian structure g on the group $G = \mathrm{Aff}_+(\mathbb{R})$ is given as follows:*

$$\begin{aligned}
&D_{X_i}X_j = \mu_{ij}X_1 + \nu_{ij}X_2, \qquad i,j = 1,2,\\
&(\mu_{11},\nu_{11}) = -\frac{1}{|A|^2}(-g_{12}g_{11}, g_{11}^2), \qquad (\mu_{12},\nu_{12}) = -\frac{1}{|A|^2}(-g_{22}g_{11}, g_{12}g_{11}),\\
&(\mu_{21},\nu_{21}) = -\frac{1}{|A|^2}(-g_{12}^2, g_{11}g_{12}), \qquad (\mu_{22},\nu_{22}) = -\frac{1}{|A|^2}(-g_{22}g_{12}, g_{12}^2),\\
&g_{11} = g(X_1) = c^2 - a^2, \qquad g_{12} = g(X_1,X_2) = cd - ab, \qquad g_{22} = g(X_2) = d^2 - b^2.
\end{aligned}$$

Proof. Immediate computation via Koszul formula. □

Theorem 4. *A left-invariant Lorentzian structure g on the group $G = \mathrm{Aff}_+(\mathbb{R})$ has constant curvature $K = \dfrac{g(X_1)}{|A|^2}$.*

Proof. Immediate computation via formula (5.1) for $P = T_qG$, $v = X_1(q)$, $w = X_2(q)$, $q \in G$. □

Corollary 1. *A left-invariant Lorentzian structure g on the group $G = \mathrm{Aff}_+(\mathbb{R})$ is locally isomorphic to the 2D Minkowski space $\mathbb{R}^2_1$ (for $K = 0$), de Sitter space $\mathbb{S}^2_1$ (for $K > 0$), or anti-de Sitter space $\widetilde{\mathbb{H}^2_1}$ (for $K < 0$).*

Remark 7. *For the case $K = 0$ we construct an explicit isometry of the group G to a half-plane of $\mathbb{R}^2_1$ in Th. 20.*

6 Attainable sets (causal futures and pasts)

Denote the set of admissible velocities $\mathcal{U} = \{u_1X_1 + u_2X_2 \mid (u_1,u_2) \in U\} \subset \mathfrak{g}$.

Theorem 5. *Let $q_0 \in G$, then*

$$J^+(q_0) = q_0 \exp(\mathcal{U}) = \{q \in G \mid \lambda_1(q) \le \lambda_1(q_0),\ \lambda_2(q) \ge \lambda_2(q_0)\}, \tag{6.1}$$

$$J^-(q_0) = q_0 \exp(-\mathcal{U}) = \{q \in G \mid \lambda_1(q) \ge \lambda_1(q_0),\ \lambda_2(q) \le \lambda_2(q_0)\}, \tag{6.2}$$

where $\exp : \mathfrak{g} \to G$ *is the exponential mapping* (3.4) *of the Lie group G.*
Moreover, $I^+(q_0) = \{q_0\} \cup \operatorname{int} J^+(q_0)$ and $I^-(q_0) = \{q_0\} \cup \operatorname{int} J^-(q_0)$.

Proof. By left-invariance of the problem, we need to prove equalities (6.1), (6.2) in the case $q_0 = \mathrm{Id}$ only.
Let us show that

$$J^+ = \exp(\mathcal{U}) = \{q \in G \mid \lambda_1(q) \le 0 \le \lambda_2(q)\}. \tag{6.3}$$

Future oriented nonspacelike one-parameter semigroups

$$\{\exp(t(u_1,u_2)) \mid t \ge 0\} = \{(x,y) \in G \mid u_1(y-1) = u_2x\}, \qquad l_1(u_1,u_2) \le 0 \le l_2(u_1,u_2),$$

fill the set $\exp(\mathcal{U})$, thus $J^+ \supset \exp(\mathcal{U})$. On the other hand, admissible trajectories of the system (4.10), (4.11) at the boundary of $\exp(\mathcal{U})$ are tangent to $\partial \exp(\mathcal{U})$ or are future directed inside $\exp(\mathcal{U})$. Thus $J^+ \subset \exp(\mathcal{U})$, and equality (6.3) follows.

A similar equality for $J^-(\mathrm{Id})$ is proved analogously. The expressions for $I^\pm(q_0)$ are straightforward. □

See the set J^+ for the problems P_1, P_2, P_3 in Figs. 8, 14, 19 respectively.

7 Existence of Lorentzian length maximizers

7.1 Existence of length maximizers for globally hyperbolic Lorentzian structures

In order to study existence of Lorentzian length maximizers we need some facts from Lorentzian geometry [4].

Let M be a Lorentzian manifold. An open subset $O \subset M$ is called causally convex if the intersection of each nonspacelike curve with O is connected. M is called strongly causally convex in any point in M has arbitrarily small causally convex neighbourhoods. Finally, a strongly causally convex Lorentzian manifold M is called globally hyperbolic if

$$J^+(p) \cap J^-(q) \text{ is compact for any } p, q \in M. \tag{7.1}$$

Theorem 6 (Th. 6.1 [4])**.** *If a Lorentzian manifold M is globally hyperbolic, then any points $q_0 \in M$, $q_1 \in J^+(q_0)$ can be connected by a Lorentzian length maximizer.*

Theorem 7. *A Lorentzian structure (g, X_0) on $\mathrm{Aff}_+(\mathbb{R})$ is globally hyperbolic iff $K \geq 0$.*

Proof. First, all left-invariant Lorentzian structures on $\mathrm{Aff}_+(\mathbb{R})$ are strongly causally convex. Indeed, $\dot{x} = u_1 y$ or $\dot{y} = u_2 y$ preserves sign and is separated from zero for $(x, y) \in O$, $u_1^2 + u_2^2 \geq C > 0$, $g(u) \leq 0$, $g_0(u) < 0$.

So we need to check condition (7.1) only. It follows from Th. 5 that for $K \geq 0$ the intersection in (7.1) is compact (it is either a parallelogram, a segment, or the empty set). The same theorem implies that for $K < 0$ there exist points $q \in G$ such that the intersection $J^+ \cap J^-(q)$ contains points from the absolute $\{y = 0\}$ in its closure, thus this intersection is not compact. □

Theorem 8. *Let $K \geq 0$. Then for any points $q_0 \in G$, $q_1 \in J^+(q_0)$ there exists a Lorentzian length maximizer from q_0 to q_1.*

Proof. Follows from Theorems 6, 7. □

7.2 Existence of length maximizers in the case $K < 0$

In this subsection we consider the remaining case $K < 0$. Introduce the decomposition

$$J^+ = D \sqcup F \sqcup E, \tag{7.2}$$
$$D = \{q \in G \mid \lambda_1(q) \leq 0 \leq \lambda_2(q),\ \lambda_3(q) > 0\}, \qquad F = \{q \in G \mid \lambda_3(q) = 0\}, \qquad E = \{q \in G \mid \lambda_3(q) < 0\},$$
$$\lambda_3(q) = \lambda_1(q) - \lambda_1(B), \qquad B = \left(\frac{d+b}{c+a}, 0\right) \in \mathbb{R}^2,$$

so that the lines $\{q \in \mathbb{R}^2 \mid \lambda_2(q) = 0\}$ and the absolute $\{y = 0\}$ intersect at the point $B \in \mathbb{R}^2_{x,y} \setminus G$, see Fig. 6 for the problem P_1.

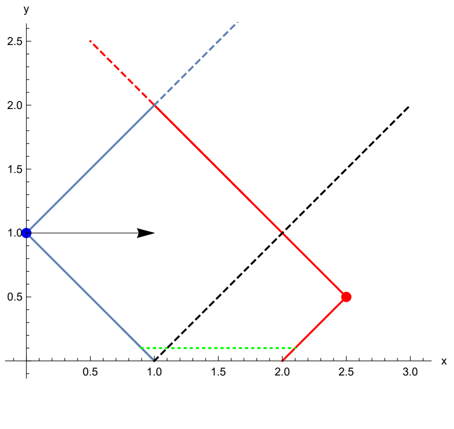

Figure 6: Case (2.2): $q_1 \in \mathcal{A} \setminus \mathrm{cl}(M)$

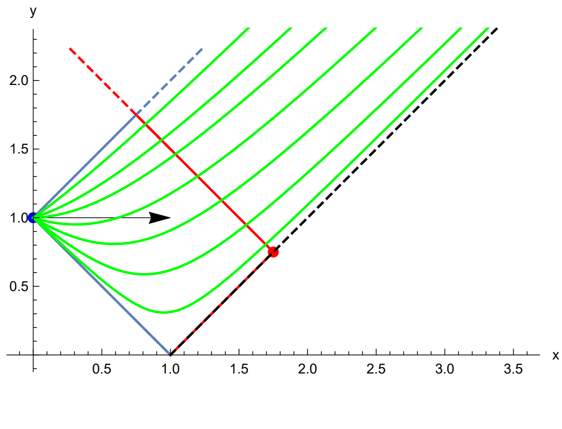

Figure 7: Case (2.3): $q_1 \in \mathcal{A} \cap \partial M$

Lemma 3. *The restriction of a negative curvature Lorentzian structure* (g, X_0) *on* $\mathrm{Aff}_+(\mathbb{R})$ *to* D *is globally hyperbolic.*

Proof. We need to check only condition (7.1).

Let $q_0, q_1 \in D$. By virtue of Th. 5, the intersection $J^+(q_0) \cap J^-(q_1)$ is either a parallelogram (if $q_1 \in \operatorname{int} J^+(q_0)$) or a segment (if $q_1 \in \partial J^+(q_0)$) or the empty set (if $q_1 \notin J^+(q_0)$), thus it is compact. $\square$

Theorem 9. *Let* $K < 0$*, and let* $q_0 = \mathrm{Id}$*,* $q_1 \in J^+$*.*

1. *If* $q_1 \in D$*, then there exists a Lorentzian length maximizer from* q_0 *to* q_1*.*
2. *If* $q_1 \in E$*, then there exist arbitrarily long trajectories from* q_0 *to* q_1*. Thus* $d(q_1) = +\infty$ *and there are no Lorentzian length maximizers from* q_0 *to* q_1*.*

Proof. Item (1) follows from Th. 6 and Lemma 3.

Item (2). Take any point $q_1 = (x_1, y_1) \in E$. Denote by $C \in \mathbb{R}^2_{x,y} \setminus G$, $C \neq B$, the intersection point of the lines $\{y = 0\}$ and $\{q \in \mathbb{R}^2 \mid \lambda_1(q) = \lambda_1(q_1)\}$, see Fig. 6 for the problem P_1. Notice that $x(C) > x(B)$. Take any $\varepsilon \in (0, 1)$. Denote by $B_\varepsilon \in G$ the intersection point of the lines $\{y = \varepsilon\}$ and $\{q \in \mathbb{R}^2 \mid \lambda_2(q) = 0\}$, and by $C_\varepsilon \in G$ the intersection point of the lines $\{y = \varepsilon\}$ and $\{q \in \mathbb{R}^2 \mid \lambda_1(q) = \lambda_1(q_1)\}$. The broken line $q_\varepsilon = q_0 B_\varepsilon C_\varepsilon q_1$ is an admissible trajectory of system (4.4), (4.5) with the cost given by the segment $B_\varepsilon C_\varepsilon$ only: $J(q_\varepsilon) = \displaystyle\int_{t(B_\varepsilon)}^{t(C_\varepsilon)} \sqrt{|g(u_1, 0)|}dt$. For $u_1 = 1$ we get $g(1, 0) = c^2 - a^2 < 0$, $x(t) = x_0 + \varepsilon t$,

$$t(C_\varepsilon) - t(B_\varepsilon) = \frac{x(C_\varepsilon) - x(B_\varepsilon)}{\varepsilon} = \frac{x(C) - x(B) + o(1)}{\varepsilon} \to +\infty \quad \text{as} \quad \varepsilon \to +0,$$

thus $J(q_\varepsilon) = \sqrt{c^2 - a^2}\ \dfrac{x(C) - x(B) + o(1)}{\varepsilon} \to +\infty$ as $\varepsilon \to +0$. So $d(q_1) = +\infty$. $\square$

Remark 8. *We prove below in Th.* 13 *that for any point* $q_1 \in F$ *there is no Lorentzian length maximizer connecting* Id *to* q_1*.*

8 Geodesics

8.1 Pontryagin maximum principle

We apply Pontryagin maximum principle (PMP) [1–3] to optimal control problem (4.4)–(4.7).

The Hamiltonian of PMP reads

$$\begin{aligned} &h_v^\nu(\lambda) = v_1 h_1(\lambda) + v_2 h_2(\lambda) - \nu\sqrt{v_1^2 - v_2^2}, \qquad \lambda \in T^*G, \qquad \nu \in \mathbb{R}, \\ &h_i(\lambda) = \langle \lambda, Y_i \rangle, \quad i = 1, 2. \end{aligned}$$

Since $[Y_1, Y_2] = -\delta Y_1 + \gamma Y_2$, then the Hamiltonian system with the Hamiltonian h_v^ν reads

$$\dot{h}_1 = -v_2(-\delta h_1 + \gamma h_2), \tag{8.1}$$
$$\dot{h}_2 = v_1(-\delta h_1 + \gamma h_2), \tag{8.2}$$
$$\dot{q} = v_1 Y_1 + v_2 Y_2.$$

8.1.1 Abnormal case

Obvious computations in the abnormal case $\nu = 0$ give the following.

Proposition 1. *Abnormal extremal trajectories are Lipschitzian reparametrizations of lightlike trajectories:*

$$\begin{aligned} &v_1 = \pm v_2 = 1, \qquad u_1 = \alpha \pm \beta, \ u_2 = \gamma \pm \delta, \\ &q(t) = \exp(t(Y_1 \pm Y_2)) = \exp(t(u_1 X_1 + u_2 X_2)), \end{aligned}$$

these are one-parameter subgroups (3.2), (3.3).

8.1.2 Normal case

Now consider the normal case $\nu = -1$. The maximality condition of PMP

$$h = v_1 h_1 + v_2 h_2 + \sqrt{v_1^2 - v_2^2} \to \max_{v_1 \geq |v_2|} \tag{8.3}$$

yields $h_1^2 - h_2^2 = v_1^2 - v_2^2 \equiv 1$. Introduce the hyperbolic coordinates

$$\begin{aligned} v_1 &= \cosh\varphi, \quad v_2 = \sinh\varphi, \qquad \varphi \in \mathbb{R}, \\ h_1 &= -\cosh\psi, \quad h_2 = \sinh\psi, \qquad \psi \in \mathbb{R}. \end{aligned}$$

Then the maximality condition (8.3) reads $h = -\cosh(\varphi - \psi) + 1 \to \max$, whence $\varphi = \psi$. Thus the maximized Hamiltonian of PMP reads $H = \frac{-h_1^2 + h_2^2}{2}$. Then the vertical subsystem (8.1), (8.2) of the Hamiltonian system of PMP reduces to the ODE $\dot{\psi} = \delta \sinh\psi + \gamma\cosh\psi$. Summing up, we have the following description of arclength-parametrized ($g = -v_1^2 + v_2^2 \equiv 1$) normal extremals.

Proposition 2. *Arclength-parametrized normal extremals satisfy the normal Hamiltonian system*

$$\begin{aligned} \dot{\lambda} &= \vec{H}(\lambda), \qquad \lambda \in T^*G, \\ H(\lambda) &= \frac{-h_1^2(\lambda) + h_2^2(\lambda)}{2} \equiv \frac{1}{2}, \qquad h_1(\lambda) < 0, \end{aligned}$$

in coordinates:

$$\dot{\psi} = \delta\cosh\psi + \gamma\sinh\psi, \tag{8.4}$$

$$\dot{q} = \cosh\psi\, Y_1 + \sinh\psi\, Y_2. \tag{8.5}$$

Normal extremals are parametrized by covectors $\lambda_0 \in C = T^*_{\mathrm{Id}}G \cap \{H(\lambda) = 1/2,\ h_1(\lambda) < 0\}$. They are given by the Lorentzian exponential mapping

$$\mathrm{Exp} : C \times \mathbb{R}_+ \to G, \qquad (\lambda_0, t) \mapsto q(t) = \pi \circ e^{t\vec{H}}(\lambda_0), \tag{8.6}$$

where $\vec{H}$ is the Hamiltonian vector field on T^*G with the Hamiltonian H, $e^{t\vec{H}} : G \to G$ is the flow of this vector field, and $\pi : T^*G \to G$, $T^*_q G \ni \lambda \mapsto q \in G$, is the canonical projection of the cotangent bundle.

8.2 Parametrization of geodesics

We integrate ODEs (8.4), (8.5) in the case $\delta \geq 0$, see (4.14). First we integrate the vertical subsystem (8.4):

$$\dot{\psi} = \delta\cosh\psi + \gamma\sinh\psi, \qquad \psi(0) = \psi_0, \qquad \delta \geq 0. \tag{8.7}$$

Proposition 3. *Cauchy problem* (8.7) *has the following solutions.*

(1) *If $K < 0$, then*

$$\psi(t) = \mu(t) - \theta, \tag{8.8}$$

$$\delta = \Delta\cosh\theta, \quad \gamma = \Delta\sinh\theta, \qquad \Delta = \sqrt{\delta^2 - \gamma^2}, \tag{8.9}$$

$$\mu(t) = \operatorname{arsinh}\tan\tau, \qquad \tau = \sigma + \rho, \tag{8.10}$$

$$\rho = \arctan\sinh(\psi_0 + \theta) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \tag{8.11}$$

$$\sigma = \Delta t \in \left(-\frac{\pi}{2} - \rho, \frac{\pi}{2} - \rho\right). \tag{8.12}$$

(2) *If $K > 0$, then*

$$\begin{aligned} &\psi(t) = \mu(t) - \theta, \\ &\gamma = s_1\Delta\cosh\theta, \quad \delta = s_1\Delta\sinh\theta, \qquad \Delta = \sqrt{\gamma^2 - \delta^2}, \qquad s_1 = \operatorname{sgn}\gamma. \end{aligned} \tag{8.13}$$

(2.1) *If* $\psi_0 + \theta = 0$, *then* $\mu(t) \equiv 0$.

(2.2) *If* $\psi_0 + \theta \neq 0$, *then*

$$\mu(t) = \operatorname{arcosh} \coth \tau, \qquad \tau = \rho - \sigma > 0, \tag{8.14}$$
$$\sigma = s_1 \Delta t < \rho, \tag{8.15}$$
$$\rho = \operatorname{artanh} \cosh(\psi_0 + \theta). \tag{8.16}$$

(3) *If* $K = 0$, *then*

$$\psi(t) = s_1 \mu(t),$$
$$\mu(t) = -\ln \tau, \qquad \tau = \rho - \gamma t > 0,$$
$$s_1 = \operatorname{sgn} \gamma, \qquad \rho = e^{-s_1 \psi_0}.$$

Proof. (1) Let $K < 0$, $\delta > 0$. Introduce variables Δ, θ according to (8.9), μ according to (8.8), and σ according to (8.12). Then Cauchy problem (8.7) transforms to

$$\frac{d\,\mu}{d\,\sigma} = \cosh \mu, \qquad \mu(0) = \mu_0 = \psi_0 + \theta,$$

which has solution (8.10) by separation of variables.

Cases (2), (3) are considered similarly. □

Now we integrate the horizontal ODE (8.5) of the Hamiltonian system for normal extremals:

$$\dot{x} = y k(\psi), \qquad k(\psi) = \alpha \cosh \psi + \beta \sinh \psi, \qquad x(0) = 0, \tag{8.17}$$
$$\dot{y} = y l(\psi), \qquad l(\psi) = \gamma \cosh \psi + \delta \sinh \psi, \qquad y(0) = 1. \tag{8.18}$$

Proposition 4. *Cauchy problem* (8.17), (8.18) *has the following solution.*

(1) *If* $K < 0$, *then*

$$x(t) = \cos \rho \left(\lambda(\tan \tau - \tan \rho) + \nu \left(\frac{1}{\cos \tau} - \frac{1}{\cos \rho} \right) \right), \tag{8.19}$$
$$y(t) = \frac{\cos \rho}{\cos \tau}, \tag{8.20}$$
$$\lambda = \frac{\alpha\delta - \beta\gamma}{\Delta^2}, \qquad \nu = \frac{\beta\delta - \alpha\gamma}{\Delta^2}, \tag{8.21}$$

where ρ, τ, Δ *are defined by* (8.9)–(8.11). *The curve* $(x(t), y(t))$ *is an arc of a hyperbola* $y^2 - (w + \sin \rho)^2 = \cos^2 \rho$, *where* $w = \frac{x - \nu(y-1)}{\lambda}$.

(2) *Let* $K > 0$.

(2.1) *If* $\psi_0 + \theta = 0$, *then*

$$x(t) = -\nu(e^\sigma - 1),$$
$$y(t) = e^\sigma,$$
$$\sigma = s_1 \Delta t, \qquad s_1 = \operatorname{sgn} \gamma, \qquad \Delta = \sqrt{\gamma^2 - \delta^2}.$$

The curve $(x(t), y(t))$ *is a line* $x + \nu(y - 1) = 0$.

(2.2) *If* $\psi_0 + \theta \neq 0$, *then*

$$x(t) = \sinh \rho \left(\nu \left(\frac{1}{\sinh \rho} - \frac{1}{\sinh \tau} \right) + s_2 \lambda (\coth \tau - \coth \rho) \right),$$
$$y(t) = \frac{\sinh \rho}{\sinh \tau},$$
$$s_2 = \operatorname{sgn} \mu_0,$$

where ρ, τ, Δ *are defined by* (8.13)–(8.16). *The curve* $(x(t), y(t))$ *is an arc of a hyperbola* $(s_2 w + \cosh \rho)^2 - y^2 = \sinh^2 \rho$, *where* $w = \frac{x + \nu(y-1)}{\lambda}$.

(3) *If $K = 0$, then*

$$x(t) = \rho\left(f(\tau - \rho) + g\left(\frac{1}{\rho} - \frac{1}{\tau}\right)\right),$$
$$y(t) = \frac{\rho}{\tau},$$
$$f = -\frac{\alpha - s_1\beta}{2\gamma}, \qquad g = -\frac{\alpha + s_1\beta}{2\gamma}, \qquad s_1 = \operatorname{sgn}\gamma,$$
$$\tau = \rho - \gamma t, \qquad \rho = e^{-s_1\psi_0}.$$

The curve $(x(t), y(t))$ is an arc of a hyperbola $w = \rho^2\left(\frac{1}{y} - 1\right)$, where $w = \frac{x+g(y-1)}{f}$.

Proof. Cauchy problem (8.17), (8.18) integrates as

$$x(t) = K(t) = \int_0^t k(s)e^{L(s)}ds, \tag{8.22}$$
$$y(t) = e^{L(t)}, \qquad L(t) = \int_0^t l(s)ds. \tag{8.23}$$

(1) Let $K < 0$, $\delta > 0$. By item (1) of Propos. 3,

$$\cosh\psi = \frac{1}{\Delta}\left(\frac{\delta}{\cos\tau} - \gamma\tan\tau\right), \qquad \sinh\psi = \frac{1}{\Delta}\left(\delta\tan\tau - \frac{\gamma}{\cos\tau}\right),$$
$$k = \Delta\tan\tau, \qquad l = \Delta\left(\frac{\lambda}{\cos\tau} + \nu\tan\tau\right), \tag{8.24}$$

and formulas (8.19), (8.20) follow from (8.22)–(8.24).

(2), (3) The cases $K \geq 0$ are treated similarly. □

8.3 Geodesic completeness

Denote the maximal domain of a solution λ_t to a Cauchy problem $\dot\lambda = \vec{H}(\lambda)$, $\lambda(0) = \lambda_0 \in C$ as $(t_{\min}(\lambda_0), t_{\max}(\lambda_0)) \ni 0$. We obtain the following explicit description of this domain from Propositions 3 and 4.

Corollary 2. (1) *If $K < 0$, then $t_{\min} = -\frac{\pi/2 + \rho}{\Delta}$, $t_{\max} = \frac{\pi/2 - \rho}{\Delta}$.*

(2) *If $K > 0$, then:*

(2.1) *if $\psi_0 + \theta = 0$, then $t_{\min} = -\infty$, $t_{\max} = +\infty$,*

(2.2) *if $\psi_0 + \theta \neq 0$, then* $\begin{cases} t_{\min} = -\infty, & t_{\max} = \frac{\rho}{\Delta} & \text{for } \gamma > 0, \\ t_{\min} = -\frac{\rho}{\Delta}, & t_{\max} = +\infty & \text{for } \gamma < 0. \end{cases}$

(3) *If $K = 0$, then* $\begin{cases} t_{\min} = -\infty, & t_{\max} = \frac{\rho}{\gamma} & \text{for } \gamma > 0, \\ t_{\min} = \frac{\rho}{\gamma}, & t_{\max} = +\infty & \text{for } \gamma < 0. \end{cases}$

We recall standard definitions of Lorentzian geometry related to geodesic completeness [4].

A timelike arclength-parametrized geodesic $q(t)$ in a Lorentzian manifold is called complete if it can be extended to be defined for $-\infty < t < +\infty$, otherwise it is called incomplete. Future and past complete (incomplete) geodesics are defined similarly.

A Lorentzian manifold M is called timelike geodesically complete if all timelike arclength-parametrized geodesics are complete, otherwise M is called timelike geodesically incomplete. Future and past timelike geodesically complete (incomplete) Lorentzian manifolds are defined similarly.

Now Corollary 2 implies the following.

Corollary 3. *If $K < 0$, then $\mathrm{Aff}_+(\mathbb{R})$ is both future and past timelike geodesically incomplete.*

Let $K \geq 0$. If $\gamma > 0$, then $\mathrm{Aff}_+(\mathbb{R})$ is past timelike geodesically complete and future timelike geodesically incomplete. If $\gamma < 0$, then $\mathrm{Aff}_+(\mathbb{R})$ is past timelike geodesically incomplete and future timelike geodesically complete.

Thus in all cases $\mathrm{Aff}_+(\mathbb{R})$ is timelike geodesically incomplete.

9 Lorentzian length maximizers

We prove that all extremal trajectories described in Sec. 8 are optimal, i.e., are Lorentzian length maximizers. The main tool is the following Hadamard's global diffeomorphism theorem.

Theorem 10 (Th. 6.2.8 [10]). *Let X, Y be smooth manifolds and let $F : X \to Y$ be a smooth mapping such that:*

1. $\dim X = \dim Y$,
2. *X and Y are arcwise connected,*
3. *Y is simply connected,*
4. *F is nondegenerate (i.e., for any $q \in X$ the differential $F_{*q} : T_qX \to T_{F(q)}Y$ is bijective),*
5. *F is proper (i.e., preimage of a compact is a compact).*

Then F is a diffeomorphism.

9.1 Diffeomorphic properties of the exponential mapping

Denote the following open subset $M \subset G$:

$$\begin{aligned} K < 0 \quad &\Rightarrow \quad M = \mathrm{int}\, D, \\ K \geq 0 \quad &\Rightarrow \quad M = \mathrm{int}\, J^+. \end{aligned}$$

The set $M \cong \mathbb{R}^2$ will serve as the domain of the exponential mapping $\mathrm{Exp} : N \to G$, in view of the following theorem.

Theorem 11. (1) $\mathrm{Exp}(N) \subset M$.

(2) $\mathrm{Exp} : N \to M$ *is a diffeomorphism.*

(3) *For any $\lambda_0 \in N$ and any $t_1 \in (0, t_{\max}(\lambda_0))$ the extremal trajectory $\mathrm{Exp}(\lambda_0, t)$, $t \in [0, t_1]$, is optimal.*

Proof. We consider only the case $K < 0$ since the case $K \geq 0$ is more simple and are treated similarly. So let $K < 0$, then

$$M = \mathrm{int}\, D = \{q \in G \mid \lambda_1(q) < 0 < \lambda_2(q),\ \lambda_3(q) > 0\}, \tag{9.1}$$

$$N = \left\{(\rho, \tau) \in \mathbb{R}^2 \mid \rho \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right),\ \tau \in \left(\rho, \frac{\pi}{2}\right)\right\}. \tag{9.2}$$

Since $\delta > 0$ by virtue of (4.14) and $\delta^2 - \gamma^2 > 0$ by virtue of $K < 0$, then $\delta > |\gamma|$. Further, we have factorizations along arclength-parametrized timelike geodesics $(x(t), y(t))$ given by item (1) of Propos. 4:

$$\lambda_1(x(t), y(t)) = \frac{2}{\delta - \gamma} \frac{\sin\left(\frac{\pi}{4} - \frac{\rho}{2}\right) \sin\left(\frac{\rho - \tau}{2}\right)}{\sin\left(\frac{\pi}{4} + \frac{\tau}{2}\right)}, \tag{9.3}$$

$$\lambda_2(x(t), y(t)) = -\frac{2}{\delta + \gamma} \frac{\sin\left(\frac{\pi}{4} + \frac{\rho}{2}\right) \sin\left(\frac{\rho - \tau}{2}\right)}{\sin\left(\frac{\pi}{4} - \frac{\tau}{2}\right)}, \tag{9.4}$$

$$\lambda_3(x(t), y(t)) = \frac{2}{\delta - \gamma} \frac{\sin\left(\frac{\pi}{4} + \frac{\rho}{2}\right) \cos\left(\frac{\rho - \tau}{2}\right)}{\sin\left(\frac{\pi}{4} + \frac{\tau}{2}\right)}. \tag{9.5}$$

(1) Factorizations (9.3)–(9.5) and equalities (9.1), (9.2) imply immediately that $\mathrm{Exp}(N) \subset M$.

(2) We apply Th. 10 to the mapping $\mathrm{Exp} : N \to M$. Both N and M are diffeomorphic to $\mathbb{R}^2$. The Jacobian of the exponential mapping is $\dfrac{\partial (x,y)}{\partial (\tau,\rho)} = -\lambda \dfrac{\cos\rho \sin(\rho-\tau)}{\cos^2\tau} < 0$ on N, thus $\mathrm{Exp} : N \to M$ is nondegenerate. Finally, factorizations (9.3)–(9.5) imply that if $(\rho,\tau) \to \partial N$, then $(x,y) = \mathrm{Exp}(\rho,\tau) \to \partial M$, thus $\mathrm{Exp} : N \to M$ is proper. Consequently, $\mathrm{Exp} : N \to M$ is a diffeomorphism.

(3) Let $\lambda_0 \in N$, and let $t_1 \in (0, t_{\max}(\lambda_0))$. Let us prove that the trajectory $q(t) = \mathrm{Exp}(\lambda_0, t)$, $t \in [0, t_1]$, is optimal. We have $q_1 = q(t_1) = \mathrm{Exp}(\lambda, t_1) \in M$. Moreover, by item (2) of this theorem $q(t)$, $t \in [0,t_1]$, is a unique arclength-parametrized geodesic connecting Id to q_1. By item (1) of Th. 9 there exists an optimal trajectory connecting these points, so it coincides with $q(t)$, $t \in [0, t_1]$.

□

9.2 Inverse of the exponential mapping and optimal synthesis

Theorem 12. *The inverse of the exponential mapping* $\mathrm{Exp}^{-1} : M \to N$, $(x_1,y_1) \mapsto (\psi_0, t_1)$ *is given as follows.*

(1) *If* $K < 0$, *then*

$$t_1 = \frac{\tau - \rho}{\Delta}, \qquad \psi_0 = \operatorname{arsinh} \tan\rho - \theta, \tag{9.6}$$

$$\tau = \arcsin\left(\frac{y_1^2 + w^2 - 1}{2y_1 w}\right), \qquad \rho = \arcsin\left(\frac{y_1^2 - w^2 - 1}{2w}\right), \qquad w = \frac{x_1 - \nu(y_1 - 1)}{\lambda}. \tag{9.7}$$

(2) *Let* $K > 0$, *and let*

$$w = \frac{x_1 + \nu(y_1 - 1)}{\lambda}, \qquad s_1 = \operatorname{sgn}\gamma. \tag{9.8}$$

(2.1) *If* $w = 0$, *then* $t_1 = s_1 \dfrac{\ln y_1}{\Delta}$, $\psi_0 = -\theta$.

(2.2) *If* $w \neq 0$, *then*

$$t_1 = s_1 \frac{\rho - \tau}{\Delta}, \qquad \psi_0 = \operatorname{arcosh} \coth\rho - \theta,$$
$$\tau = \operatorname{arcosh}\left(s_2 \frac{1 - y_1^2 - w^2}{2y_1 w}\right), \qquad \rho = \operatorname{arcosh}\left(s_2 \frac{1 - y_1^2 + w^2}{2w}\right), \qquad s_2 = \operatorname{sgn}(\lambda w). \tag{9.9}$$

(3) *If* $K = 0$, *then*

$$\begin{aligned} t_1 &= \frac{\rho - \tau}{\gamma}, \qquad \psi_0 = -s_1 \ln\rho, \\ \tau &= \sqrt{\frac{w}{y_1 - y_1^2}}, \qquad \rho = \sqrt{\frac{w y_1}{1 - y_1}}, \\ w &= \frac{x_1 - g(1 - y_1)}{f}, \qquad f = -\frac{\alpha - s_1\beta}{2\gamma}, \qquad g = -\frac{\alpha + s_1\beta}{2\gamma}, \qquad s_1 = \operatorname{sgn}\gamma. \end{aligned} \tag{9.10}$$

For any $(x_1, y_1) \in M$, *there is a unique arclength-parametrized optimal trajectory connecting* Id *to* (x_1,y_1), *and it is* $q(t) = \mathrm{Exp}(\psi_0, t)$, $t \in [0, t_1]$.

Proof. We consider only the case $K < 0$. Then the parametrization of Lorentzian geodesics given by item (1) of Propos. 4 yields

$$\begin{aligned} &\sin\rho = y_1 \sin\tau - w, \qquad \cos\rho = y_1 \cos\tau, \\ &1 = \sin^2\rho + \cos^2\rho = y_1^2 - 2y_1 w \sin\tau, \\ &\sin\tau = \frac{y_1^2 + w^2 - 1}{2y_1 w}, \qquad \sin\rho = \frac{y_1^2 - w^2 - 1}{2w}, \end{aligned}$$

and formulas of item (1) of this theorem follow since $\tau, \rho \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. □

Theorem 13. *Let $K < 0$. If $q_1 \in F$, then there is no Lorentzian length maximizer connecting q_0 to q_1.*

Proof. Lightlike extremal trajectories starting at q_0 fill the set $\partial J^+ = \{q \in G \mid \lambda_1(q)\lambda_2(q) = 0\}$. By item (1) of Th. 11, timelike extremal trajectories starting at q_0 fill the domain $\operatorname{int} D = \{q \in G \mid \lambda_1(q) < 0 < \lambda_2(q),\ \lambda_3(q) > 0\}$. Thus extremal trajectories starting at q_0 do not intersect the set $F = \{q \in G \mid \lambda_3(q) = 0\}$. By PMP, there is no optimal trajectory connecting q_0 to a point $q_1 \in F$. □

Remark 9. *The reasoning of the preceding theorem applied to the set $E = \{q \in G \mid \lambda_3(q) > 0\}$ proves once more that there are no Lorentzian length maximizers connecting q_0 to points in E, in addition to item (2) of Th. 9.*

Remark 10. *A Lorentzian metric on a manifold M is called geodesically connected [14, 15] if any two points in M can be connected by a future directed nonspacelike geodesic. For any left-invariant Lorentzian metric on $\operatorname{Aff}_+(\mathbb{R})$ we have $J^+(\operatorname{Id}) \neq \operatorname{Aff}_+(\mathbb{R})$, so such a metric is not geodesically connected. On the other hand, in the case $K \geq 0$ any point in $J^+(\operatorname{Id})$ can be connected with Id by a (length-maximizing) geodesic; in the case $K < 0$ the same holds for any point in $\operatorname{int} D \subsetneq J^+(\operatorname{Id})$.*

10 Lorentzian distance and spheres

We describe explicitly the Lorentzian distance $d(q) = d(\operatorname{Id}, q)$ and spheres $S(R) = \{q \in G \mid d(q) = R\}$, $R \in [0, +\infty]$.

10.1 The case $K < 0$

Theorem 14. *Let $K < 0$ and let $q_1 = (x_1, y_1) \in G$.*

(1) *If $q_1 \notin J^+$, then $d(q_1) = 0$.*

(2) *If $q_1 \in \partial J^+$, then $d(q_1) = 0$.*

(3) *If $q_1 \in \operatorname{int} D$, then $d(q_1) = \dfrac{\tau - \rho}{\Delta}$, where τ, ρ are given by (9.7). In particular,*

$$d(\operatorname{int} D) = \left(0, \frac{\pi}{\Delta}\right). \tag{10.1}$$

(4) *If $q_1 \in F$, then $d(q_1) = \dfrac{\pi}{\Delta}$.*

(5) *If $q_1 \in E$, then $d(q_1) = +\infty$.*

Proof. (1) follows from the definition of Lorentzian distance d.

(2) follows since the only trajectories connecting Id to $q_1 \in \partial J^+$ are lightlike by item (1) of Th. 11.

(3) follows from item (1) of Th. 11.

(4) Let $q_1 \in F$. Take any sequence $(\tau^n, \rho^n) \in N$ such that $\tau^n \to \frac{\pi}{2} - 0$, $\rho^n \to -\frac{\pi}{2} + 0$, $\frac{\tau^n + \pi/2}{\rho^n + \pi/2} \to +\infty$. Then the parametrization of the exponential mapping (8.19), (8.20) implies that the point $q^n = \operatorname{Exp}(\tau^n, \rho^n) \in \operatorname{int} D$ and $q^n \to B = \{y = \lambda_3(q) = 0\}$. By item (3) of this theorem, $d(q^n) = \frac{\tau^n - \rho^n}{\Delta} \to \frac{\pi}{\Delta}$.

Considering a trajectory of the field $X_1 = y\frac{\partial}{\partial x}$ starting at q^n and terminating at the ray F, we get the bound $d|_F \geq \frac{\pi}{\Delta}$.

Now we show that in fact $d|_F = \frac{\pi}{\Delta}$. To this end we cite the following statement of lower semicontinuity of Lorentzian distance.

Lemma 4 (Lemma 4.4 [4])**.** *For Lorentzian distance d on a Lorentzian manifold, if $d(p, q) < \infty$, $p_n \to p$, and $q_n \to q$, then $d(p, q) \leq \liminf_{n\to\infty} d(p_n, q_n)$.*

Also, if $d(p, q) = \infty$, $p_n \to p$, and $q_n \to q$, then $\lim_{n\to\infty} d(p_n, q_n) = \infty$.

Take any point $\bar{q} \in F$. Choose any sequence $\operatorname{int} D \ni q^n \to \bar{q}$. If $d(\bar{q}) = \infty$, then Lemma 4 implies $\lim_{n\to\infty} d(q_0, q^n) = \infty$, which contradicts the bound (10.1). Thus $d(\bar{q}) < \infty$. Then by Lemma 4 $d(q_0, \bar{q}) \leq \liminf_{n\to\infty} d(q_0, q^n) \leq \frac{\pi}{\Delta}$. So $d(q_0, \bar{q}) = \frac{\pi}{\Delta}$.

(5) follows from item (2) of Th. 9.

□

The explicit description of Lorentzian length maximizers given by Th. 12 implies, via transformations of elementary functions, the following characterization of Lorentzian spheres centred at Id.

Corollary 4. *Let $K < 0$.*

(1) $S(0) = \{q \in G \mid \lambda_1(q) \geq 0$ *or* $\lambda_2(q) \leq 0\}$.

(2) *If* $R \in (0, \frac{\pi}{\Delta})$, *then*

$$S(R) = \{(x, y) \in G \mid w^2 - (y - \cos\sigma)^2 = \sin^2\sigma\}, \qquad w = \frac{x - \nu(y-1)}{\lambda}, \qquad \sigma = \Delta R,$$

it is an arc of a hyperbola noncompact in both directions.

(3) $S(\frac{\pi}{\Delta}) = F$.

(4) *If* $R \in (\frac{\pi}{\Delta}, +\infty)$, *then* $S(R) = \emptyset$.

(5) $S(+\infty) = E$.

10.2 The case $K > 0$

Theorem 15. *Let $K > 0$ and let $q_1 = (x_1, y_1) \in G$.*

(1) *If* $q_1 \notin J^+$, *then* $d(q_1) = 0$.

(2) *If* $q_1 \in \partial J^+$, *then* $d(q_1) = 0$.

(3) *If* $q_1 \in \operatorname{int} J^+ \cap \{w \neq 0\}$, *then* $d(q_1) = s_1 \dfrac{\rho - \tau}{\Delta}$, *where* s_1, w, τ, ρ *are given by* (9.8), (9.9). *In particular,* $d(\operatorname{int} J^+ \cap \{w \neq 0\}) = (0, +\infty)$.

(4) *If* $q_1 \in \operatorname{int} J^+ \cap \{w = 0\}$, *then* $d(q_1) = s_1 \dfrac{\ln y_1}{\Delta}$, *where* s_1 *is given by* (9.8). *In particular,* $d(\operatorname{int} J^+ \cap \{w = 0\}) = (0, +\infty)$.

Proof. Similarly to the proof of Th. 14. □

Corollary 5. *Let $K > 0$.*

(1) $S(0) = \{q \in G \mid \lambda_1(q) \geq 0$ *or* $\lambda_2(q) \leq 0\}$.

(2) *If* $R \in (0, +\infty)$, *then*

$$S(R) = \{(x, y) \in G \mid (y - \cosh\sigma)^2 - w^2 = \sinh^2\sigma\}, \qquad w = \frac{x + \nu(y-1)}{\lambda}, \qquad \sigma = s_1 \Delta R, \qquad s_1 = \operatorname{sgn}\delta,$$

it is an arc of a hyperbola noncompact in both directions.

(3) $S(+\infty) = \emptyset$.

Proof. Similarly to the proof of Cor. 4. □

10.3 The case $K = 0$

Theorem 16. *Let $K = 0$ and let $q_1 \in G$.*

(1) *If* $q_1 \notin J^+$, *then* $d(q_1) = 0$.

(2) *If* $q_1 \in \partial J^+$, *then* $d(q_1) = 0$.

(3) *If* $q_1 \in \operatorname{int} J^+$, *then* $d(q_1) = \dfrac{\rho - \tau}{\gamma}$, *where* τ, ρ *are given by* (9.10). *In particular,* $d(\operatorname{int} J^+) = (0, +\infty)$.

Proof. Similarly to the proof of Th. 14. $\square$

Corollary 6. *Let $K = 0$.*

(1) $S(0) = \{q \in G \mid \lambda_1(q) \geq 0$ *or* $\lambda_2(q) \leq 0\}$.

(2) *If $R \in (0, +\infty)$, then*

$$S(R) = \{(x, y) \in G \mid (w + \sigma^2)y = w\}, \qquad w = \frac{x + g(y-1)}{f}, \qquad \sigma = \gamma R,$$

it is an arc of a hyperbola noncompact in both directions.

(3) $S(+\infty) = \emptyset$.

Proof. Similarly to the proof of Cor. 4. $\square$

10.4 Regularity of Lorentzian distance

Corollary 7. *We have $d \in C^\omega(M) \cap C(\operatorname{cl} D)$.*

Proof. We consider only the case $K < 0$. If $q_1 \in M = \operatorname{int} D$, then item (3) of Th. 14 gives $d(q_1) = \frac{\tau_1 - \rho_1}{\Delta}$, and the functions τ_1, ρ_1 are real-analytic since $\operatorname{Exp}^{-1} : \operatorname{int} D \to N$ is real-analytic by virtue of the inverse function theorem for real-analytic mappings.

In order to show the inclusion $d \in C(\operatorname{cl} D)$, it remains to prove continuity of d on the boundary $\partial D = \partial J^+ \cup F$. If $\operatorname{int} D \ni q^n \to q_1 \in \partial J^+$, then by virtue of items (2), (3) of Th. 14 we have $d(q^n) \to 0 = d(q_1)$. And if $\operatorname{int} D \ni q^n \to q_1 \in F$, then similarly $d(q^n) \to \frac{\pi}{\Delta} = d(q_1)$. $\square$

Now we study asymptotics of the Lorentzian distance $d(q)$ near the boundary of the domain M. For a point $q \in M$, denote by $d_M(q)$ the Euclidean distance from q to ∂M. The explicit expression for the Lorentzian distance in the domain M given by Theorems 14–16 implies that near smoothness points of ∂M the distance $d(q)$ is Hölder with exponent $\frac{1}{2}$ of the distance $d_M(q)$, similarly to the Minkowski plane.

Corollary 8. *Let $\bar{q} \in \partial M$ be a point of smoothness of the curve ∂M. Then*

$$\begin{aligned} &d(q) = d(\bar{q}) + f(\bar{q})\sqrt{d_M(q)} + O(d_M(q))^{3/2}, \\ &M \ni q \to \bar{q}, \qquad f(\bar{q}) \neq 0. \end{aligned}$$

Remark 11. *Alternative proofs of Corollaries 7, 8 follow by local isometry of $\operatorname{Aff}_+(\mathbb{R})$ with standard constant curvature Lorentzian manifolds $\mathbb{R}^2_1$, $\mathbb{S}^2_1$, $\widetilde{\mathbb{H}^2_1}$.*

11 Isometries

11.1 Infinitesimal isometries of Lorentzian manifolds

We recall some necessary facts of Lorentzian (in fact, pseudo-Riemannian geometry) [13].

A vector field X on a Lorentzian manifold (M, g) is called a Killing vector field (or an infinitesimal isometry) if $L_X g = 0$.

Proposition 5 ([13], Propos. 23)**.** *A vector field X is Killing iff the mappings ψ_t of its local flow satisfy $\psi_t^* g = g$, where $\psi_t : M \to M$ is the shift of M along X by time t.*

Corollary 9. *A vector field X is Killing iff $d(q_1, q_2) = d(\psi_t(q_1), \psi_t(q_2))$ for all $q_1, q_2 \in M$ and all t for which the right-hand side is defined.*

Proposition 6 ([13], Propos. 25)**.** *A vector field X is Killing iff*

$$Xg(V, W) = g([X, V], W) + g(V, [X, W]), \qquad V, W \in \operatorname{Vec}(M). \tag{11.1}$$

Denote by $i(M)$ the set of Killing vector fields on a Lorentzian manifold M. The set $i(M)$ is a Lie algebra over $\mathbb{R}$ w.r.t. Lie bracket of vector fields.

Lemma 5 ([13], Lemma 28)**.** *The Lie algebra $i(M)$ on a connected Lorentzian manifold M, $\dim M = n$, has dimension at most $\frac{n(n+1)}{2}$.*

Remark 12. *Let M be a connected Lorentzian manifold of dimension n. Then $\dim i(M) = \frac{n(n+1)}{2}$ iff M has constant curvature (Exercises 14, 15 [13]).*

Denote by $I(M)$ the set of all isometries of a Lorentzian manifold M.

Theorem 17 ([13], Theorem 32)**.** *$I(M)$ is a Lie group.*

Denote by $ci(M)$ the set of all complete Killing vector fields on M.

Proposition 7 ([13], Propos. 33)**.** (1) *$ci(M)$ is a Lie subalgebra of $i(M)$.*

(2) *There is a Lie anti-isomorphism between the Lie algebra of the Lie group $I(M)$ and the Lie algebra $ci(M)$.*

Denote by $I_0(M)$ the connected component of the identity in the Lie group $I(M)$.

11.2 Killing vector fields and isometries of $\mathrm{Aff}_+(\mathbb{R})$

We compute the Lie algebra of Killing vector fields for left-invariant Lorentzian structures on $G = \mathrm{Aff}_+(\mathbb{R})$.

By Th. 4, such Lorentzian structures have constant curvature. By Remark 12,

$$\dim i(G) = 3. \tag{11.2}$$

Left translations on the Lie group G are obvious isometries. They are generated by right-invariant vector fields on G:

$$\widetilde{X}_1(q) = R_{q*}X_1(\mathrm{Id}) = \frac{\partial}{\partial x}, \qquad \widetilde{X}_2(q) = R_{q*}X_2(\mathrm{Id}) = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y},$$

where $R_q\ :\ \bar{q} \mapsto \bar{q}q$ is the right translation on G. Since $[\widetilde{X}_i, X_j] = 0$, Propos. 11.1 implies that $\widetilde{X}_1$, $\widetilde{X}_2$ are Killing vector fields. By virtue of (11.2), in order to describe the 3D Lie algebra $i(G)$ it remains to find just one Killing vector field linearly independent on $\widetilde{X}_1$, $\widetilde{X}_2$.

Lemma 6. *If $X \in \mathrm{Vec}(G)$ is a Killing vector field such that $X(\mathrm{Id}) = 0$, then X is tangent to Lorentzian spheres $S(R)$, $R \in [0, +\infty]$.*

Proof. Local flow of X preserves the Lorentzian distance $d(\mathrm{Id}, q)$, thus the Lorentzian spheres as well. □

Lemma 7. *The following vector field is tangent to Lorentzian spheres $S(R)$, $R \in (0, +\infty)$:*

(1) $K < 0 \quad \Rightarrow \quad X_- = \left(y^2 + w^2\right)\frac{\partial}{\partial w} + 2wy\frac{\partial}{\partial y} = \left(\lambda(y^2 + w^2 - 1) + 2\nu wy\right)\frac{\partial}{\partial x} + 2wy\frac{\partial}{\partial y}$, $w = \frac{x - \nu(y-1)}{\lambda}$,

(2) $K > 0 \quad \Rightarrow \quad X_+ = \left(y^2 + w^2\right)\frac{\partial}{\partial w} + 2wy\frac{\partial}{\partial y} = \left(\lambda(y^2 + w^2 - 1) - 2\nu wy\right)\frac{\partial}{\partial x} + 2wy\frac{\partial}{\partial y}$, $w = \frac{x + \nu(y-1)}{\lambda}$,

(3) $K = 0 \quad \Rightarrow \quad X_0 = w\frac{\partial}{\partial w} + y(1-y)\frac{\partial}{\partial y} = \left(x + g(y^2 - 1)\right)\frac{\partial}{\partial x} + y(1-y)\frac{\partial}{\partial y}$, $w = \frac{x + g(y-1)}{f}$.

Proof. Follows from the explicit parametrization of the spheres $S(R)$, $R \in (0, +\infty)$, see Corollaries 4, 5, 6 respectively. □

Theorem 18. *Let $K \neq 0$. Then $i(G) = \mathrm{span}(\widetilde{X}_1, \widetilde{X}_2, X_\pm)$, where $\pm = \operatorname{sgn} K$, and $X_\pm$ is given by items (1), (2) of Lemma 7. The table of Lie brackets in this Lie algebra is $[\widetilde{X}_1, \widetilde{X}_2] = \widetilde{X}_1$, $[\widetilde{X}_1, X_\pm] = \mp\frac{2\nu}{\lambda}\widetilde{X}_1 + \frac{2}{\lambda}\widetilde{X}_2$, $[\widetilde{X}_2, X_\pm] = \frac{2(\lambda^2 - \nu^2)}{\lambda}\widetilde{X}_1 \pm \frac{2\nu}{\lambda}\widetilde{X}_2 + X_\pm$. The Lie algebra $i(G)$ is isomorphic to the Lie algebra $\mathfrak{sl}(2)$ of the Lie group $\mathrm{SL}(2)$ of unimodular 2×2 matrices.*

Proof. The vector field $X_\pm$ satisfies identity (11.1), thus it is Killing. Since $X_\pm$ is linearly independent of $\widetilde{X}_1$, $\widetilde{X}_2$ and $\dim i(G) = 3$, it follows that $i(G) = \operatorname{span}(\widetilde{X}_1, \widetilde{X}_2, X_\pm)$. The table of Lie brackets in this Lie algebra is verified immediately. Moreover, these Lie brackets imply that the Lie algebra $i(G)$ is simple, thus it is isomorphic to $\mathfrak{sl}(2)$ or $\mathfrak{so}(3)$, see the classification of 3D Lie algebras in [7]. But $i(G)$ contains a 2D Lie subalgebra spanned by $\widetilde{X}_1$, $\widetilde{X}_2$, which is impossible in $\mathfrak{so}(3)$. Thus $i(G) \cong \mathfrak{sl}(2)$. □

Theorem 19. *Let $K = 0$. Then $i(G) = \operatorname{span}(\widetilde{X}_1, \widetilde{X}_2, X_0)$, where X_0 is given by item (3) of Lemma 7. The table of Lie brackets in this Lie algebra is $[\widetilde{X}_1, \widetilde{X}_2] = \widetilde{X}_1$, $[\widetilde{X}_1, X_0] = \widetilde{X}_1$, $[\widetilde{X}_2, X_0] = 2g\widetilde{X}_1 - \widetilde{X}_2 + X_0$. The Lie algebra $i(G)$ is isomorphic to the Lie algebra $\mathfrak{sh}(2)$ of the Lie group $\mathrm{SH}(2)$ of hyperbolic motions of the plane.*

Proof. Similarly to the proof of Th. 18. □

Proposition 8. (1) $ci(G) = \operatorname{span}(\widetilde{X}_1, \widetilde{X}_2)$.

(2) $I_0(\mathrm{Aff}_+(\mathbb{R})) = \{L_q \mid q \in \mathrm{Aff}_+(\mathbb{R})\} \cong \mathrm{Aff}_+(\mathbb{R})$.

Proof. Item (1). The vector fields $\widetilde{X}_1$, $\widetilde{X}_2$ are complete. Although, each vector field X_0, $X_\pm$ is not complete.

Item (2). By virtue of Propos. 7 and item (1) of this proposition, the Lie algebra of the Lie group $I_0(\mathrm{Aff}_+(\mathbb{R}))$ is anti-isomorphic to $ci(G) = \operatorname{span}(\widetilde{X}_1, \widetilde{X}_2)$. □

11.3 Isometric embedding of $\mathrm{Aff}_+(\mathbb{R})$ into $\mathbb{R}^2_1$ in the case $K = 0$

Theorem 20. *Let $K = 0$. The mapping $i : \mathrm{Aff}_+(\mathbb{R}) \to \Pi \subset \mathbb{R}^2_1$, $\Pi = \{(\widetilde{x}, \widetilde{y}) \in \mathbb{R}^2_1 \mid s_1\widetilde{y} + \widetilde{x} < 1/\gamma\}$,*

$$i(x, y) = (\widetilde{x}, \widetilde{y}) = \left(\frac{1}{2}\left(\frac{y-1}{y} - \frac{w}{\gamma}\right), \frac{s_1}{2}\left(\frac{y-1}{y} + \frac{w}{\gamma}\right)\right), \tag{11.3}$$

is an isometry.

Proof. We give a proof for the problem P_3, in the general case $K = 0$ the proof is similar.

For the problem P_3 we have $\Pi = \{(\widetilde{x}, \widetilde{y}) \in \mathbb{R}^2_1 \mid \widetilde{y} + \widetilde{x} < 1\}$,

$$i(x, y) = (\widetilde{x}, \widetilde{y}) = \left(\frac{1}{2}\left(1 - \frac{1}{y} + x\right), \frac{s_1}{2}\left(1 - \frac{1}{y} + x\right)\right). \tag{11.4}$$

Let $q_j = (x_j, y_j) \in \mathrm{Aff}_+(\mathbb{R})$, $\widetilde{q}_j = i(q_j) = (\widetilde{x}_j, \widetilde{y}_j) \in \mathbb{R}^2_1$, $j = 1, 2$. Immediate computation on the basis of (11.4) shows that $\widetilde{q}_j \in \Pi$, $j = 1, 2$. We prove that

$$\widetilde{d}(\widetilde{q}_1, \widetilde{q}_2) = d(q_1, q_2), \tag{11.5}$$

where d and $\widetilde{d}$ are the Lorentzian distances in $\mathrm{Aff}_+(\mathbb{R})$ and $\mathbb{R}^2_1$ respectively.

First we show that

$$d(q_1, q_2) \neq 0 \iff \widetilde{d}(\widetilde{q}_1, \widetilde{q}_2) \neq 0. \tag{11.6}$$

Denote $\bar{q} = q_1^{-1}q = (\bar{x}, \bar{y}) = ((x_2 - x_1)/y_1, y_2/y_1)$. Then

$$d(q_1, q_2) \neq 0 \iff d(\mathrm{Id}, \bar{q}) \neq 0 \iff \bar{x} > 0,\ \bar{y} > 1 \iff x_2 > x_1,\ y_2 > y_1.$$

On the other hand,

$$\widetilde{d}(\widetilde{q}_1, \widetilde{q}_2) \neq 0 \iff \widetilde{x}_2 - \widetilde{x}_1 > |\widetilde{y}_2 - \widetilde{y}_1| \iff \begin{cases} \widetilde{x}_2 - \widetilde{x}_1 > \widetilde{y}_2 - \widetilde{y}_1, \\ \widetilde{x}_2 - \widetilde{x}_1 > \widetilde{y}_1 - \widetilde{y}_2 \end{cases}$$

$$\iff \begin{cases} x_2 - \frac{1}{y_2} - x_1 + \frac{1}{y_1} > -x_1 + x_2 - \frac{1}{y_1} + \frac{1}{y_2}, \\ x_2 - \frac{1}{y_2} - x_1 + \frac{1}{y_1} > x_1 - x_2 + \frac{1}{y_1} - \frac{1}{y_2} \end{cases} \iff \begin{cases} \frac{1}{y_1} > \frac{1}{y_2}, \\ x_1 - x_2 < 0, \end{cases}$$

and (11.6) follows.

Now let $d(q_1, q_2) \neq 0$, $\widetilde{d}(\widetilde{q}_1, \widetilde{q}_2) \neq 0$, and we prove equality (11.5). We have

$$d^2(q_1, q_2) = d^2(\mathrm{Id}, \bar{q}) = \left(\sqrt{\frac{\bar{x}\bar{y}}{\bar{y}-1}} - \sqrt{\frac{\bar{x}}{\bar{y}(\bar{y}-1)}}\right)^2 = \frac{\bar{x}(\bar{y}-1)}{\bar{y}} = \frac{(x_2 - x_1)(y_2 - y_1)}{y_1 y_2},$$

$$\widetilde{d}^2(\widetilde{q}_1, \widetilde{q}_2) = (\widetilde{x}_2 - \widetilde{x}_1)^2 - (\widetilde{y}_2 - \widetilde{y}_1)^2 = \frac{1}{4}\left(-\frac{1}{y_2} + x_2 + \frac{1}{y_1} - x_1\right)^2 - \frac{1}{4}\left(-\frac{1}{y_2} - x_2 + \frac{1}{y_1} + x_1\right)^2$$
$$= \frac{(x_2 - x_1)(y_2 - y_1)}{y_1 y_2},$$

and equality (11.5) follows. □

Remark 13. *The explicit formulas* (11.3) *for the isometry* $i : \mathrm{Aff}_+(\mathbb{R}) \to \Pi$ *were discovered as follows. The exponential mappings for* $\mathrm{Aff}_+(\mathbb{R})$ *in the case* $K = 0$ *and for the Minkowski plane* $\mathbb{R}^2_1$ *have respectively the form:*

$$\mathrm{Exp} : \begin{pmatrix} \psi \\ t \end{pmatrix} \mapsto \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \rho(f(\tau - \rho) + g\left(\frac{1}{\rho} - \frac{1}{\tau}\right) \\ \frac{\rho}{\tau} \end{pmatrix}, \qquad \widetilde{\mathrm{Exp}} : \begin{pmatrix} \widetilde{\psi} \\ \widetilde{t} \end{pmatrix} \mapsto \begin{pmatrix} \widetilde{x} \\ \widetilde{y} \end{pmatrix} = \begin{pmatrix} \widetilde{t}\cosh\widetilde{\psi} \\ \widetilde{t}\sinh\widetilde{\psi} \end{pmatrix}. \quad (11.7)$$

We set in these formulas $t = \widetilde{t}$, $\psi = \widetilde{\psi}$, *and obtain* (11.4).

Remark 14. *In the case* $K = 0$ *the group* $\mathrm{Aff}_+(\mathbb{R})$ *cannot be isometric to the whole Minkowski space* $\mathbb{R}^2_1$ *since the first is not geodesically complete (see Cor.* 3*), while the second is.*

It would be interesting to construct isometric embeddings of $\mathrm{Aff}_+(\mathbb{R})$ *to* $\mathbb{S}^2_1$ *(*$\widetilde{\mathbb{H}^2_1}$*) in the case* $K > 0$ *(resp.* $K < 0$*). This is more complicated since in this case the formulas analogous to* (11.7) *are more involved.*

12 Examples

In this section we present detailed results for the problems P_1–P_3 defined in Example 5.

12.1 Problem P_1

In this case $K < 0$. The causal future of the point Id is $J^+ = \exp(\mathcal{U}) = \{(x, y) \in G \mid x \geq |y - 1|\}$, see Fig. 8.

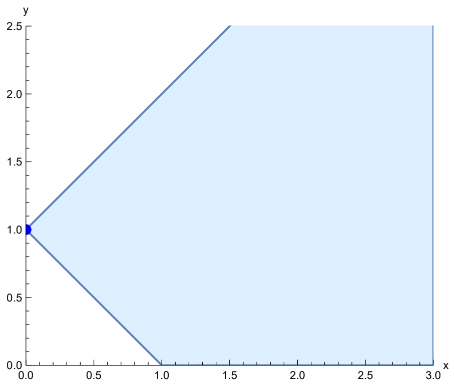

Figure 8: J^+ for the problem P_1

The group G is not globally hyperbolic since for $q_1 = (x_1, y_1) \in G$ with $x_1 > y_1 + 1$ the intersection $J^+(\mathrm{Id}) \cap J^-(q_1)$ is not compact, see Fig. 9. Although, the domain $\mathrm{int}\, D = \{(x, y) \in G \mid x > |y - 1|,\ x < y + 1\}$ is globally hyperbolic, see Fig. 10.

Theorem 21. *Let* $q_1 = (x_1, y_1) \in M \setminus \{\mathrm{Id}\}$ *for the problem* P_1.

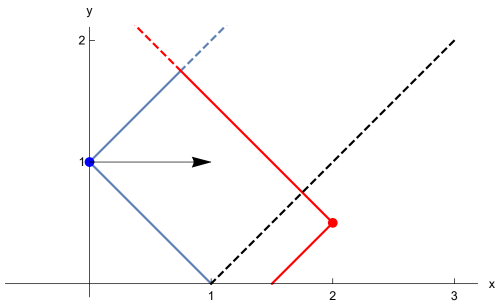

Figure 9: Problem P_1: G is not globally hyperbolic

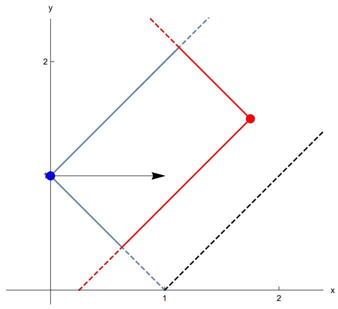

Figure 10: Problem P_1: D is globally hyperbolic

(1) *If* $x_1 = |y_1 - 1|$, *then* $x(t) = \pm(e^{\pm t} - 1)$, $y(t) = e^{\pm t}$, $\pm = \operatorname{sgn}(y_1 - 1)$, $t_1 = \pm \ln y_1$, $d(q_1) = 0$.

(2) *If* $x_1 > |y_1 - 1|$, *then*

$$x(t) = \cos\rho(\tan\tau - \tan\rho), \qquad y(t) = \frac{\cos\rho}{\cos\tau}, \qquad \tau = \rho + t, \qquad t_1 = \tau - \rho = d(q_1),$$
$$\tau = \arcsin\frac{x_1^2 + y_1^2 - 1}{2x_1 y_1}, \qquad \rho = \arcsin\frac{y_1^2 - x_1^2 - 1}{2x_1},$$

the curve $(x(t), y(t))$ *is an arc of the hyperbola* $y^2 - (x - \sin\rho)^2 = \cos^2\rho$.

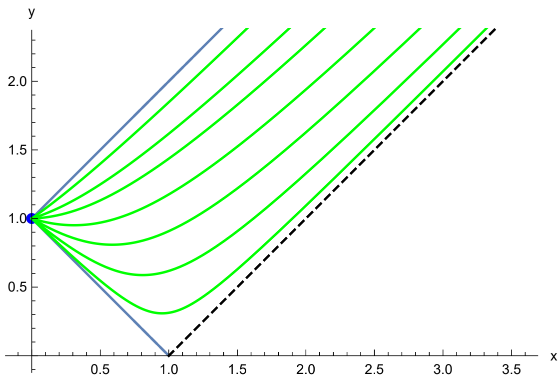

Figure 11: Lorentzian length maximizers in P_1

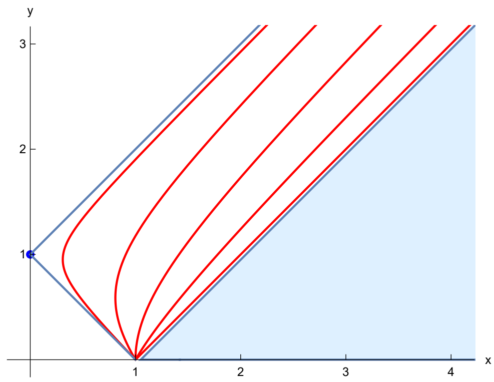

Figure 12: Lorentzian spheres in P_1

12.2 Problem P_2

In this case $K > 0$.

Theorem 22. *Let* $q_1 = (x_1, y_1) \in J^+ \setminus \{q_0\}$ *for the problem* P_2.

(1) *If* $y_1 - 1 = |x_1|$, *then* $x(t) = \pm(e^t - 1)$, $y(t) = e^t$, $\pm = \operatorname{sgn} x_1$, $t_1 = \ln y_1$, $d(q_1) = 0$.

(2) *If* $x_1 = 0$, *then* $x(t) \equiv 0$, $y(t) = e^t$, $t_1 = \ln y_1 = d(q_1)$.

(3) *If* $0 < |x_1| < y_1 - 1$, *then*

$$x(t) = \pm(\sinh\rho\coth\tau - \cosh\rho), \quad y(t) = \frac{\sinh\rho}{\sinh\tau}, \qquad \pm = \operatorname{sgn} x_1, \qquad \tau = \rho - t,$$
$$\rho = \operatorname{arcosh}\frac{1 + x_1^2 - y_1^2}{2|x_1|}, \qquad \tau = \operatorname{arcosh}\frac{1 - x_1^2 - y_1^2}{2|x_1|y_1}, \qquad t_1 = \rho - \tau = d(q_1),$$

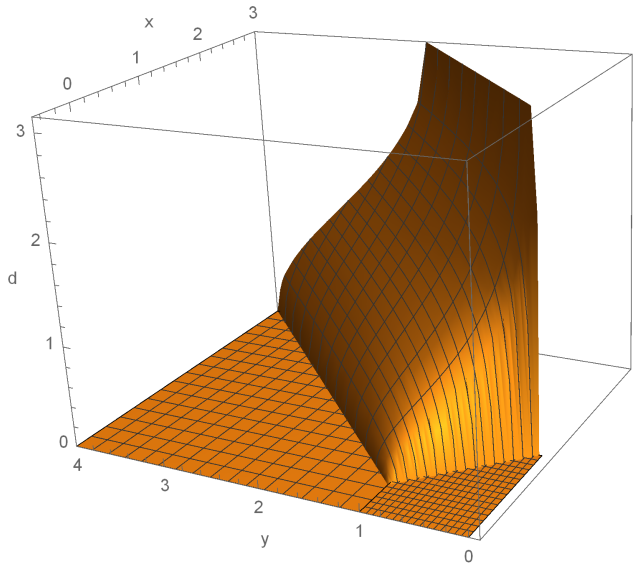

Figure 13: Plot of Lorentzian distance in P_1

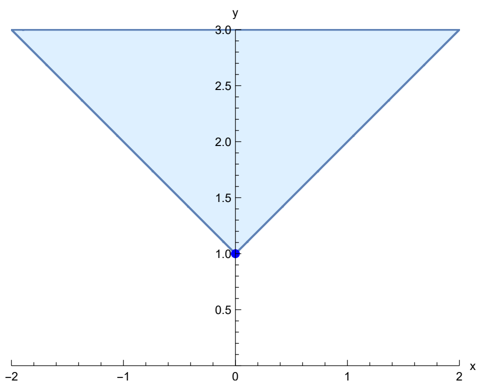

Figure 14: J^+ for the problem P_2

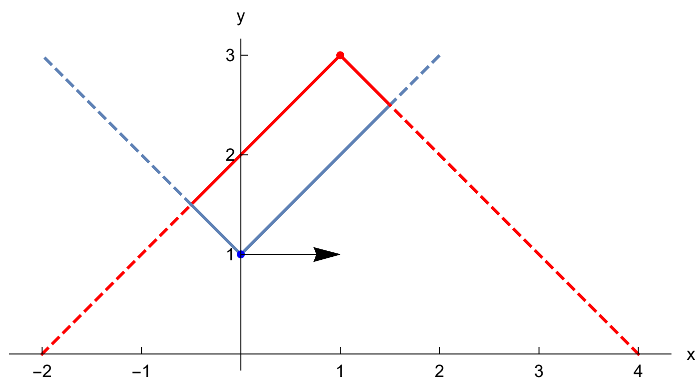

Figure 15: Problem P_2: G is globally hyperbolic

is the arc of the hyperbola $(\pm x + \cosh \rho)^2 - y^2 = \sinh^2 \rho$.

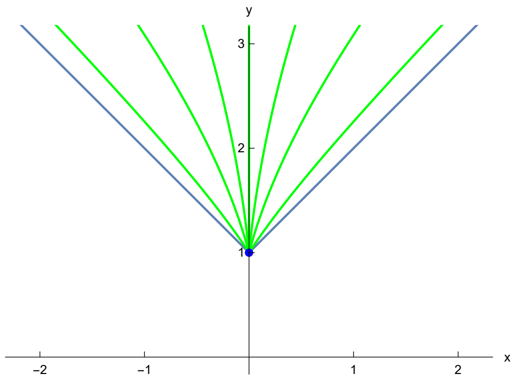

Figure 16: Lorentzian length maximizers in P_2

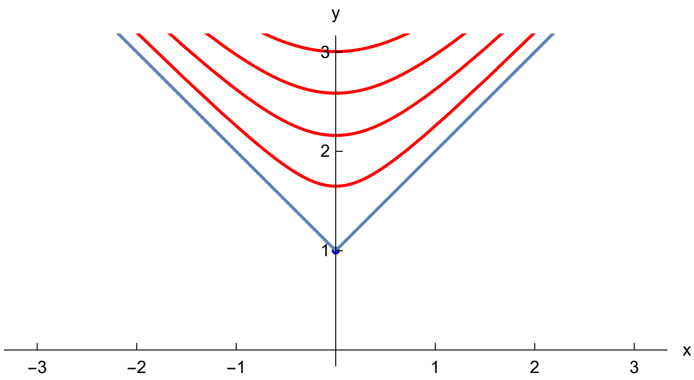

Figure 17: Lorentzian spheres in P_2

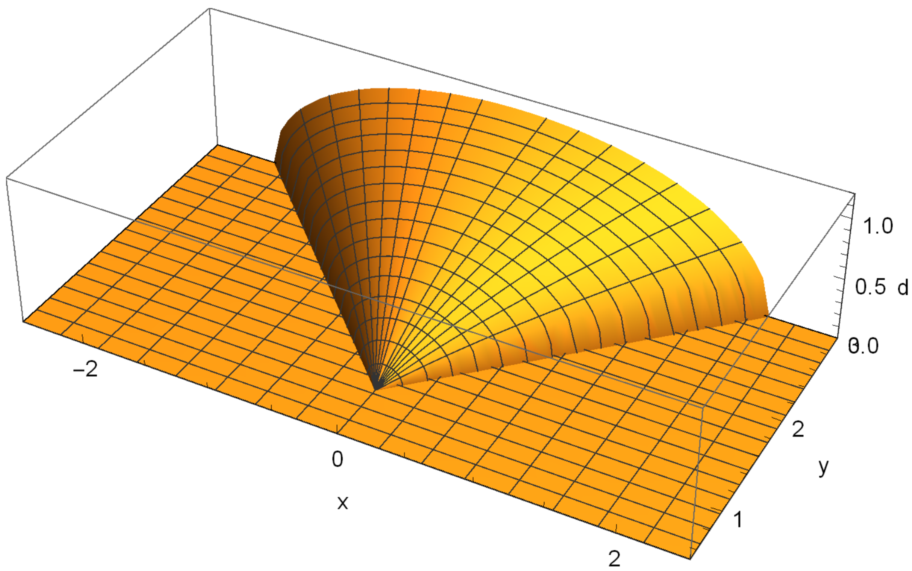

Figure 18: Plot of Lorentzian distance in P_2

12.3 Problem P_3

In this case $K = 0$.

Theorem 23. *Let* $q_1 = (x_1, y_1) \in J^+ \setminus \{q_0\}$ *for the problem* P_3.

(1) *If* $x_1 = 0$, *then* $x(t) \equiv 0$, $y(t) = e^t$, $t_1 = \ln y_1$, $d(q_1) = 0$.

(2) *If* $y_1 = 1$, *then* $x(t) = t$, $y(t) \equiv 1$, $t_1 = x_1$, $d(q_1) = 0$.

(3) *If* $x_1 > 0$ *and* $y_1 > 1$, *then* $x(t) = \rho(\rho - \tau)$, $y(t) = \frac{\rho}{\tau}$,

$$\tau = \rho - t, \qquad \rho = \sqrt{\frac{x_1 y_1}{y_1 - 1}}, \qquad \tau = \sqrt{\frac{x_1}{y_1(y_1 - 1)}}, \qquad t_1 = \rho - \tau = d(q_1),$$

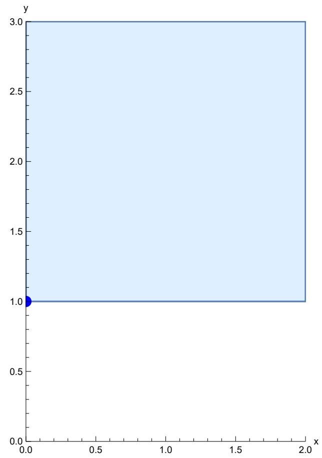

Figure 19: J^+ for the problem P_3

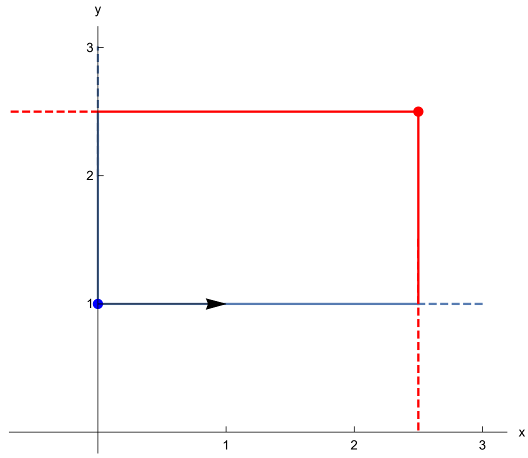

Figure 20: Problem P_3: G is globally hyperbolic

is the arc of the hyperbola $x = \rho^2 \left(1 - \frac{1}{y}\right)$.

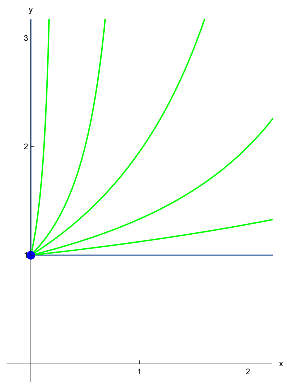

Figure 21: Lorentzian length maximizers in P_3

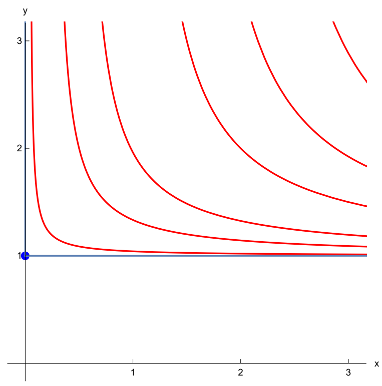

Figure 22: Lorentzian spheres in P_3

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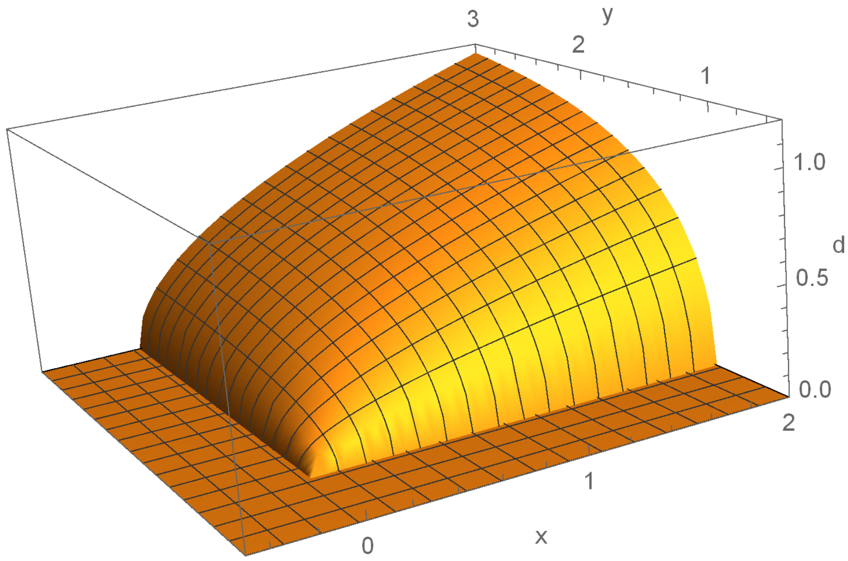

Figure 23: Plot of Lorentzian distance in P_3

References

[1] A. Agrachev, Yu. Sachkov, *Control theory from the geometric viewpoint*, Berlin Heidelberg New York Tokyo. Springer-Verlag, 2004.

[2] Yu. Sachkov, *Introduction to geometric control*, Springer, 2022.

[3] L.S. Pontryagin, V. G. Boltyanskii, R. V. Gamkrelidze, E.F. Mishchenko, *Mathematical Theory of Optimal Processes*, New York/London. John Wiley & Sons, 1962.

[4] Beem, J.K., Ehrlich, P.E., Easley, K.L.: *Global Lorentzian Geometry*. Monographs Textbooks Pure Appl. Math. 202, Marcel Dekker Inc. (1996)

[5] Müller, O., Sànchez, M. An Invitation to Lorentzian Geometry. *Jahresber. Dtsch. Math. Ver.* 115, 153–183 (2014)

[6] Wald, R.M.: *General Relativity*, Univ. Chicago Press (1984)

[7] N. Jacobson, *Lie Algebras*, Dover Publications, New York, 1979

[8] H. S. M. Coxeter, *Non-Euclidean Geometry*, Mathematical Association of America, 1998

[9] S. Stahl, *The Poincaré Half-Plane*, Jones and Barlett Publishers, 1993

[10] Krantz S. G., Parks H. R., *The Implicit Function Theorem: History, Theory, and Applications*, Birkauser, 2001.

[11] Yu. Sachkov, Lorentzian geometry on the Lobachevsky plane, *Math. Notes*, 114:1 (2023), 127–130

[12] J.A. Wolf, *Spaces of constant curvature*, AMS, 2011.

[13] B. O'Neill, *Semi-Riemannian geometry*, Academic Press, 1983.

[14] I. P. Costa e Silva and J. L. Flores. Geodesic connectedness of affine manifolds. *Ann. Mat. Pura Appl.* (4), 200(3):1135–1148, 2021.

[15] M. Sànchez, Geodesic connectedness of semi-Riemannian manifolds, *Nonlinear Analysis* 47, 3085–3102 (2001)

[16] M. Grochowski, Geodesics in the sub-Lorentzian geometry. *Bull. Polish. Acad. Sci. Math.*, 50 (2002).

[17] M. Grochowski, Reachable sets for the Heisenberg sub-Lorentzian structure on $\mathbb{R}^3$. An estimate for the distance function. *Journal of Dynamical and Control Systems*, vol. 12, 2006, 2, 145–160.

[18] Yu. L. Sachkov, E.F. Sachkova, Sub-Lorentzian distance and spheres on the Heisenberg group, *Journal of Dynamical and Control Systems*, volume 29, pages 1129–1159 (2023)