Geodesic Flow of the Sub-Riemannian Structure of Engel Type with Strictly Abnormal Extremals

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***Abstract*—We consider a left-invariant sub-Riemannian problem of Engel type on the central extension of the special linear group. Interest in this problem comes from the fact that it has strictly abnormal trajectories, and the normal geodesic flow is Liouville integrable. An extremal trajectory is called strictly abnormal if it is not present among normal geodesics. It is known that the most complicated singularities of the sub-Riemannian metric arise near abnormal trajectories. The presence of a strictly abnormal trajectory in combination with the integrability of the normal geodesic flow makes the problem under consideration a model example for studying the singularities of the sub-Riemannian metric. We apply to the problem the invariant formulation of Pontryagin maximum principle (PMP), in which the vertical subsystem (for adjoint variables) of the Hamiltonian system of PMP is independent of the state variables. We show that the vertical subsystem is reduced to the equation of a skewed pendulum. The first integrals of the system are found and an explicit solution is obtained in a special case. In the general case, we carry out a qualitative analysis of the phase flow of the Hamiltonian system.**

***Index Terms*—sub-Riemannian geometry, Engel type sub-Riemannian structure, strict abnormal geodesic, Pontryagin maximum principle**

I. Introduction

This paper makes a small step in the study of singularities for sub-Riemannian metrics. Strong interest in the investigation of sub-Riemannian structures is due not only to mathematical reasons but also to applications for nonholonomic mechanics and modelling of the human vision. For general introduction to sub-Riemannian geometry we refer to the recent book [1]. Below we recall some necessary definitions and give a motivation for this work.

A sub-Riemannian manifold is a triple (M, Δ, g), where M is a connected smooth manifold, $\Delta \subset TM$ is a subbundle of the tangent bundle (called *a distribution*) and g is a scalar product on Δ. A Lipschitzian curve $\gamma : [0, T] \to M$ is called *an admissible curve* if it is tangent to the distribution Δ a.e. *A sub-Riemannian length* of an admissible curve $\gamma(\,\cdot\,)$ is

$$l(\gamma) = \int\limits_0^T \sqrt{g_{\gamma(t)}(\dot{\gamma}(t), \dot{\gamma}(t))}\, \mathrm{d}t. \tag{1}$$

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We assume that the distribution Δ satisfies the Hörmander condition, i.e., the Lie brackets of the vector fields tangent to Δ span the whole tangent bundle TM. In this case any two points can be connected by an admissible curve due to the Rashevskii-Chow theorem. Define the sub-Riemannian distance between points $x, y \in M$ as the infimum for the lengths of the admissible curves connecting the point x and y. Thus, the manifold M becomes a metric space. Despite the fact that the corresponding topology is equivalent to the topology defined by a Riemannian structure, the sub-Riemannian metric is quite different from the Riemannian one. In particular, sub-Riemannian spheres even of small radius have singularities. One of the reasons for this phenomenon is the existence of so called abnormal geodesics. These geodesics do not depend on the scalar product on the sub-Riemannian distribution, but depend on the distribution itself. Notice that an admissible curve can be simultaneously a normal and an abnormal geodesic. So, we are interested in strict abnormal geodesics for the study of singularities.

Let us introduce a kind of a complexity measure for a sub-Riemannian structure. Define recursively $\Delta^1 = \Delta$, $\Delta^{i+1} = \Delta^i + [\Delta, \Delta^i]$. Since the distribution satisfies the Hörmander condition, there exists $s \in \mathbb{N}$ such that $\Delta^s = TM$. The tuple of numbers $(\dim \Delta^i)_{i=1}^s$ is called *the growth vector* of the distribution Δ. Important particular cases of sub-Riemannian structures are left-invariant sub-Riemannian structures on Lie groups. Obviously, in the left-invariant case the growth vector does not depend on the point of a sub-Riemannian manifold.

A left-invariant sub-Riemannian structure with the growth vector $(2, 3, 4)$ is called *an Engel type sub-Riemannian structure*. Such structures were classified by D. Almeida [2] . Then I. Beschastnyi and A. Medvedev [3] found an Engel type sub-Riemannian structure such that the corresponding geodesic flow is Liouville integrable and there exists a strict abnormal geodesic. This is the simplest sub-Riemannian structure having these properties. This allows us to study singularities of the geodesic flow near the strict abnormal geodesic in this particular case. Hopefully, this will help to shed some light on one of the most important problems in sub-Riemannian geometry: the study of possible singularities of the geodesic flow, the sub-Riemannian metric and the spheres.

The paper has the following structure. In Section II we introduce the sub-Riemannian structure under consideration

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and formulate the corresponding optimal control problem. Next, we use the Hamiltonian approach to describe geodesics. Namely, we apply the Pontryagin maximum principle and derive the Hamiltonian system of ODEs for geodesics in Section III. We analyze the adjoint subsystem of the Pontryagin Hamiltonian system in Section IV. Then we obtain an explicit solution of the adjoint subsystem in a special case in Section V.

II. Problem formulation

Let G be a connected Lie group, and $\mathfrak{L}$ be the Lie algebra of left-invariant vector fields on G. A left-invariant sub-Riemannian (SR) structure can be defined via an orthonormal frame $X_1,\ X_2 \in \mathfrak{L}$ as

$$\Delta_q = \operatorname{span}(X_1(q), X_2(q)), \quad \langle X_i, X_j \rangle = \delta_{ij},\ i,j = 1,2,$$

where $q \in G$ and δ_{ij} is the Kronecker delta.

A length minimizer can be found as a solution to the following optimal control problem [4]:

$$\begin{aligned} &\dot\gamma(t) = u_1(t)\,X_1(\gamma(t)) + u_2(t)\,X_2(\gamma(t)),\\ &\gamma(0) = q^0, \quad \gamma(T) = q^1, \quad l(\gamma) \to \min, \end{aligned} \tag{2}$$

where X_1, X_2 are basis left-invariant vector fields tangent to the distribution Δ that are orthonormal w.r.t. the scalar product $g(\,\cdot\,,\,\cdot\,)$, the controls u_1, u_2 are real-valued L_∞-functions. and the length $l(\gamma)$ defined by (1) is equal to

$$l(\gamma) = \int_0^T \sqrt{u_1^2(t) + u_2^2(t)}\,\mathrm{d}t.$$

Remark 1: Since problem (2) is left–invariant, without loss of generality one can chose q^0 as identity of the group. The solution for arbitrary q^0 is obtained by a left shift.

Remark 2: Due to the Cauchy-Schwarz inequality [4], minimization of $l(\gamma)$ is equivalent to minimization of the action

$$E(\gamma) = \frac{1}{2}\int_0^T (u_1^2(t) + u_2^2(t))\,\mathrm{d}t.$$

In this work, we study the problem of finding length minimizers on the Lie group $\overline{\mathrm{SL}}_2$ that is the central extension of the special linear group SL_2 over real numbers $\mathbb{R}$:

$$q \in \overline{\mathrm{SL}}_2 = \left\{ \left(\begin{array}{ccc} x & y & 0 \\ z & w & 0 \\ 0 & 0 & C \end{array} \right) \middle| \begin{array}{l} x,y,z,w,C \in \mathbb{R}, \\ xw - yz = 1 \end{array} \right\}.$$

We consider the following basis left-invariant vector fields:

$$X_1(q) = (L_q)_* \left(\partial_y - \frac{1}{2}\partial_C \right)\Big|_e,\ X_2(q) = (L_q)_* \left(\frac{T_4}{2}\partial_z \right)\Big|_e,$$

where $(L_q)_*$ is push–forward under left translation $L_q h = qh$, e is identity of the group, and $T_4 > 0$ is a constant parameter.

In coordinates $(x,y,z,w,C) \in \mathbb{R}^5$ we have

$$X_1(q) = x\partial_y + z\partial_w + \frac{C}{2}\partial_C,\ X_2(q) = \frac{T_4}{2}y\partial_x + \frac{T_4}{2}w\partial_z.$$

Thus, we study the following optimal control problem:

$$\begin{cases} \dot x = \frac{T_4}{2} y u_2, & x(0) = w(0) = C(0) = 1,\\ \dot y = x u_1, & y(0) = z(0) = 0,\\ \dot z = \frac{T_4}{2} w u_2, & x(T) = x^1,\ y(T) = y^1,\ z(T) = z^1,\\ & w(T) = w^1,\ C(T) = C^1,\\ \dot w = z u_1, & \\ \dot C = \frac{1}{2} C u_1, & \frac{1}{2}\int_0^T (u_1^2(t) + u_2^2(t))\,\mathrm{d}t \to \min. \end{cases} \tag{3}$$

Proposition 1: The control system (3) is completely controllable: for any point $q^1 = (x^1, y^1, z^1, w^1, C^1)$, satisfying $x^1 w^1 - y^1 z^1 = 1$, there exists controls $u_1(t), u_2(t)$, such that the corresponding trajectory arrives to q^1 for the time $T > 0$.

Proof: Complete controllability of the system is guaranteed by the Rashevskii–Chow theorem [4]. Indeed, the vector fields X_1, X_2, $X_3 = [X_1, X_2]$, $X_4 = [X_1, X_3] + T_4 X_1$ form a basis of the Lie algebra $\mathfrak{L}$ of $\overline{\mathrm{SL}}_2$. ∎

Further, a question of existence of optimal trajectories arises: does there always exist an admissible trajectory satisfying the boundary conditions, on which the minimizing functional reaches its minimum value? For our problem (3) the answer is positive. Existence of optimal trajectories is guaranteed by the Filippov theorem [4].

III. Pontryagin maximum principle

Introduce the following family of functions on the cotangent bundle T^*G depending on the parameters $u = (u_1, u_2)$ and ν:

$$H_u^\nu(\lambda) = u_1 h_1(\lambda) + u_2 h_2(\lambda) - \frac{\nu}{2}(u_1^2 + u_2^2), \qquad \lambda \in T^*G,$$

where $h_i = \langle\,\cdot\,, X_i\rangle$ for $i = 1, \ldots, 4$ are linear on the fibers of the cotangent bundle functions. The Pontryagin maximum principle [4], [5] gives necessary conditions of optimality.

Theorem 1: If $\tilde q(\,\cdot\,)$ and $\tilde u(\,\cdot\,)$ are an optimal process for problem (3), then there exist a curve $\lambda \in \operatorname{Lip}([0,T], T^*G)$, $\pi(\lambda(t)) = \tilde q(t)$ and a number $\nu \in \{0,1\}$ such that for a.e. $t \in [0,T]$ we have

$$\begin{aligned} &\dot\lambda(t) = \vec H_{\tilde u(t)}(\lambda(t)),\\ &H^\nu_{\tilde u(t)}(\lambda(t)) = \max_{u \in \mathbb{R}^2} H_u^\nu(\lambda(t)),\\ &(\lambda(t), \nu) \neq 0, \end{aligned}$$

where $\pi : T^*G \to G$ is the natural projection and $\vec H$ is the Hamiltonian vector field corresponding to a Hamiltonian H.

The curve $\lambda(\,\cdot\,)$ is called *an extremal* and its projection $\pi(\lambda(\,\cdot\,))$ is called *an extremal trajectory*. If sufficient small arcs of an extremal trajectory are optimal, then this trajectory is called *a geodesic*. In the case $\nu = 0$, the corresponding extremal and the corresponding geodesic are called *abnormal*. In the case $\nu = 1$, the corresponding extremal and geodesic are called *normal*. An abnormal geodesic that is not equal to the projection of any normal extremal is called *a strict abnormal geodesic*.

The vertical (adjoint) subsystem of the Hamiltonian system of the Pontryagin maximum principle reads as $\dot h_i = \{H_u^\nu, h_i\}$ for $i = 1, \ldots, 4$, where $\{\,\cdot\,,\,\cdot\,\}$ is the canonical Poisson bracket on T^*G. Computation of the Poisson brackets gives

$$\begin{aligned} &\dot h_1 = -u_2 h_3, \quad \dot h_3 = u_1 h_4 - T_4(u_1 h_1 - u_2 h_2),\\ &\dot h_2 = u_1 h_3, \qquad \dot h_4 = 0. \end{aligned} \tag{4}$$

From the results of paper [3] it follows that the abnormal extremal trajectory with $u_1 = 0$ and $u_2 = 1$ is strict abnormal geodesic. Thus, from (3) we get its parametrization:

$$x = 1, \quad y = 0, \quad z = e^{\frac{T_4}{2}t} - 1, \quad w = 1, \quad C = 1. \tag{5}$$

IV. Qualitative analysis of the Hamiltonian system in the general case

It follows from the maximum condition that in the normal case $u_1 = h_1$, $u_2 = h_2$. So, using (4) we obtain that the vertical subsystem of the normal Hamiltonian system reads as

$$\begin{aligned} \dot{h}_1 &= -h_2 h_3, \quad & \dot{h}_3 &= h_1 h_4 - T_4(h_1^2 - h_2^2), \\ \dot{h}_2 &= h_1 h_3, & \dot{h}_4 &= 0. \end{aligned} \tag{6}$$

This system has two obvious first integrals: $h_4 = \text{const}$ and the Hamiltonian $H = \frac{1}{2}(h_1^2 + h_2^2)$. Trajectories with unit velocities correspond to the level surface of the Hamiltonian $H = \frac{1}{2}$. Introduce the polar angle $\theta \in S^1$ as $h_1 = \cos\theta$, $h_2 = \sin\theta$. System (6) reduces to the system of a skewed pendulum

$$\begin{cases} \dot{\theta} = h_3, & \theta(0) = \theta^0 \\ \dot{h}_3 = h_4 \cos\theta - T_4 \cos 2\theta, & h_3(0) = h_3^0. \end{cases} \tag{7}$$

System (7) is a conservative system of one degree of freedom [6]. Such systems have the first integral – the total energy

$$G = \frac{1}{2}h_3^2 - (h_4 - T_4\cos\theta)\sin\theta = \frac{1}{2}h_3^2 - h_4 h_2 + T_4 h_1 h_2.$$

Proposition 2: The trajectories of (6) are the curves of intersection of the cylinder $H = \frac{1}{2}$ and the surface $G = \text{const}$, which is a two-sheeted hyperboloid for $G > 0$, a one-sheeted hyperboloid for $G < 0$ and a cone for $G = 0$. The apex of the cone is at the point $h_1 = \frac{h_4}{T_4}$, $h_2 = h_3 = 0$ and the generatrix is parallel to the axis h_2.

Proof: The quadratic form $\frac{1}{2}h_3^2 - h_4 h_2 + T_4 h_1 h_2$ is reduced to canonical form $\frac{1}{2}h_3^2 + T_4(h_1' - \frac{h_4}{2T_4})^2 - T_4(h_2' - \frac{h_4}{2T_4})^2$, where $h_1' = \frac{1}{2}(h_1 + h_2)$ and $h_2' = \frac{1}{2}(h_1 - h_2)$. ■

Fixed points of system (7) are determined by the condition $\dot{h}_3 = \dot{\theta} = 0$. By solving the equation $h_4\cos\theta - T_4\cos 2\theta = 0$, we see that the fixed points have the form

$$h_3 = 0, \quad \theta = \arctan\left(\frac{s_1\sqrt{B}}{h_4 + s_2 A}\right), \tag{8}$$

where $A = \sqrt{h_4^2 + 8T_4^2} > 0$, $B = 8T_4^2 + 2s_1 h_4(A - s_1 h_4) \geq 0$ and s_1, s_2 take values ± 1.

Note that $B = 0$ iff $|h_4| = T_4$. By analyzing the condition $B \geq 0$ and the type of extremum of G at the fixed point we obtain the following proposition.

Proposition 3: System (6) has the following fixed points (see. Fig. 1):

(1) Two centers and two saddles for $|h_4| < T_4$.
(2) Center, saddle and cusp for $|h_4| = T_4$.
(3) A center and a saddle for $|h_4| > T_4$.

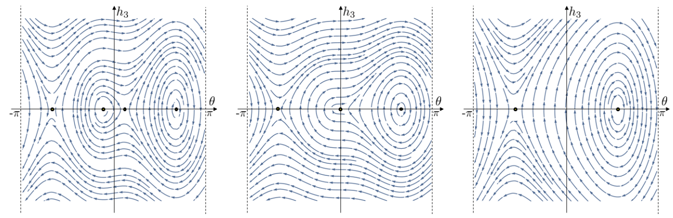

Fig. 1. Phase portrait of a skewed pendulum. From left to right: four ($|h_4| < T_4$), three ($|h_4| = T_4$) and two ($|h_4| > T_4$) fixed points.

V. Special case $h_4 = 0$

In this section we provide an explicit formula for extremal controls in the special case $h_4 = 0$. Note, that in this case system (6) is reduced to the system of mathematical pendulum

$$\begin{cases} \dot{\theta} = h_3, & \theta(0) = \theta^0 = \arg(h_1^0 + \boldsymbol{i} h_2^0) \\ \dot{h}_3 = -T_4\cos 2\theta, & h_3(0) = h_3^0. \end{cases} \tag{9}$$

This system is integrated in Jacobi elliptic functions. Explicit solution in terms of rectified coordinates can be found in [7].

Alternatively, one can derive explicit expression for $h_3(t)$ as a solution to the polynomial ODE and then obtain $\theta(t)$ as integral of $h_3(t)$. By virtue of (9), we have

$$\ddot{h}_3 - 2Gh_3 + h_3^3 = 0,$$

with initial conditions $h_3(0) = h_3^0$, $\dot{h}_3(0) = -T_4\cos 2\theta^0$.

An explicit solution of this Cauchy problem in general case, see [9, App. A], is given by $h_3(t) = A\,\mathrm{cn}(\Omega(t - t_0), k)$, where the constant parameters A, Ω, t_0 and k are determined by the initial conditions and the constant T_4.

The polar angle $\theta(t)$ is obtained by

$$\theta(t) = \theta^0 + \int_0^t h_3(\tau)\mathrm{d}\tau.$$

The last integral admits expression in Jacobi elliptic functions.

The extremal controls are given by $u_1(t) = \cos\theta(t)$, $u_2(t) = \sin\theta(t)$.

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